The Action of the Handlebody Group on the First Homology Group of the Surface

Susumu Hirose
Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840-8502, Japan
e-mail: hirose@ms.saga-u.ac.jp

Dedicated to Professor Takao Matumoto on his sixtieth birthday

Abstract. The action of mapping class groups of a handlebody on the first homology group is investigated. A homological analogy of mapping class groups of a handlebody is defined and its system of generators is given.

1. Introduction

Let \( \Sigma_g \) be a closed orientable surface of genus \( g \), bounding a 3-dimensional handlebody \( H_g \). Griffiths [1] showed that a homeomorphism of \( \Sigma_g \) extends to a homeomorphism of \( H_g \) exactly when its induced automorphism on \( \pi_1(\Sigma_g) \) preserves the normal closure of the set of boundary circles of a standard collection of meridian disks for \( H_g \). In this case, its induced automorphism on \( H_1(\Sigma_g) \) is a symplectic block matrix of the form 
\[
\begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}
\]
with respect to a standard basis of \( H_1(\Sigma_g) \) whose first \( g \) elements are the boundary circles of the meridian disks. Our first main result is that every such matrix is induced by a homeomorphism that extends to \( H_g \). We also consider the mapping classes of \( \Sigma_g \) whose induced automorphisms preserve the subgroup of \( H_1(\Sigma_g) \) generated by these boundary circles. These might be called homologically extendible homeomorphisms. Our second main result gives an efficient generating set for the group of isotopy classes of homologically extendible homeomorphisms (which we call homological handlebody group), when \( g \geq 3 \).

2. Preliminaries and results

A 3-dimensional handlebody \( H_g \) is an orientable 3-manifold constructed from a 3-ball by attaching \( g \) 1-handles. The boundary of \( H_g \) is an orientable closed surface

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Figure 1:

\[ \Sigma_g \text{ of genus } g. \text{ Let } M_g \text{ be the mapping class group of } \Sigma_g \text{ and } H_g \text{ be the mapping class group of } H_g, \text{ for short, we call this group the handlebody group. For elements } a, b \text{ and } c \text{ of a group, we write } c = c^{-1}, \text{ and } a * b = ab\pi. \]

Let \( P_g \) be a planar surface constructed from a 2-disk by removing \( g \) copies of disjoint 2-disks. As indicated in Figure 1, the boundary components of \( P_g \) are denoted by \( \gamma_0, \gamma_2, \ldots, \gamma_{2g} \), and some properly embedded arcs of \( P_g \) by \( \gamma_1, \gamma_3, \ldots, \gamma_{2g-1}, \beta_1, \ldots, \beta_{2g-2} \) and \( \beta'_1, \ldots, \beta'_{2g-2} \). The 3-manifold \( P_g \times [-1, 1] \) is homeomorphic to \( H_g \). On \( \partial(P_g \times [-1, 1]) = \Sigma_g \), we define \( c_{2i-1} = \partial(\gamma_{2i-1} \times [-1, 1]) \) (\( 1 \leq i \leq g+1 \)), \( b_{2j} = \partial(\beta_{2j} \times [-1, 1]) \), \( b'_{2j} = \partial(\beta'_{2j} \times [-1, 1]) \) (\( 2 \leq j \leq g-1 \)), and \( c_{2k} = \gamma_{2k} \times \{0\} \) (\( 1 \leq k \leq g \)). In Figures 2 and 3, these circles are illustrated and oriented.

Figure 2:

For a simple closed curve \( a \) on \( \Sigma_g \), we define the Dehn twist \( T_a \) about \( a \) as indicated in Figure 4. For short, we write \( C_i \) for \( T_{c_i} \), and \( B_{2j} \) for \( T_{b_{2j}} \). As elements of \( H_1(\Sigma_g, \mathbb{Z}) \), we take

\[
\begin{align*}
x_1 &= -c_1, & y_1 &= -c_2 \\
x_i &= b_{2i}, & y_i &= -c_{2i}, & \text{where } 2 \leq i \leq g-1, \\
x_g &= c_{2g+1}, & y_g &= -c_{2g}.
\end{align*}
\]
Then, \( \{x_1, y_1, \cdots, x_g, y_g\} \) is a basis of \( H_1(\Sigma_g, \mathbb{Z}) \), and satisfy \( (x_i, y_j) = \delta_{i,j} \), \( (x_i, x_j) = (y_i, y_j) = 0 \) for the intersection form \( (, ) \). Let \( E_g \) be a identity \( g \times g \) matrix, and \( J = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix} \).

We define \( \text{Sp}(2g) = \{ M \in GL(2g, \mathbb{Z}) \mid M'JM = J \} \), where \( M' \) means the transpose of \( M \).

We can characterize the handlebody group \( \mathcal{H}_g \) by the actions of its elements on the fundamental group \( \pi_1(\Sigma_g, p) \), where \( p \) is a point on \( \Sigma_g \). Let \( l_1 \) be an arc on \( \Sigma_g \) which begins from \( p \) and ends on \( c_1 \), \( l_i \) (\( 2 \leq i \leq g-1 \)) be an arc on \( \Sigma_g \) which begins from \( p \) and ends on \( b_{2i} \), and \( l_g \) be an arc on \( \Sigma_g \) which begins from \( p \) and ends on \( c_{2g} \). We denote by \( \mathcal{N} \) the normal closure of \( \{l_1c_1l_1^{-1}, l_2b_4l_2^{-1}, \cdots, l_{g-1}b_{2g-2}l_{g-1}^{-1}, l_gb_{2g+1}l_g^{-1}\} \), then \( \mathcal{H}_g = \{ \phi \in M_g \mid \phi(\mathcal{N}) = \mathcal{N} \} \).

We define a homological analogue of \( \mathcal{H}_g \). Let \( N \) be the \( \mathbb{Z} \)-submodule of \( H_1(\Sigma_g, \mathbb{Z}) \) generated by \( \{x_1, \cdots, x_g\} \), and \( \mathcal{H}\mathcal{H}_g \) be a subgroup of \( M_g \) defined by \( \mathcal{H}\mathcal{H}_g = \{ \phi \in M_g \mid \phi_*(\mathcal{N}) = \mathcal{N} \} \). We call \( \mathcal{H}\mathcal{H}_g \) the homological handlebody group of genus \( g \).

For each element \( \phi \) of \( M_g \), we define a \( 2g \times 2g \) matrix \( M_\phi \) by
\[
(\phi(x_1), \phi(x_2), \cdots, \phi(x_g), \phi(y_1), \phi(y_2), \cdots, \phi(y_g)) = (x_1, x_2, \cdots, x_g, y_1, y_2, \cdots, y_g)M_\phi.
\]
Then, $M_{\phi}$ is an element of $Sp(2g)$, and the map $\mu$ from $M_{\phi}$ to $Sp(2g)$ defined by mapping $\phi$ to $M_{\phi}$ is a surjection. On the other hand, $\mu|_{\mathcal{H}_g}$ is not a surjection. We define a subgroup $urSp(2g)$ of $Sp(2g)$ by

$$urSp(2g) = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp(2g) \right\},$$

where $A$, $B$, and $D$ are $g \times g$ matrices, $0$ is a $g \times g$ zero matrix, and $ur$ stands for “upper right”. We show the following theorem

**Theorem 2.1.** $\mu(\mathcal{H}_g) = urSp(2g)$.

Since $urSp(2g)$ is an infinite index subgroup, we have

**Corollary 2.2.** $\mathcal{H}_g$ is an infinite index subgroup of $M_g$.

**Remark 2.3.** An embedding of $\Sigma_g$ into the 3-sphere $S^3$ is called trivial if $S^3 \setminus \Sigma_g$ is the disjoint union of the interior of 3-dimensional handlebodies of genera $g$. For this embedding, we define

$$\mathcal{E}(S^3, \Sigma_g) = \left\{ \phi \in M_g \mid \text{there is an element } \Phi \in \text{Diff}^+(S^3) \text{ such that } \Phi|_{\Sigma_g} \text{ represents } \phi \right\}.$$

Since $\mathcal{E}(S^3, \Sigma_g) \subset \mathcal{H}_g$, $\mathcal{E}(S^3, \Sigma_g)$ is an infinite index subgroup of $M_g$. This is much different from the case for surfaces trivially embedded in $S^4$. In this case, $\mathcal{E}(S^4, \Sigma_g)$ is a finite index subgroup of $M_g$ [2].

By definition, $\mathcal{H}_{\mathcal{H}_g} = \mu^{-1}(urSp(2g))$. Let $[a]$ be the largest integer $n$ which satisfies $n \leq a$, and $d_j$, $d'_j$, $e_k$, $e'_k$ are indicated in Figures 5 and 6. We show

**Theorem 2.4.** If $g \geq 3$, $\mathcal{H}_{\mathcal{H}_g}$ is generated by $C_1$, $C_2C_1^2C_2$, $C_2C_1C_3C_2$, $C_2C_{2i}B_2C_{2i}$, $C_{2i}C_{2i+1}B_2C_{2i}$ $(2 \leq i \leq g - 1)$, $C_2C_{2g-1}C_{2g+1}C_{2g}$, $T_{d_j}T_{d'_j}$ $(1 \leq j \leq \lfloor \frac{g-1}{2} \rfloor)$, and $T_{e_k}T_{e'_k}$ $(1 \leq k \leq \lfloor \frac{g-2}{2} \rfloor)$.

The author does not know whether $\mathcal{H}_{\mathcal{H}_2}$ is finitely generated, and whether $\mathcal{H}_{\mathcal{H}_g}$ are finitely presented.
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3. Proof of Theorem 2.1

By definition $\mu(\mathcal{H}_g) \subset \text{wrSp}(2g)$. We show that $\text{wrSp}(2g) \subset \mu(\mathcal{H}_g)$. Let $S_0$ be the $g \times g$ symmetric matrix, and $U_1, U_2, U_3$ the $g \times g$ unimodular matrices given by

$$S_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

**Lemma 3.1.** The group $\text{wrSp}(2g)$ is generated by

$$\left\{ \begin{pmatrix} E_g & S_0 \\ 0 & E_g \end{pmatrix}, \begin{pmatrix} U_i & 0 \\ 0 & (U_i')^{-1} \end{pmatrix}, \text{ where } i = 1, 2, 3 \right\}.$$

**Proof.** If we set $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, then the condition $M'JM = J$ is equivalent to $A'D = E_g, B'D = D'B$. Let $S = B'D$, then $S$ is symmetric, for $S' = D'B'' = D'B = B'D = S$. Since $A'D = E_g, A = U$ is unimodular, and $D = (U')^{-1}$. By the above equations, we see $US = US' = AD'B = E_gB = B$, therefore,

$$\begin{pmatrix} U & 0 \\ 0 & (U')^{-1} \end{pmatrix} \begin{pmatrix} E_g & S \\ 0 & E_g \end{pmatrix} = \begin{pmatrix} U & US \\ 0 & (U')^{-1} \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

This equation shows that $\text{wrSp}(2g)$ is generated by the following types of elements:
(I) \( \begin{pmatrix} E_g & S \\ 0 & E_g \end{pmatrix} \), where \( S \) is symmetric.

(II) \( \begin{pmatrix} U & 0 \\ 0 & (U')^{-1} \end{pmatrix} \), where \( U \) is unimodular.

By applying the argument by Hua and Reiner [3, p.420 and p.421], we see that elements of (I) and (II) are generated by four elements listed in the statement of this lemma.

Suzuki [6] introduced elements \( \rho \) (cyclic translation of handles), \( \omega_1 \) (twisting a knob), \( \rho_{12} \) (interchanging two knobs), and \( \theta_{12} \) (sliding) of \( H_g \). In [6], their actions on the fundamental group of \( \Sigma_g \) were listed. Using this list, we have

\[
\mu(C_1) = \begin{pmatrix} E_g & S_0 \\ 0 & E_g \end{pmatrix}, \quad \mu(\rho) = \begin{pmatrix} U_1 & 0 \\ 0 & (U_1')^{-1} \end{pmatrix},
\]

\[
\mu(\rho_{12} \theta_{12} \rho_{12}^{-1}) = \begin{pmatrix} U_2 & 0 \\ 0 & (U_2')^{-1} \end{pmatrix}, \quad \mu(\omega_1) = \begin{pmatrix} U_3 & 0 \\ 0 & (U_3')^{-1} \end{pmatrix}.
\]

Lemma 3.1 now implies that \( urSp(2g) \subset \mu(H_g) \).

4. Proof of Theorem 2.4

Let \( I_g \) be the kernel of \( \mu \). This group \( I_g \) is called the *Torelli group*. Theorem 2.1 shows that \( \mathcal{H}H_g \) is generated by \( \mathcal{H}_g \cup I_g \). Johnson [4] showed that, when \( g \) is larger than or equal to 3, \( I_g \) is finitely generated.

We review results in [4]. For the oriented simple closed curves shown in Figure 2, we refer to \( (c_1, c_2, \cdots, c_{2g+1}) \) and \( (c_\beta, c_5, \cdots, c_{2g+1}) \) as *chains*. For oriented simple closed curves \( d \) and \( e \) which intersect transversely in one point, we construct an oriented simple closed curve \( d + e \) from \( d \cup e \) as follows: choose a disk neighborhood of the intersection point and in it make a replacement as indicated in Figure 7. For a consecutive subset \( \{c_i, c_{i+1}, \cdots, c_j\} \) of a chain, let \( c_i + \cdots + c_j \) be the oriented simple closed curve constructed by repeated applications of the above operations.

Let \( (i_1, \cdots, i_{r+1}) \) be a subsequence of \((1, 2, \cdots, 2g+1)\) (Resp. \((\beta, 5, \cdots, 2g+1)\)). We construct the union of circles \( C = c_{i_1} + \cdots + c_{i_2 - 1} \cup c_{i_2} + \cdots + c_{i_3 - 1} \cup \cdots \cup c_{i_r} + \cdots + c_{i_{r+1} - 1} \). If \( r \) is odd, a regular neighborhood of \( C \) is an oriented compact
surface with 2 boundary components. Let $\phi$ be the element of $\mathcal{M}_g$ defined as the composition of the positive Dehn twist along the boundary curve to the left of $C$ and the negative Dehn twist along the boundary curve to the right of $C$. Then, $\phi$ is an element of $\mathcal{I}_g$. We denote $\phi$ by $[i_1, \ldots, i_{r+1}]$, and call this the odd subchain map of $(c_1, c_2, \ldots, c_{2g+1})$ (resp. $(c_3, c_5, \ldots, c_{2g+1})$). Johnson [4] showed the following theorem:

**Theorem 4.1 (4, Main Theorem).** For $g \geq 3$, the odd subchain maps of the two chains $(c_1, c_2, \ldots, c_{2g+1})$ and $(c_3, c_5, \ldots, c_{2g+1})$ generate $\mathcal{I}_g$.

From now on, we assume that $g \geq 3$. Since $c_3 = B_4(c_4)$ and $B_4 \in \mathcal{H}_g$, it suffices for our purpose to consider the odd subchain maps of $(c_1, c_2, \ldots, c_{2g+1})$. By the word ”an odd chain map”, we mean an odd subchain map of $(c_1, c_2, \ldots, c_{2g+1})$. We use the following results by Johnson [4].

**Lemma 4.2 (4, Lemma 1).**

(a) $C_j$ commutes with $[i_1, i_2, \ldots]$ if and only if $j$ and $j+1$ are either both contained in or are disjoint from the $i$’s.

(b) If $i \neq j+1$, then $\overline{C_j} \ast [\ldots, j, i, \ldots] = [\ldots, j+1, i, \ldots]$.

(c) If $k \neq j$, then $C_j \ast [\ldots, k, j+1, \ldots] = [\ldots, k, j, \ldots]$.

**Lemma 4.3.** For any sequence of integers $5 \leq k_3 < k_6 < \cdots < k_{2n} \leq 2g + 1$,

\[
[1, 2, 3, 4][1, 2, k_3, k_6, \ldots, k_{2n}]B_4 \ast [3, 4, k_5, \ldots, k_{2n}]
= [k_5, k_6, \ldots, k_{2n}][1, 2, 3, 4, k_5, k_6, \ldots, k_{2n}].
\]

*Proof.* At first, we can show,

\[
[1, 2, 3, 4][1, 2, 5, 6, \ldots, 2n]B_4 \ast [3, 4, 5, \ldots, 2n]
= [5, 6, \ldots, 2n][1, 2, 3, 4, 5, 6, \ldots, 2n].
\]

Johnson showed this equation only in the case where $n = g$. But we can apply the proof of Lemma 10 of [4] for the case where $3 \leq n < g$ because we can regard each surfaces in Figure 18 of [4] as a surface of genus $n$ which is a submanifold of $\Sigma_g$.

We take a proper conjugation of the above equation by $C_5, C_6, \ldots, C_{2g+1}$, then we have the equation which we need to show. \hfill \Box

Odd chain maps $och_1, och_2$ are $\mathcal{H}_g$-equivalent (denoted by $och_1 \sim_{\mathcal{H}_g} och_2$), if there is an element $\phi$ of $\mathcal{H}_g$ such that $\phi \ast och_1 = och_2$. For an odd chain $[n_1, n_2, \ldots, n_k]$, we introduce a sequence of integers $m_1, m_2, \ldots, m_g, m_{g+1}$ as follows

\[
m_i = \#\{n_1, n_2, \ldots, n_k\} \cap \{2i - 1, 2i\}, \text{ when } 1 \leq i \leq g,
= \#\{n_1, n_2, \ldots, n_k\} \cap \{2g + 1\}, \text{ when } i = g + 1.
\]
We define $\Phi([n_1, n_2, \cdots, n_k]) = [[m_1, \cdots, m_{g+1}]]$, and call $[[m_1, \cdots, m_{g+1}]]$ a block sequence. By applying Lemma 4.2 to $C_1, C_3, C_5, \cdots, C_{2g+1} \in \mathcal{H}_g$, we show the following

**Lemma 4.4.** If odd chain maps $och_1$, $och_2$ satisfy $\Phi(och_1) = \Phi(och_2)$, then $och_1 \sim_{\mathcal{H}_g} och_2$.

For any $[[m_1, m_2, \cdots, m_{g+1}]]$ so that $m_i = 0, 1, 2$, for $1 \leq i \leq g$ and $m_{g+1} = 0, 1$, there are odd chain maps $och$ such that $\Phi(och) = [[m_1, m_2, \cdots, m_{g+1}]]$ and we call these $och$ representatives of $[[m_1, m_2, \cdots, m_{g+1}]]$. Lemma 4.4 says that, for any $[[m_1, m_2, \cdots, m_{g+1}]]$ satisfying the above condition, any pair of representatives of it are $\mathcal{H}_g$-equivalent. We show

**Lemma 4.5.**

(a) Odd chain maps $och_1$ and $och_2$ such that $\Phi(och_1) = [[\cdots, 0, 2, o o o]]$ and $\Phi(och_2) = [[\cdots, 2, o o o]]$, where the parts indicated by $\cdots$ and $o o o$ are equal to each other, are $\mathcal{H}_g$-equivalent.

(b) Odd chain maps $och_1$ and $och_2$ such that $\Phi(och_1) = [[\cdots, 1, 2, o o o]]$ and $\Phi(och_2) = [[\cdots, 2, 1, o o o]]$, where the parts indicated by $\cdots$ and $o o o$ are equal to each other, are $\mathcal{H}_g$-equivalent.

(c) Odd chain maps $och_1$ and $och_2$ such that $\Phi(och_1) = [[\cdots, 0, 1, o o o]]$ and $\Phi(och_2) = [[\cdots, 1, 0, o o o]]$, where the parts indicated by $\cdots$ and $o o o$ are equal to each other, are $\mathcal{H}_g$-equivalent.

**Proof.** For each $k = 1, \cdots, g$, we can show that $C_{2k}C_{2k-1}C_{2k+1}C_{2k}$ are elements of $\mathcal{H}_g$ by checking their actions on the fundamental group of $\Sigma_k$.

(a). As a representative of $[[\cdots, 0, 2, \cdots]]$, we take $[\cdots, i, 2k + 1, 2k + 2, \cdots]$, where $i \leq 2k - 2$. By Lemma 4.2, $(C_{2k}C_{2k-1}C_{2k+1}C_{2k})^{*}[\cdots, i, 2k + 1, 2k + 2, \cdots] = [\cdots, i, 2k - 1, 2k, \cdots]$. The odd chain map $[\cdots, i, 2k - 1, 2k, \cdots]$ is a representative of $[[\cdots, 2, 0, \cdots]]$.

(b). As a representative of $[[\cdots, 1, 2, \cdots]]$, we take $[\cdots, i, 2k - 1, 2k + 1, 2k + 2, \cdots]$, where $i \leq 2k - 2$. By Lemma 4.2, $(C_{2k}C_{2k-1}C_{2k+1}C_{2k})^{*}[\cdots, i, 2k - 1, 2k + 1, 2k + 2, \cdots] = [\cdots, i, 2k - 1, 2k, 2k + 1, \cdots]$. The odd chain map $[\cdots, i, 2k - 1, 2k, 2k + 1, \cdots]$ is a representative of $[[\cdots, 2, k + 1, \cdots]]$.

(c). As a representative of $[[\cdots, 0, 1, \cdots]]$, we take $[\cdots, i, 2k + 2, \cdots]$, where $i \leq 2k - 2$. By Lemma 4.2, $(C_{2k}C_{2k-1}C_{2k+1}C_{2k})^{*}[\cdots, i, 2k + 2, \cdots] = [\cdots, i, 2k, \cdots]$. The odd chain map $[\cdots, i, 2k, \cdots]$ is a representative of $[[\cdots, 1, 0, \cdots]]$.

By this lemma, we show that any odd chain map is $\mathcal{H}_g$-equivalent to a odd chain map $och$ such that $\Phi(och) = [[2, \cdots, 2, 1, \cdots, 1, 0, \cdots, 0]]$. 

\[\square\]
Lemma 4.6. Any odd subchain map representing a block sequence $\{2, \cdots, 2, 1, \cdots, 0, \cdots, 0\}$ is a product of $B_4 \in \mathcal{H}_g$ and odd chain maps representing $\{2, 2, 0, \cdots, 0\}$, $\{1, \cdots, 1, 0, \cdots, 0\}$ and $\{2, 1, \cdots, 1, 0, \cdots, 0\}$.

Proof. For a block sequence $\{2, \cdots, 2, 1, \cdots, 1, 0, \cdots, 0\}$, the length of sequence of 2’s is called $h$-length of the block sequence. By Lemma 4.3, we see,

$$\{2, 2, 0, \cdots, 0\} \{2, 0, \cdots, 0\} B_4 \{0, 2, \cdots, 0\} = \{0, 0, \cdots, 0\} \{2, 2, \cdots, 0\},$$

where $\cdots$ indicate the part which are the same each other. This equation shows that, if $h$-length of a block sequence is at least two, then this is a product of $B_4$, block sequences whose $h$-length are shorter than this, and $\{2, 2, 0, \cdots, 0\}$. By the induction on $h$-length, we show this lemma. $\square$

The block sequence $\{2, 2, 0, \cdots, 0\}$ is represented by $\{1, 2, 3, 4\} = T_{b_4}T_{b_4}$, which is an element of $\mathcal{H}_g$. As a representative of $\{1, \cdots, 2j+2, 1, 0, 0, \cdots, 0\}$, we take \[ 2, 3, 6, 7, \cdots, 4l-2, 4l-1, \cdots, 4j+2, 4j+3 = T_{d_j}T_{d_j}^{-1}, \] and as a representative of $\{2, 1, \cdots, 1, 0, 0, \cdots, 0\}$, we take $\{1, 2, 4, 5, \cdots, 4l, 4l+1, \cdots, 4k+4k+5\} = T_{e_k}T_{e_k}^{-1}$. Therefore, $\mathcal{H}_g \cup \{T_{d_j}T_{d_j}^{-1}, T_{e_k}T_{e_k}^{-1} \mid 1 \leq j \leq \lfloor \frac{2j+1}{2} \rfloor, 1 \leq k \leq \lfloor \frac{4k+5}{4} \rfloor \}$ generates $\mathcal{H}_g$. Takahashi [7] showed

Theorem 4.7 ([7]). $\mathcal{H}_g$ is generated by $C_1, C_2C_1^2C_2, C_2C_1C_3C_2, C_2C_{2i-1}B_{2i}C_{2i}, C_{2i}B_{2i+1}B_{2i}C_{2i}$ $(2 \leq i \leq g-1)$, $C_{2g}C_{2g-2}C_{2g+1}C_{2g}$.

Theorem 2.4 follows from this theorem and the above observation.

Remark 4.8. McCullough and Miller [5] showed that $\mathcal{I}_2$ is not finitely generated. So, the author does not know whether $\mathcal{H}_2$ is finitely generated or not.

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