The Uniform Convergence of a Sequence of Weighted Bounded Exponentially Convex Functions on Foundation Semigroups

HODA A. ALI

Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo, Egypt

Abstract. In the present paper we shall prove that on a foundation ∗-semigroup S with an identity and with a locally bounded Borel measurable weight function ω, the pointwise convergence and the uniform convergence of a sequence of ω-bounded exponentially convex functions on S which are also continuous at the identity are equivalent.

1. Introduction

In [6] Okb El-Bab proved that if S is a foundation topological ∗-semigroup with an identity e and with a Borel measurable weight function ω such that 0 < ω ≤ 1 and 1/ω is locally bounded (i.e., bounded on compact subsets of S) and if \( P_e(S,\omega) \) is the set of ω-bounded Borel measurable exponentially convex functions on S which are continuous at e. Then a sequence \((\phi_n)\) in \( P_e(S,\omega) \), converges pointwise on S to a function \( \phi \in P_e(S,\omega) \) if and only if \((\phi_n)\) converges to \( \phi \) uniformly on compact subsets of S. He proved also that this result remains valid for any Borel measurable weight function ω such that ω and 1/ω are locally bounded.

2. Preliminaries

A topological semigroup S is called a ∗-semigroup if there is a continuous mapping ∗ : S → S such that \((x^*)_* = x\) and \((xy)_* = y_* x_*\) for all \( x, y \in S \). A locally bounded (i.e., bounded on compact subsets of S) mapping \( \omega : S \to \mathbb{R}^+ \) (\( \mathbb{R}^+ \) denote the set of positive real numbers) is called a weight function on S if \( \omega(x^*) = \omega(x) \) and \( \omega(xy) \leq \omega(x) \omega(y) \) for all \( x, y \in S \). A function \( f : S \to \mathbb{R} \) is called ω-bounded if there is a positive number \( K \) such that \( |f(x)| \leq K \omega(x) (x \in S) \). A real valued function \( \phi \) on S is called exponentially convex if it satisfies

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \phi(x_i, x_j) \geq 0
\]

for all \( \{x_1, \cdots, x_n\} \) from S and \( \{c_1, \cdots, c_n\} \) from \( \mathbb{R} \). We denote by \( P_e(S,\omega)(P(S,\omega),\omega) \).

Received February 1, 2005.
2000 Mathematics Subject Classification: 43xx.
Key words and phrases: exponentially convex function, foundation semigroups.

337
respectively) the set of $\omega$-bounded, Borel measurable, continuous at $e$ and exponentially convex function on $S$ (the set of $\omega$-bounded continuous exponentially convex function on $S$, respectively). A $*$-representation of $S$ by bounded operators on a Hilbert space $H$ is a homomorphism: $x \to \pi(x)$ of $S$ into $L(H)$, the space of all bounded linear operators on $H$, such that $\pi(x^*) = (\pi(x))^*$ for all $x \in S$ and $\pi(e)$ is the identity operator on $H$. A representation $\pi$ is called cyclic if there is a (cyclic) vector $\xi \in H$ such that the set $\{\pi(x)\xi : x \in S\}$ is dense in $H$, and $\pi$ is called $\omega$-bounded if there is a positive number $K$ such that $\|\pi(x)\| \leq K\omega(x)(x \in S)$. Note that a $*$-representation $\pi$ is $\omega$-bounded if and only if $\|\pi(x)\| \leq \omega(x)(x \in S)$. For further information see [2], [4], [5].

Recall that (see for example, [1]) $\hat{L}(S)$ or $M_0(S)$ denotes the set of all measures $\mu \in M(S)$ (the convolution measure algebra of bounded complex measure on $S$ with the total variation norm $\|\cdot\|$), for which the mapping $x \to \delta_x * \mu$ and $x \to |\mu| * \delta_x$ (where $\delta_x$ denotes the point mass at $x$ for $x \in S$) from $S$ into $M(S)$ are weakly continuous. If $\omega$ is a locally bounded Borel measurable weight function on $S$, then we denote by $M^k_{\omega}(S, \omega)$ the set of all complex regular measures $\mu$ on $S$ such that $\omega \mu \in M_0(S)$, where $M^k_{\omega}(S)$ denote the set of all measures in $M_{\omega}(S)$ with compact support. We observe that $M^k_{\omega}(S, \omega)$ with convolution

$$(\mu * \nu)(f) = \int f(xy) d\mu(x) d\nu(y) \quad (f \in C_c(S)),$$

where $C_c(S)$ denote the space of all continuous complex valued functions on $S$ with compact support.

A semigroup $S$ is called foundation if $U(\text{supp}(\mu) : \mu \in M_0(S))$ is dense in $S$. It is well known that $M_0(S)$ is a two sided closed $L$-ideal of $M(S)$ and if $S$ is also foundation semigroup with identity, then both mapping $x \to \delta_x * \mu$ and $x \to \mu * \delta_x (\mu \in M_0(S))$ from $S$ into $M_0(S)$ are norm continuous (see [7]). We observe that if $S$ is a foundation semigroup with identity and with a locally bounded Borel measurable weight function $\omega$, then both the mappings $x \to \delta_x * \mu$ and $x \to \mu * \delta_x (\mu \in M^k_{\omega}(S, \omega))$ from $S$ into $M^k_{\omega}(S, \omega)$ are $\|\cdot\|_\omega$ norm continuous, where $\|\mu\|_\omega = \|\omega \mu\|$ for every $\mu \in M^k_{\omega}(S, \omega)$.

Now we introduce two new topologies $\tau_U$ and $\tau_J$ on $P(S, \omega)$.

3. The $\tau_U$-topology and the $\tau_J$-topology on $P(S, \omega)$

The following two definitions are needed for the proof of the main result.

**Definition 3.1.** For each compact subset $F$ of $S$, positive numbers $\alpha, \beta$, and $\phi_0 \in P(S, \omega)$ of a foundation $*$-semigroup $S$ with an identity $e$ and with a locally bounded Borel measurable weight function $\omega$ we define,

$$U_{F;\alpha,\beta}(\phi_0) = \{ \phi \in P(S, \omega) : |\phi(x) - \phi_0(x)| < \alpha \}$$

and

$$(1) \quad |\phi(x^2) - \phi_0(x^2)| < \beta \quad \text{for all} \quad x \in F.$$
The family of the sets of the form (1) define a base for a topology on $P(S, \omega)$ which is denoted by $\tau_\omega$.

**Definition 3.2.** For $\mu_1, \cdots, \mu_m \in M_\omega^k(S, \omega)$, positive real numbers $\alpha, \beta, \gamma$, and $\phi_0 \in P(S, \omega)$ let

$$(2) \quad \{ \phi \in P(S, \omega) : |\int_S [\phi(y) - \phi_0(y)]d\mu_j(y)| < \alpha, \quad |\int_S [\phi(y^2) - \phi_0(y^2)]d\mu_j(y)| < \beta, \quad \text{for } j = 1, \cdots, m \text{ and } |\phi(e) - \phi_0(e)| < \gamma \}$$

the family of the sets of the form (2) define a base for a topology $\tau_J$ on $P(S, \omega)$.

**Lemma 3.1.** Let $S$ be a *-semigroup (not necessarily topological) with an identity and with a weight $\omega$. Then every $\omega$-bounded exponentially convex function $\phi$ on $S$ satisfies the following inequality

$$(3) \quad |\phi(x) - \phi(xy)|^2 \leq \phi(e)\omega^2(x)[\phi(e) - 2\phi(y) + \phi(y^2)] \quad (x, y \in S).$$

**Proof.** Since $\phi$ is $\omega$-bounded, from [3] it follows that there exists a $\omega$-bounded cyclic *-representation $\pi$ of $S$ by bounded operators on a Hilbert space $H$ with a cyclic vector $\xi$. Such that $||\xi||^2 = \phi(e)$ and $\phi(x) = (\pi(x)\xi, \xi)$ \quad ($x \in S$).

For every $x, y \in S$, we have

$$|\phi(x) - \phi(xy)|^2 = |(\pi(x)\xi, \xi) - (\pi(xy)\xi, \xi)|^2$$

$$= |(\pi(x)\xi, \xi) - (\pi(y)\xi, \xi)|^2$$

$$\leq ||\pi(x)\xi||^2||\xi - \pi(y)\xi||^2$$

$$= (\pi(x)\xi, \pi(x)\xi)||\xi||^2 - 2\phi(y) + \phi(y^2))\]$$

$$= \phi(x^2)[\phi(e) - 2\phi(y) + \phi(y^2)]$$

$$\leq \phi(e)\omega^2(x)[\phi(e) - 2\phi(y) + \phi(y^2)]. \quad \square$$

**Lemma 3.2.** Let $S$ be a foundation *-semigroup with identity and with a locally bounded Borel measurable weight function $\omega$. Then $P_e(S, \omega) = P(S, \omega)$.

**Proof.** Let $\phi \in P_e(S, \omega)$. Take a fixed $x_0 \in S$ and let $W$ be a fixed compact neighborhood of $x_0$. Since $\omega$ is locally bounded, there exists a positive real number $M$ such that $\omega(x) \leq M$ for all $x \in W$. Given $\varepsilon > 0$, by the continuity of $\phi$ at $e$ there exists a neighborhood $U$ of $e$ such that

$$|\phi(e) - 2\phi(u) - \phi(u^2)|^2 < \frac{\varepsilon}{2M[|\phi(e)|^2 + 1]} \quad (u \in U).$$

Let

$$W_1 = [U^{-1}(Ux) \cap (xU)U^{-1}] \cap W.$$
define a neighborhood of \(e\). Let \(z \in W_1\), then \(uz = vx\) for some \(u, v \in U\), so by (3)
\[
|\phi(z) - \phi(x)| \leq |\phi(z) - \phi(ux)| + |\phi(ux) - \phi(x)|
\leq (\phi(e))^{\frac{1}{2}} \omega(z)(|\phi(e) - 2\phi(u) + \phi(u^2)|)^{\frac{1}{2}}
+ (\phi(e))^{\frac{1}{2}} \omega(x)(|\phi(e) - 2\phi(v) + \phi(v^2)|)^{\frac{1}{2}}
\leq 2M(\phi(e))^{\frac{1}{2}} \frac{\varepsilon}{2M(\phi(e))^{\frac{1}{2}} + 1} < \varepsilon
\]
so \(\phi \in P(S, \omega)\) and the proof is complete. \(\square\)

The following theorem is the main result of this paper and it generalizes Theorem 1 of [6]. Note that \(P_\tau(S, \omega) = P(S, \omega)\), by Lemma 3.2.

**Theorem 3.1.** Let \(S\) be a foundation \(*\)-semigroup with identity and with a locally bounded Borel measurable weight function \(\omega\). Then the \(\tau_\mathcal{U}\)-topology and the \(\tau_\mathcal{F}\)-topology are identical on \(P(S, \omega)\).

**Proof.** Take \(\phi_0\) fixed in \(P(S, \omega)\). Let \(J_{\mu_1, \cdots, \mu_n; \beta, \gamma, \lambda}(\phi_0)\) be an arbitrary basic \(\tau_\mathcal{F}\)-neighborhood of \(\phi_0\). Choose a positive number \(\eta\) such that \(\eta < \lambda\) and
\[
2\eta + \eta \max\{||\mu_1||, \cdots, ||\mu_n||\} < \min(\beta, \gamma).
\]
Choose a compact set \(F_0\) such that \(e \in F_0\) with
\[
\int_{S \setminus F_0} (\omega(y))^2d|\mu_j|(y) < \eta, \text{ and } \int_{S \setminus F_0} \omega(y)d|\mu_j|(y) < \eta(j = 1, \cdots, m).
\]
Then it is clear that
\[
\mathcal{U}_{F_0, \eta, \eta}(\phi_0) \subset J_{\mu_1, \cdots, \mu_n; \beta, \gamma, \lambda}(\phi_0).
\]

Conversely, suppose that \(\mathcal{U}_{F, \alpha_0, \beta_0}(\phi_0)\) is an arbitrary \(\tau_\mathcal{U}\)-neighborhood of \(\phi_0\). Let \(\beta = \min\{\alpha_0, \beta_0\}\) and \(M\) be a positive number such that \(\omega(x) \leq M\) for all \(x \in F\). Put
\[
\gamma = \min\left\{\frac{\beta^2}{81M^2(1 + \phi_0(e))}, \frac{\beta^2}{81M^2(1 + \phi_0(e))}\right\}
\]
\[
\delta = \min\left\{\frac{\beta}{6(\phi_0(e) + 1)}, 1\right\}.
\]
By the continuity of \(\phi_0\) at \(e\) there exists a compact neighborhood \(U\) of \(e\) such that for all \(y \in U\)
\[
|\phi_0(y) - \phi_0(e)| < \gamma \text{ and } |\phi_0(y^2) - \phi_0(e)| < \gamma.
\]
Now choose a positive measure \(\mu \in M^b(S, \omega)\) such that \(\mu(S) = 1\) and \(e \in \text{supp}(\mu) \subseteq U\).

By the \(\|\cdot\|_\omega\) norm continuity of the mapping \(x \to \delta_x * \mu\) from \(S\) into \(M^b(S, \omega)\) and
the compactness of $F$ we can find a finite subset $\{x_1, \cdots, x_n\}$ of $F$ such that the set $\{\delta_x \ast \mu : x \in F\}$ can be covered by $\{N_{x_1}, \cdots, N_{x_n}\}$, where

$$\begin{align*}
N_{x_i} = \{ \lambda \in M^k(S, \omega) : \|\lambda - \delta_{x_i} \ast \mu\|_{\omega} < \delta \} \quad \text{for} \quad i = 1, \cdots, n.
\end{align*}$$

Again by the $\|\cdot\|_{\omega}$-norm continuity of the mapping $x \to \delta_{x} \ast \mu$ from $S$ into $M^k(S, \omega)$, we can find $s_1, s_2, \cdots, s_\ell \in S$ such that the set $\{\delta_{s_j} \ast \mu : x \in F\}$ can be covered by $\{N'_{s_1}, \cdots, N'_{s_\ell}\}$, where

$$\begin{align*}
N'_{s_j} = \{ \lambda \in M^k(S, \omega) : \|\lambda - \delta_{s_j} \ast \mu\|_{\omega} < \delta \} \quad (j = 1, \cdots, \ell).
\end{align*}$$

Put

$$z_i = x_i, i = 1, \cdots, n, z_{n+j} = s_j s_j^* s_j^2 \quad \text{for} \quad 1 \leq j \leq \ell.$$

Put $p = n + \ell$ and let $\mu_k = \delta_{x_k} \ast \mu \ast (k = 1, 2, \cdots, p)$. We shall prove that

$$\begin{align*}
J_{\mu_1, \mu_2, \cdots, mp; \delta, \delta}(\phi_0) \cap J_{\mu, \gamma, \gamma}(\phi_0) \subseteq U_{\beta, \gamma}(\phi_0).
\end{align*}$$

To prove this we choose $\phi \in J_{\mu_1, \cdots, \mu_p; \delta, \delta}(\phi_0)$. Let $x$ be a fixed but arbitrary element in $F$. We have $\|\delta_x \ast \mu - \delta_{x_j} \ast \mu\|_{\omega} < \delta$ and $\|\delta_{x_k} \ast \mu \ast (k = 1, 2, \cdots, p) < \delta$ for some $j$ and $q \in \{1, 2, \cdots, p\}$. Therefore

$$\begin{align*}
|\delta_x \ast \mu(\phi) - \delta_x \ast \mu(\phi_0)| &= \left| \int \phi(y) - \phi_0(y) d\delta_x \ast \mu(y) \right| \\
&\leq \left| \int \phi(y) d(\delta_x \ast \mu - \delta_{x_j} \ast \mu)(u) \right| \\
&\quad + \left| \int [\phi(y) - \phi_0(y)] d\mu_j(y) \right| + \left| \int \phi_0(y) d(\delta_{x_j} \ast \mu - \delta_x \ast \mu)(y) \right| \\
&\leq \phi(e) \|\delta_x \ast \mu - \delta_{x_j} \ast \mu\|_{\omega} + \phi_0(e) \|\delta_{x_j} \ast \delta_x \ast \mu\|_{\omega} \\
&< \delta(\phi(e) + \phi_0(e) + 1) < \frac{\beta}{3}.
\end{align*}$$

(In the above we have used [3]). Similarly by using the inequality $\|\delta_x \ast \mu - \delta_{x_j} \ast \mu\|_{\omega} < \delta$, we can prove that

$$\begin{align*}
|\delta_{x_k} \ast \mu(\phi) - \delta_{x_k}(\phi_0)| < \frac{\beta}{3}.
\end{align*}$$

Suppose now that

$$\begin{align*}
\phi \in J_{\mu; \gamma, \gamma}(\phi_0).
\end{align*}$$
Then for every $x \in F$ by (3) and the Holder inequality we have

$$|\delta_x * \mu(\phi) - \phi(x)| \leq |\int_S \phi(xy)d\mu(y) - \int_S \phi(x)d\mu(y)|$$

$$\leq \int_S |\phi(xy) - \phi(x)|d\mu(y)$$

$$\leq \omega(x)(\phi(e))^{\frac{1}{2}}(\int_U |\phi(e) - 2\phi(y) + \phi(y^2)|d\mu(y))^{\frac{1}{2}}$$

$$\leq M\phi(e)^{\frac{1}{2}}(\int_U |\phi(e) - 2\phi(y) + \phi(y^2)|d\mu(y))^{\frac{1}{2}}$$

by (4) we obtain

$$\int_S [\phi(e) - 2\phi(y) + \phi(y^2)]d\mu(y)$$

$$\leq 2|\int_U [\phi(e) - \phi(y)]d\mu(y)| + |\int_U [\phi(y^2) - \phi(e)]d\mu(y)|$$

$$\leq 2|\int_U [\phi(e) - \phi_0(e)]d\mu(y)| + \int_U |\phi_0(y) - \phi(y)|d\mu(y)$$

$$+ \int_U |\phi_0(y^2) - \phi_0(y)|d\mu(y) + \int_U |\phi(y^2) - \phi_0(y^2)|d\mu(y)$$

so for every $x \in F$

$$|\delta_x * \mu(\phi) - \phi(x)| \leq 3M(\phi(e)\gamma)^{\frac{1}{2}} < \frac{\beta}{3},$$

(7)

Similarly for every $x \in F$

$$|\delta_x^2 * \mu(\phi) - \phi(x^2)| \leq \omega(x^2)\phi(e)^{\frac{1}{2}}(\int_U |\phi(e) - 2\phi(y) + \phi(y^2)|d\mu(y))^{\frac{1}{2}}$$

$$< 3M^2(\phi(e)\gamma)^{\frac{1}{2}} < \frac{\beta}{3},$$

(8)

Finally, for every $\phi \in J_{\mu_1,\ldots,\mu_p,\delta,\delta}(\phi_0) \cap J_{\mu,\gamma,\gamma}(\phi_0)$ and every $x \in F$ from (5) and (7).

We have

$$|\phi(x) - \phi_0(x)| \leq |\phi(x) - \delta_x * \mu(\phi)|$$

$$+ |\delta_x * \mu(\phi) - \delta_x * \mu(\phi_0)| + |\delta_x * \mu(\phi_0) - \delta_x * \mu(\phi)|$$

$$< \frac{\beta}{3} + \frac{\beta}{3} + \frac{\beta}{3} = \beta$$
similarly for every $x \in F$ from (6) and (8) we have
\[ |\phi(x^2) - \phi_0(x^2) < \beta. \]
That is $\phi \in \mathcal{U}_{F,\beta,\beta}(\phi_0)$. The proof is now complete. Since $\mathcal{U}_{F,\beta,\beta} \subset \mathcal{U}_{F,\alpha,\beta_0}$. \hfill \Box

**Theorem 3.2.** Let $S$ be a foundation topological $\ast$-semigroup with an identity and with a locally bounded measurable $\omega$. Then a sequence $\{\phi_n\}$ of $\omega$-bounded continuous exponential convex function on $S$ converges pointwise to a continuous function $\phi$ if and only if $\{\phi_n\}$ converges to $\phi$ in the topology of uniform convergence on compact subset of $S$.

**Proof.** It is known that the pointwise limit of a sequence of exponentially convex function is an exponentially convex. Suppose that $\{\phi_n\}$ converges to $\phi$ pointwise on $S$ and $\phi$ is also continuous. Then $\phi \in \mathcal{P}(S,\omega)$. From the Lebesgue dominated convergence Theorem it follows that $\phi_n \to \phi$ in the $\tau_J$-topology. So by Theorem 3.1, $\phi_n \to \phi$ in the $\tau_U$-topology. The converse is obvious. \hfill \Box

**References**


