Negative Definite Functions on Hypercomplex Systems

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Abstract. We present a concept of negative definite functions on a commutative normal hypercomplex system $L_1(Q, m)$ with basis unity. Negative definite functions were studied in [5] and [4] for commutative groups and semigroups respectively. The definition of such functions on $Q$ is a natural generalization of that defined on a commutative hypergroups.

1. Preliminaries

Let $Q$ be a complete separable locally compact metric space of points $p, q, r, \cdots$; $\beta(Q)$ is the $\sigma$-algebra of Borel subsets, and $B_0(Q)$ is the subring of $B(Q)$, which consists of sets with compact closure. We shall consider the Borel measures; i.e., positive regular measures on $B(Q)$, finite on compact sets. We denote by $C(Q)$ the space of continuous functions on $Q$: $C_b(Q)$, $C_\infty(Q)$ and $C_0(Q)$ consists respectively of bounded, tending to zero at infinity and compactly supported functions from $C(Q)$.

A hypercomplex system with the basis $Q$ is defined by its structure measure $c(A, B, r)$ ($A, B \in B(Q); r \in Q$). A structure measure $c(A, B, r)$ is a Borel measure in $A$ (respectively $B$) if we fix $B, r$ (respectively $A, r$) which satisfies the following properties:

(H1) \( \forall A, B \in \beta_0(Q), \) the function $c(A, B, r) \in C_0(Q)$,

(H2) \( \forall A, B \in \beta_0(Q) \) and $s, r \in Q$, the following associativity relation holds

\[
\int_Q c(A, B, r) d_r c(E_r, C, s) = \int_Q C(B, C, r) d_r C(A, E_r, s), \quad C \in B(Q).
\]
The structure measure is said to be commutative if
\[ c(A, B, r) = c(B, A, r), \quad (A, B \in B_0(Q)). \]

A measure \( m \) is said to be a multiplicative measure if
\[ \int_Q c(A, B, r)\,dm(r) = m(A)m(B); \quad A, B \in \beta_0(Q). \]

We will suppose the existence of a multiplicative measure. Under certain restrictions imposed on the commutative structure measure, multiplicative measure exists. (See [10]).

Consider the space \( L_1(Q, m) = L_1 \) of functions on \( Q \) with respect to the multiplicative measure \( m \).

**Theorem 1.1.** For any \( f, g \in L_1(Q, m) \), the convolution
\[ (f \ast g)(r) = \int_Q f(p)d_p \int_Q g(q)d_q c(E_p, E_q, r) \]
\[ = \int_Q \int_Q f(p)g(q)c(p, q, r)\,dm(p)\,dm(q) \]
\[ = \int_Q \int_Q f(p)g(q)d_{r}(p, q) \]
is well defined. (See [2]).

The space \( L_1(Q, m) \) with the convolution (1.1) is a Banach algebra which is commutative if (H3) holds. This Banach algebra is called the hypercomplex system with the basis \( Q \).

It is obvious that \( C(A, B, r) = (K_A \ast K_B)(r); \ A, B \in \beta_0(Q) \) and \( K_A \) is the characteristic function of the set \( A \).

A hypercomplex system may or may not have a unity. If a unity not included in \( L_1(Q, m) \), then it is convenient to join it formally to \( L_1 \).

A non zero measurable and bounded almost everywhere function \( Q \ni r \to \chi(r) \in C \) is said to be a character of the hypercomplex system \( L_1 \) if \( \forall A, B \in \beta_0(Q) \)
\[ \int_Q c(A, B, r)\chi(r)dm(r) = \chi(A)\chi(B), \]
\[ \int \chi(r)dm(r) = \chi(C), \quad C \in \beta_0(Q). \]

A hypercomplex system is said to be normal, if there exists an involution homomorphism \( Q \ni r \to r^* \in Q \), such that \( m(A) = m(A^*) \) and \( c(A, B, C) = c(C, B^*, A), c(A, B, C) = c(A^*, C, B), (A, B \in \beta_0(Q)) \), where
\[ c(A, B, C) = \int_C c(A, B, r)dm(r). \]
(H6) A normal hypercomplex system possesses a basis unity if there exists a point \( e \in Q \) such that \( e^* = e \) and
\[
c(A, B, e) = m(A^* \cap B), \quad A, B \in \beta(Q).
\]
We should remark that, for a normal hypercomplex system, the mapping
\[
L_1(Q, m) \ni f(r) \rightarrow f^*(r) \in L_1(Q, m)
\]
is an involution in the Banach algebra \( L_1 \), the multiplicative measure is unique and the characters of such a system are continuous. (See [1]). A character \( \chi \) of a normal hypercomplex system is said to be Hermitian if
\[
\chi(r^*) = \overline{\chi(r)}, \quad (r \in Q).
\]
Denote the set of all bounded Hermitian characters by \( X_h \), i.e.,
\[
X_h = \{ \chi \in C_b(Q) : \chi \neq 0, \int c(A, B, r)\chi(r)dm(r) = \chi(A)\chi(B), \overline{\chi(r)} = \chi(r^*) \}.
\]
Let \( L_1(Q, m) \) be a hypercomplex system with compact basis, \( \hat{Q} \) be a dual countable basis (collection of all characters \( \chi, \phi, \psi, \cdots \)), and \( \hat{m} \) be a Plancherel measure. The space \( L_1(\hat{Q}, \hat{m}) = l_1(\hat{m}) \) becomes a hypercomplex system with discrete basis if we define a dual structure measure \( \hat{c} \) by the formula
\[
\hat{c}(\chi, \phi, \psi) = \hat{m}(\chi)\hat{m}(\phi)\int_Q \chi(r)\phi(r)\psi(r)dm(r), \quad (\chi, \phi, \psi \in \hat{Q})
\]
and assume that the integral in (1.2) is nonnegative.

This dual hypercomplex system is normal if we set \( \chi^* = \overline{\chi} \), and it has a basis unity \( \hat{e} \equiv 1 \). See [2].

2. Generalized translation operators and hypercomplex system

In a series of works originated as early as in 1938, J. Delsarte [7], [8], and then B. M. Levitan [11], [12] noticed that some facts of classical harmonic analysis can be generalized by replacing exponential functions \( e^{i\lambda q} \) (\( q, \lambda \in \mathbb{R}^1 \)) by some family of complex-valued functions \( \chi(q, \lambda) \) which inherit the following property of the indicated exponential functions. The exponential functions are connected with the family of ordinary translation operators \( R_p \) (\( p \in \mathbb{R}^1 \)) acting upon complex-valued functions \( f(q) \) (\( q \in \mathbb{R}^1 \)) according to the rule
\[
(R_pf)(q) = f(p + q),
\]
i.e.,
\[
R_pe^{i\lambda q} = e^{i\lambda p}e^{i\lambda q}
\]
(2.1)
for any \( \lambda \).

For functions \( \chi(q, \lambda) \), where \( q \) varies in some set \( Q \) and \( \lambda \) in another set \( \hat{Q} \), there should exist a family of linear “generalized translation” operators \( R_p \) \((p \in Q)\), acting on the functions of the variable \( q \in Q \) such that an equality of (2.1) type is valid.

\[
(R_p \chi(., \lambda))(q) = \chi(p, \lambda)\chi(q, \lambda) \quad (p, q \in Q, \lambda \in \hat{Q}).
\]

It is natural that the family of such operators \( R_p \) should have some additional properties, similar to a usual shift. It was clear from [8], [12] that it is important to study not only generalized translations but a convolution of functions associated with these translations. So by analogy with the usual convolution

\[
(f \ast g)(q) = \int_R f(p)g(q - p)dp = \int_R f(p)(R_{-p}g)(q)dp,
\]

it is possible to introduce a generalized convolution \( \ast \) similar to (2.2), associated with the generalized translation operators:

\[
(f \ast g)(q) = \int_Q f(p)(R_{p}^\ast g)(q)dm(p)
\]

which is equivalent to the form (1.1).

In (2.3), the involution \( \ast \) in \( Q \) is used instead of the inverse in \( R \), and \( m \) is the multiplicative measure.

Let \( L_1(Q, m) \) be a hypercomplex system with a basis \( Q \) and \( \Phi \) be a space of complex-valued functions on \( Q \). Assume that an operator valued function \( Q \ni p \to R_p : \Phi \to \Phi \) is given such that the function \( g(p) = (R_pf)(q) \) belongs to \( \Phi \) for any \( f \in \Phi \) and any fixed \( q \in Q \). The operators \( R_p \) \((p \in Q)\) are called generalized translation operators, provided that the following axioms are satisfied:

1. **Associativity axiom:** The equality
   \[
   (R_q(R_pf))(r) = (R_q(R_pf))(r)
   \]
   holds for any elements \( p, q \in Q \).

2. **Existence of the identity element:** There exists an element \( e \in Q \) such that \( R_e \) is the identity in \( \Phi \). (See [3]).

### 3. Positive and negative definite functions on hypercomplex system

Let \( L_1(Q, m) \) be a commutative normal hypercomplex system with basis unity.

**Definition 3.1.** A continuous bounded function \( \varphi(r) \) \((r \in Q)\) is called positive definite if the inequality

\[
\sum_{i,j=1}^N \lambda_i \overline{\lambda_j}(R_{r_i \ast}^\ast \varphi)(r_j) \geq 0
\]
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holds for all \( r_1, \ldots, r_n \in Q \), and \( \lambda_1, \ldots, \lambda_n \in C \), \((n \in N)\).

If the generalized translation operators \( R_t \) extended to \( L_\infty \) map \( C_b(Q) \) into \( C_b(Q \times Q) \), then the inequality (3.1) of positive definiteness is equivalent for the functions \( \varphi(r) \in C_b(Q) \) to

\[
\int_Q \int_{Q} (R_t \varphi)(s^*) x(t) \overline{x(s)} dt ds \geq 0, \quad x \in L_1(Q, m).
\]

By \( P(Q) \), we shall denote the set of all continuous positive definite functions on \( Q \).

**Theorem 3.2.** Every function \( \varphi \in P(Q) \) admits a unique representation in the form of an integral

\[
\varphi(r) = \int_{X_h} \chi(r) d\mu(\chi), \quad \chi \in X_h,
\]

where \( \mu \) is a nonnegative finite regular measure on the space \( X_h \). Conversely, each function of the form (3.2) belongs to \( P(Q) \).

For the proof, see [1].

Theorem 3.2 is an analog of the Bochner theorem for hypercomplex systems.

**Corollary 3.3.** If \( \varphi \in P(Q) \); then the following properties holds:

1. \( \varphi(e) \geq 0; \)
2. \( \varphi(r^*) = \overline{\varphi(r)} \quad \forall r \in Q; \)
3. \( |\varphi(r)| \leq \varphi(e) \quad \forall r \in Q; \)
4. \( |R_s \varphi(t)|^2 \leq (R_{s^*} \varphi)(s)(R_{t^*} \varphi)(t); \)
5. \( |\varphi(s) - \varphi(t)|^2 \leq 2 \varphi(e)[\varphi(e) - \text{Re}(R_{s^*} \varphi)(t^*)] \quad (s, t \in Q). \)

**Definition 3.4.** A continuous bounded function \( \psi : Q \to C \) is called negative definite if for any \( r_1, \ldots, r_n \in Q \) and \( c_1, \ldots, c_n \in C \)

\[
\sum_{i,j=1}^{n} [\psi(r_i) + \overline{\psi(r_j)} - (R_{r_i^*} \psi)(r_j)] c_i \overline{c_j} \geq 0.
\]

For example each constant function, \( c \geq 0 \) is negative definite. Obviously the following holds for a negative definite function \( \psi \):

\[
\psi(e) \geq 0, \quad \psi(r) = \psi(r^*), \quad (R_r \ast \psi)(r) \in R \quad \text{and} \quad \psi(r) + \psi(r^*) \geq (R_{r^*} \psi)(r).
\]

Let us abbreviate the set of negative definite functions on \( Q \) by \( N(Q) \).

We note that \( \psi = \psi^* \), and \( \text{Re} \ \psi \) is non negative if \( (R_r \psi)(r) \geq 0. \)
Theorem 3.5. A function $\psi : Q \to C$ is negative definite if and only if the following conditions are satisfied:

\begin{align*}
(3.5) \quad & (i) \quad \psi(e) \geq 0, \quad \psi \text{ is continuous bounded function} \\
(3.6) \quad & (ii) \quad \psi(r) = \psi(r^*) \quad \text{for each} \quad r \in Q, \quad \text{and} \\
(3.7) \quad & (iii) \quad \text{if} \quad r_1, \ldots, r_n \in Q, \quad \text{and} \quad c_1, \ldots, c_n \in C \quad \text{with} \quad \sum_{i=1}^{n} c_i = 0, \quad \text{then} \\
& \sum_{i,j=1}^{n} (R_{r_j} \psi)(r_i)c_i \bar{c}_j \leq 0 \quad (3.7)
\end{align*}

holds.

Proof. Suppose first that $\psi \in N(Q)$. It is clear that (i) and (ii) are satisfied. Let $n \in \mathbb{N}; r_1, \ldots, r_n \in Q$ and $c_1, \ldots, c_n \in C$ be such that

$$\sum_{i=1}^{n} c_i = 0.$$

Then we find

$$0 \leq \sum_{i,j=1}^{n} \left( \psi(r_i) + \psi(r_j) - (R_{r_j} \psi)(r_i) \right) c_i \bar{c}_j$$

$$= \left( \sum_{j=1}^{n} c_j \right) \left( \sum_{i=1}^{n} \psi(r_i) c_i \right) + \left( \sum_{i=1}^{n} c_i \right) \left( \sum_{j=1}^{n} \psi(r_j) c_j \right) - \sum_{i,j=1}^{n} \left( R_{r_j} \psi \right)(r_i)c_i \bar{c}_j$$

$$= - \sum_{i,j=1}^{n} \left( R_{r_j} \psi \right)(r_i)c_i \bar{c}_j.$$

Then (iii) is satisfied.

Conversely, suppose that $\psi$ satisfies (i)-(iii), and consider $r_1, \ldots, r_n \in Q$ and $c_1, \ldots, c_n \in C$. For the $(n+1)$-tuples $e, r_1, \ldots, r_n \in Q$ and $c_0, c_1, \ldots, c_n \in C$, where

$$\sum_{i=0}^{n} c_i = 0$$

i.e.,

$$c_0 = - \sum_{i=1}^{n} c_i,$$
we get by (iii)

\[ 0 \geq \sum_{i,j=0}^n \left( R_{r^*_j} \psi \right) (r_i) c_i \overline{c_j} \]

\[ = \sum_{i,j=1}^n \left( R_{r^*_j} \psi \right) (r_i) c_i \overline{c_j} + c_0 \sum_{i=1}^n \psi (r_i) c_i + c_0 \sum_{j=1}^n \psi (r^*_j) \overline{c_j} + \psi (e) |c_0|^2 \]

\[ = \sum_{i,j=1}^n \left( (R_{r^*_j} \psi)(r_i) - \psi (r_i) - \psi (r^*_j) \right) c_i \overline{c_j} + \psi (e) |c_0|^2 \]

hence, using (i) and (ii), that

\[ \sum_{i,j=1}^n \left[ \psi (r_i) + \overline{\psi (r_j)} - (R_{r^*_j} \psi) (r_i) \right] c_i \overline{c_j} \geq \psi (e) |c_0|^2 \geq 0. \]

**Corollary 3.6.** Let \( \psi \) be a function on \( Q \)

(i) If \( \psi \in N(Q) \), then \( r \mapsto \psi (r) - \psi (e) \) is negative definite.

(ii) If \( \varphi \in P(Q) \), then \( r \mapsto \varphi (e) - \varphi (r) \) is negative definite.

**Proof.** We will use the Theorem 3.5.

(i). Conditions (3.5) and (3.6) are clearly satisfied. For the condition (3.7), let \( r_1, \cdots, r_n \in Q \) and \( c_1, \cdots, c_n \in C \) be given satisfying

\[ \sum_{i=1}^n c_i = 0. \]

Then we find

\[ \sum_{i,j=1}^n R_{r^*_j} (\psi (r_j) - \psi (e)) c_i \overline{c_j} = \sum_{i,j=1}^n \left( R_{r^*_j} \psi \right) (r_i) c_i \overline{c_j} - \psi (e) \left| \sum_{i=1}^n c_i \right|^2 \]

\[ = \sum_{i,j=1}^n \left( R_{r^*_j} \psi \right) (r_i) c_i \overline{c_j} - 0 \]

\[ = \sum_{i,j=1}^n \left( R_{r^*_j} \psi \right) (r_i) c_i \overline{c_j} \]

\[ \leq 0 \]

which proves the negative definiteness of \( \psi (r) - \psi (e) \).
(ii). Let \( r_1, \cdots, r_n \in Q \) and \( c_1, \cdots, c_n \in C \) be given satisfying
\[
\sum_{i=1}^{n} c_i = 0.
\]
Then we find
\[
\sum_{i,j=1}^{n} \left( \varphi(e) - (R_{r_i^*}\varphi)(r_j) \right) c_i c_j = - \sum_{i,j=1}^{n} \left( R_{r_i^*}\varphi(r_i) - \varphi(e) \right) c_i c_j
\]
\[
= - \sum_{i,j=1}^{n} \left( R_{r_i^*}\varphi \right)(r_i) c_i c_j + \varphi(e) \left| \sum_{i=1}^{n} c_i \right|^2
\]
\[
= - \sum_{i,j=1}^{n} \left( R_{r_i^*}\varphi \right)(r_i) c_i c_j
\]
\[
\leq 0
\]
because \( \varphi \in P(Q) \), and since the function \( \varphi(e) - \varphi(r) \) clearly satisfies (i) and (ii) of Theorem 3.5, it is negative definite. \( \square \)

Now, we state the definition of negative definiteness in another form.

A continuous bounded function \( \psi(r) \ (r \in Q) \) is called negative definite if the inequality
\[
(3.8) \quad \int_Q \int_Q \left( \psi(r) + \overline{\psi(s)} - (R_{s^*}\psi)(r) \right) x(r) \overline{x(s)} dr ds \geq 0
\]
holds for all \( x \in L_1 \).

If the generalized operators \( R_t \) extends to \( L_\infty \) map \( C_b(Q) \) into \( C_b(Q \times Q) \), then the definitions of negative definiteness (3.3) and (3.8) are equivalent for the function \( \psi(r) \in C_b(Q) \).

By the condition, we have \( (R_t \phi)(s^*) \in C_b(Q \times Q) \), then the last inequality (3.8) clearly implies (3.3). Let us prove the converse assertion. Let \( Q_n \) be an increasing sequence of compact sets covering the entire \( Q \). We consider a function \( y(r) \in C_0(Q) \) and set \( \lambda_i = y(r_i) \) in (3.7). This yields
\[
\sum_{i,j=1}^{n} \left( R_{r_i^*}\psi \right)(r_j) y(r_i) \overline{y(r_j)} \leq 0.
\]

By integrating this inequality with respect to each \( r_1, \cdots, r_n \) over the sets \( Q_k \) \((k \in N)\) and collecting similar terms we conclude that
\[
nm(Q_k) \int_{Q_k} (R_{r^*}\psi)(r) |y(r)|^2 dr + n(n-1) \int_{Q_k} \int_{Q_k} (R_{r^*}\psi)(s) y(r) \overline{y(s)} dr ds \leq 0.
\]
Further, we divide this inequality by $n^2$ and pass to the limit as $n \to \infty$. We get
\[
\int_{Q_k} \int_{Q_k} (R_{r^*} \psi)(s) y(r) \overline{y(s)} dr ds \leq 0
\]
for each $k \in \mathbb{N}$. By passing to the limit as $k \to \infty$ and applying Lebesgue Theorem [9], we see that the inequality
\[
\int_{Q} \int_{Q} (R_{r^*} \psi)(s) y(r) \overline{y(s)} dr ds \leq 0
\]
holds for all functions from $C_0(Q)$. Approximating an arbitrary function from $L_1$ by finite continuous functions we arrive at (3.8) for all $x \in L_1$. Theorem 3.5 completes the conclusion of the desired equivalence.

**Theorem 3.7.** Let $\psi \in N(Q)$ with $\text{Re}\psi \geq 0$. Then
\[
\sqrt{(R_r(\psi))(s)} \leq \sqrt{|\psi(r)| + |\psi(s)|} ; \quad r, s \in Q.
\]

**Proof.** Let $\psi \in N(Q)$, then the $n \times n$ matrix
\[
\begin{pmatrix}
\psi(r_i) + \overline{\psi(r_j)} - (R_{r^*}\psi)(r_i) \\
\psi(r_i) + \overline{\psi(s)} - (R_{r^*}\psi)(r_i)
\end{pmatrix}
\]
is positive Hermitian for any $i, j = 1, \ldots, n$.

Take $n = 2$, and $r, s \in Q$. Since the matrix
\[
\begin{pmatrix}
\psi(r) + \overline{\psi(r)} - (R_{r^*}\psi)(r) & \psi(r) + \overline{\psi(s)} - (R_{r^*}\psi)(r) \\
\psi(s) + \overline{\psi(r)} - (R_{s^*}\psi)(s) & \psi(s) + \overline{\psi(s)} - (R_{s^*}\psi)(s)
\end{pmatrix}
\]
has non-negative determinant, we find, using $(R_{r^*}\psi)(s) = (R_{r^*}\psi)(r)$, and properties (3.4).

We get
\[
|\psi(r) + \overline{\psi(s)} - (R_{s^*}\psi)(r)|^2 \leq \left(2\text{Re}\psi(r) - (R_{r^*}\psi)(r)\right).
\]

\[
\begin{align*}
2\text{Re}\psi(s) - (R_{s^*}\psi)(s) & \leq 4\text{Re}\psi(r)\text{Re}\psi(s) \\
& \leq 4|\psi(r)||\psi(s)|.
\end{align*}
\]

Then
\[
|(R_{s^*}\psi)(r) - \psi(r) - \overline{\psi(s)}| \leq 2\sqrt{|\psi(r)||\psi(s)|}
\]
and

\[ |(R_s \psi)(r)| \leq \left( \sqrt{|\psi(r)|} + \sqrt{|\psi(s)|} \right)^2. \]

\[ \square \]

**Theorem 3.8.** Let \( \psi : Q \to C \) be a function on \( Q \). Assume that

(i) \( \psi \) is continuous bounded and \( \psi(e) \geq 0 \).

(ii) \( \varphi_t : r \to \exp(-t \psi(r)) \) are positive definite for each \( t > 0 \).

Then \( \psi \) is negative definite.

**Proof.** By (i) the functions \( \varphi_t \) are continuous and \( \varphi_t(e) \leq 1 \). Therefore Corollary 3.6 (ii) implies that \( r \mapsto \frac{1}{t}(1 - \varphi_t(r)) \) is negative definite for any \( t > 0 \). Since

\[ \left| \psi(r) - \frac{1}{t}(1 - \varphi_t(r)) \right| \leq t \exp|\psi(t)|, \quad \text{for } 0 < t < 1. \]

We obtain that

\[ \lim_{t \to 0} \frac{1}{t}(1 - \varphi_t) = \psi \]

uniformly on compact subsets of \( Q \). Then it is easy to prove that \( \psi \) satisfy (3.3). \( \square \)

We do not know whether the inverse assertion of Theorem 3.8 does hold in general.

4. Negative definite functions on hypergroups

Let \( K \) be a commutative hypergroup (see [1], [2]). We define the action of generalized invariant operators \( R_r(r \in Q) \) upon arbitrary Borel functions \( f \) on \( K \) by the formula

\[ (R_r f)(s) = (\delta_s * \delta_r)(f), \]

where the convolution

\[ K \times K \ni (r, s) \mapsto \delta_r * \delta_s \in M(K) \]

is continuous. \( M(k) \) is equipped with weak topology, and \( \delta_r \) is the Dirac measure.

A continuous function \( \psi : K \to C \) is negative definite on \( K \) if for \( x_1, \ldots, x_n \in K, c_1, \ldots, c_n \in C \), the inequality

\[ \sum_{i,j=1}^{n} \left( \psi(x_i + \overline{\psi(x_j)} - \delta_{x_i} * \delta_{x_j}(\psi) \right) c_i \overline{c_j} \geq 0 \]

holds.
It is fairly easy to observe that all our studied properties and theorems of negative definite functions on hypercomplex system are easily established for the above case (see [6]).

References