Factor Rank and Its Preservers of Integer Matrices

Seok-Zun Song and Kyung-Tae Kang
Department of Mathematics, Cheju National University, Jeju 690-756, Korea
e-mail: szsong@cheju.ac.kr and kangkt@cheju.ac.kr

Abstract. We characterize the linear operators which preserve the factor rank of integer matrices. That is, if $M$ is the set of all $m \times n$ matrices with entries in the integers and $\min(m, n) > 1$, then a linear operator $T$ on $M$ preserves the factor rank of all matrices in $M$ if and only if $T$ has the form either $T(X) = UXV$ for all $X \in M$, or $m = n$ and $T(X) = UX^tV$ for all $X \in M$, where $U$ and $V$ are suitable nonsingular integer matrices. Other characterizations of factor rank-preservers of integer matrices are also given.

1. Introduction

The research of Linear Preserver Problems is an active area of matrix theory (see [1]-[7]). Many researchers have studied on the ranks and their preservers of matrices over fields ([1]-[5]). Also (nonnegative) integer matrices are combinatorially interesting matrices and hence it has been a subject of many research works ([6], [7]).

If $F$ is an algebraically closed field, which linear operators $T$ on the space of $m \times n$ matrices over $F$ preserve the rank of each matrix? Evidently if $U$ and $V$ are $m \times m$ and $n \times n$ nonsingular matrices, respectively, then $X \rightarrow UXV$ is a rank-preserving linear operator. When $m = n$, $X \rightarrow UX^tV$ is also. Already in 1957 Marcus and Moyls [4] found that such $(U, V)$-operators were the only rank preservers. Later they [5] obtained that $T$ preserves all ranks if and only if $T$ preserves rank 1. In 1981, Lautemann [3] extended these results to an arbitrary field, and found that $T$ preserves all ranks if and only if $T$ is bijective and preserves rank 1 if and only if $T$ is a $(U, V)$-operator.

In this paper, we characterize linear operators which preserve the factor ranks of all matrices over the ring of integers. That is, if $M$ is the set of all $m \times n$ matrices with entries in the integers and $\min(m, n) > 1$, then a linear operator $T$ on $M$ preserves the factor rank of all matrices in $M$ if and only if $T$ has the form either $T(X) = UXV$ for all $X \in M$, or $m = n$ and $T(X) = UX^tV$ for all $X \in M$, where $U$ and $V$ are suitable nonsingular integer matrices. Other characterizations of factor rank-preservers of integer matrices are also given.

Received September 2, 2005.
2000 Mathematics Subject Classification: 15A03, 15A04, 15A36.
Key words and phrases: factor rank preserver, $(U, V)$-operator.
2. Preliminaries and basic results

Let \( \mathcal{M}_{m \times n}(\mathbb{Z}) \) denote the set of all \( m \times n \) matrices with entries in the ring, \( \mathbb{Z} \) of integers. Addition, multiplication by scalars, and the product of matrices are defined as if \( \mathbb{Z} \) were a field. Let \( \mathcal{E}_{m,n} = \{ E_{ij} \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, n \} \), where \( E_{ij} \) is the \( m \times n \) matrix whose \((i,j)\)th entry is 1 and whose other entries are 0. We call each member of \( \mathcal{E}_{m,n} \) a cell.

Lowercase, boldface letters will represent vectors, a vector \( \mathbf{u} \) is column vector (\( \mathbf{u}^t \) is a row vector). A nonzero vector \( \mathbf{p} = [p_i] \) in \( \mathbb{Z}^n \) is irreducible if the greatest common divisor of nonzero \( p_i \)’s is 1 (that is, \( \gcd(p_1, \ldots, p_n) = 1 \)). A subset \( S = \{ \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_d \} \) of \( \mathbb{Z}^n \) is called linearly independent if there exist \( \alpha_1, \alpha_2, \ldots, \alpha_d \) in \( \mathbb{Z} \), not all zeros, such that \( \sum_{i=1}^{d} \alpha_i \mathbf{s}_i = \mathbf{0} \); \( S \) is called linearly independent if it is not linearly dependent.

An \( n \times n \) integer matrix \( A \) is called nonsingular if for any vector \( \mathbf{x} \) in \( \mathbb{Z}^n \), \( A\mathbf{x} = \mathbf{0} \) implies that \( \mathbf{x} = \mathbf{0} \). We note that nonsingularity and invertibility of a square integer matrix are not equivalent. For example, consider a matrix \( A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \) in \( \mathcal{M}_{2 \times 2}(\mathbb{Z}) \).

Then we can easily show that \( A \) is nonsingular but not invertible in \( \mathcal{M}_{2 \times 2}(\mathbb{Z}) \).

**Lemma 2.1.** Let \( \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n \) be linearly independent vectors in \( \mathbb{Z}^n \). Then for any nonzero vector \( \mathbf{b} \) in \( \mathbb{Z}^n \), there exist nonzero integer \( \beta \) and integers \( \alpha_i \), not all zero, such that \( \beta \mathbf{b} = \alpha_1 \mathbf{p}_1 + \alpha_2 \mathbf{p}_2 + \cdots + \alpha_n \mathbf{p}_n \).

**Proof.** Let \( A \) be the \( n \times n \) matrix whose columns are \( \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n \). Then \( A \) is nonsingular, and hence \( \det(A) \) is a nonzero integer. Consider a system \( A\mathbf{x} = \mathbf{b} \) of \( n \) linear equations in \( n \) unknowns. By Cramer’s rule, this system has a unique solution \( x_i = \frac{\det(A_i)}{\det(A)} \) in the rational numbers for all \( i = 1, 2, \ldots, n \), where \( A_i \) is the matrix obtained by replacing the entries in the \( i \)th column of \( A \) by the entries in \( \mathbf{b} \). Then we have

\[
\mathbf{b} = \frac{\det(A_1)}{\det(A)} \mathbf{p}_1 + \frac{\det(A_2)}{\det(A)} \mathbf{p}_2 + \cdots + \frac{\det(A_n)}{\det(A)} \mathbf{p}_n.
\]

If we take \( \beta = \det(A) \) and \( \alpha_i = \det(A_i) \), then the result follows. \( \square \)

If \( \mathbf{a} \) and \( \mathbf{b} \) are nonzero vectors in \( \mathbb{Z}^n \), we denote \( \mathbf{a} \simeq \mathbf{b} \) if \( \mathbf{a} \) and \( \mathbf{b} \) have an irreducible common factor. That is, \( \mathbf{a} \simeq \mathbf{b} \) if and only if there exists an irreducible vector \( \mathbf{p} \) in \( \mathbb{Z}^n \) such that \( \mathbf{a} = \alpha \mathbf{p} \) and \( \mathbf{b} = \beta \mathbf{p} \) for some nonzero integers \( \alpha \) and \( \beta \). Then we can easily show that \( \simeq \) is an equivalence relation in \( \mathbb{Z}^n \).

**Proposition 2.2.** If \( \mathbf{a} \) and \( \mathbf{b} \) are nonzero vectors in \( \mathbb{Z}^n \) with \( \alpha \mathbf{a} = \beta \mathbf{b} \) for some nonzero integers \( \alpha \) and \( \beta \), then we have \( \mathbf{a} \simeq \mathbf{b} \).

**Proof.** Let \( \mathbf{a} = [a_1, \ldots, a_n] \), \( \mathbf{b} = [b_1, \ldots, b_n] \) and \( \alpha' = \gcd(a_1, \ldots, a_n) \). Then there exists an irreducible vector \( \mathbf{p} \) in \( \mathbb{Z}^n \) such that \( \mathbf{a} = \alpha' \mathbf{p} \). Thus \( \alpha \mathbf{a} = \beta \mathbf{b} \) becomes

\[
\alpha \alpha' \mathbf{p} = \beta \mathbf{b}
\]
Let $\gamma = \gcd(\alpha \alpha', \beta)$, $\gamma_1 = \frac{\alpha \alpha'}{\gamma}$ and $\gamma_2 = \frac{\beta}{\gamma}$. Then $\gamma_1$ and $\gamma_2$ are nonzero in $\mathbb{Z}$, and (2.1) becomes

$$
(2.2) \quad \gamma_1 p = \gamma_2 b.
$$

Therefore we have that $\gamma_1$ divides every $\gamma_2 b_i$ for all $i = 1, \cdots, n$. Since $\gcd(\gamma_1, \gamma_2) = 1$ and $p$ is an irreducible vector, $\gamma_2 = \pm 1$ so that $b = \pm \gamma_1 p$. Therefore $a$ and $b$ have an irreducible common factor $p$, and thus $a \simeq b$. \hfill \Box

The factor rank, $fr(A)$, of a nonzero matrix $A \in \mathcal{M}_{m \times n}(\mathbb{Z})$ is defined as the least integer $k$ for which there exist $m \times k$ and $k \times n$ matrices $B$ and $C$, respectively, with $A = BC$. If the matrices were considered as matrices in the real field, then the factor ranks of them are the same as their ranks. The factor rank of a zero matrix is zero.

It is obvious that for a matrix $A$ in $\mathcal{M}_{m \times n}(\mathbb{Z})$, $fr(A) = 1$ if and only if there exist two nonzero vectors $a \in \mathbb{Z}^m$ and $x \in \mathbb{Z}^n$ such that $A = ax^t$. We call $a$ the left factor, and $x$ the right factor of $A$.

For any index $i \in \{1, \cdots, n\}$, we denote $e^{(n)}_i$ as the irreducible vector in $\mathbb{Z}^n$ with “1” in $i^{th}$ position and zero elsewhere.

**Lemma 2.3.** Let $A$ and $B$ be factor rank-1 matrices in $\mathcal{M}_{m \times n}(\mathbb{Z})$ with factorizations $A = ax^t$ and $B = by^t$, where $A + B \neq 0$. Then $fr(A + B) = 1$ if and only if $a \simeq b$ or $x \simeq y$.

**Proof.** Suppose that $fr(A + B) = 1$. Let

$$
A = ax^t = [x_1 a, \cdots, x_n a] = [a_1 x^t, \cdots, a_m x^t]^t
$$

and

$$
B = by^t = [y_1 b, \cdots, y_n b] = [b_1 y^t, \cdots, b_m y^t]^t.
$$

If $A + B$ has exactly one nonzero $i^{th}$ row or exactly one nonzero $j^{th}$ column, so do $A$ and $B$. In this case, $A$ and $B$ have an irreducible common left factor $e^{(m)}_i$ or an irreducible common right factor $e^{(n)}_j$. Thus we can assume that $A + B$ has at least two nonzero rows and at least two nonzero columns. Furthermore, without loss of generality, we may assume that columns of $A + B$ are all nonzero.

Case 1) $x_i y_i = 0$ for some $i \in \{1, \cdots, n\}$. If $x_i = 0$, then $y_i \neq 0$ because $A + B$ has no zero column. Since $A$ is not a zero matrix, there exists an index $j$ different from $i$ such that $x_j \neq 0$. Therefore, the $i^{th}$ and $j^{th}$ columns of $A + B$ are $y_i b$ and $x_j a + y_j b$, respectively. Since $fr(A + B) = 1$, there exist nonzero scalars $\alpha, \beta$ in $\mathbb{Z}$ such that $\alpha y_i b = \beta (x_j a + y_j b)$, equivalently $\beta x_j a = (\alpha y_i - \beta y_j) b$. Since $\beta x_j \neq 0$, we have $\alpha y_i - \beta y_j \neq 0$. It follows from Proposition 2.2 that $a \simeq b$. Similarly, a parallel argument holds if $y_i = 0$.

Case 2) $x_i y_i \neq 0$ for all $i = 1, \cdots, n$. Consider any distinct $i^{th}$ and $j^{th}$ columns of $A + B$. Since $fr(A + B) = 1$, there exist two nonzero scalars $\alpha$ and $\beta$ in $\mathbb{Z}$ such
that $\alpha(x; a + y; b) = \beta(x; a + y; b)$, equivalently $(\alpha x_i - \beta x_j) a = (\beta y_j - \alpha y_i) b$. If $\alpha x_i - \beta x_j \neq 0$, then we have $\beta y_j - \alpha y_i \neq 0$. By Proposition 2.2, we have $a \simeq b$. Now, if $\alpha x_i - \beta x_j = 0$, then $\alpha x_i - \beta x_j = \beta y_j - \alpha y_i = 0$. Thus,

\[
\alpha x_i = \beta x_j \quad \text{and} \quad \beta y_j = \alpha y_i.
\]

This shows that $x_i y_j = x_j y_i$ for all $i, j = 1, \ldots, n$. Thus there exist nonzero integers $s$ and $t$ such that $sx_i = ty_i$ for all $i = 1, \ldots, n$. Therefore we have $sx = ty$. It follows from Proposition 2.2 that $x \simeq y$. Thus we have shown the sufficiency.

The necessity is an immediate consequence. \hfill \Box

3. Factor rank-1 preserver

Suppose that $T$ is a linear operator on $\mathcal{M}_{m \times n}(\mathbb{Z})$. Then $T$ is a

(i) $(U, V)$-operator if there exist nonsingular matrices $U$ in $\mathcal{M}_{m \times m}(\mathbb{Z})$ and $V$ in $\mathcal{M}_{n \times n}(\mathbb{Z})$ such that $T(X) = UXV$ for all $X$ in $\mathcal{M}_{m \times n}(\mathbb{Z})$, or $m = n$ and $T(X) = UX^tV$ for all $X$ in $\mathcal{M}_{m \times n}(\mathbb{Z})$, where $X^t$ denotes the transpose of $X$;

(ii) factor rank preserver if $fr(T(X)) = fr(X)$ for all $X$ in $\mathcal{M}_{m \times n}(\mathbb{Z})$;

(iii) factor rank-$k$ preserver if $fr(T(X)) = k$ whenever $fr(X) = k$ for all $X$ in $\mathcal{M}_{m \times n}(\mathbb{Z})$.

Lemma 3.1. If $T$ is a $(U, V)$-operator on $\mathcal{M}_{m \times n}(\mathbb{Z})$, then $T$ is an injective factor rank preserver.

Proof. It follows directly from the definition of a $(U, V)$-operator. \hfill \Box

Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and a linear operator $T$ on $\mathcal{M}_{2 \times 2}(\mathbb{Z})$ defined by $T(X) = AX$ for all $X$ in $\mathcal{M}_{2 \times 2}(\mathbb{Z})$. Then $T$ is a $(U, V)$-operator because $A$ is nonsingular. Clearly, $T$ is injective. But $T$ is not surjective: for any cell $E_{ij}$ in $\mathbb{E}_{2, 2}$, there is not a matrix $X$ in $\mathcal{M}_{2 \times 2}(\mathbb{Z})$ such that $T(X) = E_{ij}$. Therefore a $(U, V)$-operator on $\mathcal{M}_{m \times n}(\mathbb{Z})$ may not be invertible.

For any matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathcal{M}_{m \times n}(\mathbb{Z})$, let $A \circ B$ denote the Hadamard (or Schur) product, the $(i, j)^{th}$ entry of $A \circ B$ is $a_{ij}b_{ij}$.

Lemma 3.2. Let $B = [b_{ij}]$ be a factor rank-1 matrix in $\mathcal{M}_{m \times n}(\mathbb{Z})$. Then there exist diagonal matrices $D$ in $\mathcal{M}_{m \times m}(\mathbb{Z})$ and $E$ in $\mathcal{M}_{n \times n}(\mathbb{Z})$ such that $X \circ B = DXE$ for all $X$ in $\mathcal{M}_{m \times n}(\mathbb{Z})$.

Proof. If $fr(B) = 1$, then there exist vectors $d = [d_1, d_2, \ldots, d_m]^t$ and $e = [e_1, e_2, \ldots, e_n]^t$ such that $B = de^t$, equivalently $b_{ij} = d_i e_j$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let $D = diag(d_1, \ldots, d_m)$ and $E = diag(e_1, \ldots, e_n)$. Now, the $(i, j)^{th}$ entry of $X \circ B$ is $x_{ij}b_{ij}$ and the $(i, j)^{th}$ entry of $DXE$ is $d_i x_{ij} e_j = x_{ij}b_{ij}$. Therefore we have the results. \hfill \Box
**Theorem 3.3.** Let $T$ be a linear operator on $\mathbb{M}_{m \times n}$. Then $T$ is an injective factor rank-1 preserver if and only if $T$ is a $(U,V)$-operator.

**Proof.** The sufficiency follows from Lemma 3.1. So, we shall show the necessity. For any cell $E_{ij}$ in $\mathbb{M}_{m,n}$, we can write $T(E_{ij}) = u_{ij}^T v_{ij}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, where $u_{ij} \in \mathbb{Z}^n$ and $v_{ij} \in \mathbb{Z}^n$ are nonzero vectors. Let $j$ and $k$ be arbitrary integers in $\{1, \ldots, n\}$. Since $E_{ij} + E_{ik}$ is of factor rank-1, the factor rank of $T(E_{ij} + E_{ik}) = u_{ij}^T v_{ij} + u_{ik}^T v_{ik}$ must be 1. It follows from Lemma 2.3 that $u_{ij} \simeq u_{ik}$ or $v_{ij} \simeq v_{ik}$. Now, we will show that for a fixed $i$ in $\{1, \ldots, m\}$, either

$$u_{i1} \simeq u_{i2} \simeq \cdots \simeq u_{in} \quad \text{or} \quad v_{i1} \simeq v_{i2} \simeq \cdots \simeq v_{in}. \quad (3.1)$$

Suppose that $v_{i1} \not\simeq v_{ij}$ for some index $j$. By Lemma 2.3, we have $u_{i1} \simeq u_{ij}$ because $fr(T(E_{i1} + E_{ij})) = 1$. If $u_{i1} \not\simeq u_{ik}$ for some index $k$, then we have $v_{i1} \simeq v_{ik}$ by Lemma 2.3. Therefore $v_{ij} \not\simeq v_{ik}$ because $\simeq$ is an equivalence relation. But then $u_{ij} \simeq u_{ik}$ and this would imply $u_{i1} \simeq u_{ik}$ because $u_{ij} \simeq u_{ij}$. This contradicts to $u_{i1} \not\simeq u_{ik}$, and thus $(3.1)$ is established.

Similarly, we can show that for a fixed $j$ in $\{1, \ldots, n\}$, either

$$u_{1j} \simeq u_{2j} \simeq \cdots \simeq u_{mj} \quad (3.2)$$

or

$$v_{1j} \simeq v_{2j} \simeq \cdots \simeq v_{mj}. \quad (3.3)$$

If $u_{i1} \simeq u_{i2} \simeq \cdots \simeq u_{in}$, there exist an irreducible vector $p_i$ in $\mathbb{Z}^m$ and nonzero integers $c_j$ such that $u_{ij} = c_j p_i$ for all $j = 1, \ldots, n$. Thus we have $T(E_{ij}) = p_i(c_j v_{ij})$ for all $j = 1, \ldots, n$. We can therefore restate $(3.1)$ as follows. For a fixed $i$ in $\{1, \ldots, m\}$, either

$$u_{i1} = u_{i2} = \cdots = u_{in} = p_i \quad (3.4)$$

or

$$v_{i1} = v_{i2} = \cdots = v_{in} = q_i, \quad (3.5)$$

where $p_i$ and $q_i$ are irreducible vectors.

Assume that $(3.4)$ holds for some $i$. If $v_{i1}, v_{i2}, \ldots, v_{in}$ are linearly dependent, then there exist $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $\mathbb{Z}$, not all zeros, such that $\sum_{j=1}^{n} \alpha_j v_{ij} = 0$. Consider a factor rank-1 matrix $X = \sum_{j=1}^{n} \alpha_j E_{ij}$. Then we have

$$T(X) = T\left(\sum_{j=1}^{n} \alpha_j E_{ij}\right) = p_i \left(\sum_{j=1}^{n} \alpha_j v_{ij}\right) = 0,$$
exist an irreducible vectors

follows from (3.2) that there exist nonzero integers

for all $j = 1, \ldots, n$, and consequently (3.4) must hold for all $i$. Suppose that (3.2) holds for some $j = 1, \ldots, n$. Then $u^{ij} (= p_i)$ appears both in (3.4) and (3.2). It follows from (3.2) that there exist nonzero integers $\alpha_s$ such that $u^{ij} = \alpha_s p_i$ for all $s = 1, \ldots, m$. Note that $v_{i1}, v_{i2}, \ldots, v_{in}$ are linearly independent since (3.4) is satisfied. By Lemma 2.1, there exist nonzero integer $\beta_s$ and integers $\beta_{sk}$, not all zero, such that $\beta_s v_{sj} = \sum_{k=1}^n \beta_{sk} v_{ik}$ for all $s = 1, \ldots, m$. Then we have

$$\beta_s u^{ij} v_{sj}^t = \sum_{k=1}^n \beta_{sk} u^{ij} v_{ik} = \sum_{k=1}^n \beta_{sk} \alpha_s p_i v_{ik}^t = \sum_{k=1}^n \beta_{sk} \alpha_s u^{ik} v_{ik}^t,$$

equivalently $T(\beta_s E_{ij}) = T\left( \sum_{k=1}^n \beta_{sk} \alpha_s E_{ik} \right)$ for all $s \in \{1, \ldots, m\} \setminus \{i\}$. This contradicts to the fact that $T$ is injective. Thus we have established that either

(3.6)

$$u^{ij} = p_i \quad \text{and} \quad v_{ij} = b_{ij} q_j$$

for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, where $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ are linearly independent irreducible vectors and $b_{ij}$ are nonzero integers, or

(3.7)

$$v_{ij} = q_i \quad \text{and} \quad u^{ij} = b_{ij} p_j$$

for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, where $q_1, \ldots, q_m$ and $p_1, \ldots, p_n$ are linearly independent irreducible vectors and $b_{ij}$ are nonzero integers.

If $m \neq n$, (3.7) is not possible. For, if $m < n$, then the set $\{p_1, \ldots, p_m\}$ would be linearly dependent by Lemma 2.1. Similar conclusion follows if $m > n$. Hence, if $m \neq n$, only (3.6) is possible.

Assume that (3.6) holds. Let $U'$ be the $m \times m$ matrix whose columns are $p_1, \ldots, p_m$ and let $V'$ be the $n \times n$ matrix whose rows are $q_1, \ldots, q_n$. Then $U'$ and $V'$ are nonsingular, and

$$T(E_{ij}) = u^{ij} v_{ij}^t = p_{i1} b_{ij} q_j^t = U'(b_{ij} E_{ij}) V'$$

for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. It follows that for any matrix $X$ in $\mathcal{M}_{m \times n}(\mathbb{Z})$, we have $T(X) = U'(X \circ B) V'$, where $B = [b_{ij}]$ as above. Now, we claim $fr(B) = 1$.

If not, there exists a $2 \times 2$ submatrix $B' = \begin{bmatrix} b_{ij} & b_{ik} \\ b_{ij} & b_{ik} \end{bmatrix}$ of $B$ such that $fr(B') = 2$. Consider a factor rank-1 matrix $Y = E_{ij} + E_{ik} + E_{ij} + E_{ik}$. Then the factor rank of

$$T(Y) = p_i (b_{ij} q_j + b_{ik} q_k)^t + p_l (b_{lj} q_j + b_{lk} q_k)^t$$

must be 1. Since $p_i \neq p_l$, it follows that $b_{ij} q_j + b_{ik} q_k \simeq b_{lj} q_j + b_{lk} q_k$. Therefore there exist an irreducible vectors $q$ and nonzero integers $\alpha$ and $\beta$ such that $b_{ij} q_j + b_{ik} q_k =$
\[ \alpha q \text{ and } b_{ij}q_{ij} + b_{ik}q_{ik} = \beta q, \text{ equivalently } (b_{ij}\beta - b_{ij}\alpha)q_{ij} = (b_{ik}\alpha - b_{ik}\beta)q_{ik}. \] It follows from \( q_{ij} \neq q_{ik} \) that \( b_{ij}\beta - b_{ij}\alpha = b_{ik}\alpha - b_{ik}\beta = 0 \) so that \( b_{ij}b_{ik} = b_{ik}b_{ij} \). This implies that the factor rank of \( B' \) is 1, a contradiction. Therefore we have \( fr(B) = 1 \). By Lemma 3.2, there exist diagonal matrices \( D \) in \( \mathcal{M}_{m \times m}(\mathbb{Z}) \) and \( E \) in \( \mathcal{M}_{n \times n}(\mathbb{Z}) \) such that \( X \circ B = DXE \) for all \( X \) in \( \mathcal{M}_{m \times n}(\mathbb{Z}) \). Since \( B \) has no zero entries, it follows that \( D \) and \( E \) are nonsingular. Let \( U = U'D \) and \( V = EV' \). Then \( U \) and \( V \) are nonsingular. Furthermore, we have \( T(X) = UXV \) for all matrix \( X \) in \( \mathcal{M}_{m \times n}(\mathbb{Z}) \). Therefore \( T \) is a \((U, V)\)-operator.

If (3.7) holds, then \( m = n \) and we can easily establish that for any matrix \( X \) in \( \mathcal{M}_{m \times n}(\mathbb{Z}) \), \( T(X) = UXV \) for some \( n \times n \) nonsingular matrices \( U \) and \( V \). Therefore \( T \) is a \((U, V)\)-operator. \( \square \)

4. Factor rank preserver

In this section, we characterize the linear operators which preserve the factor rank of all matrices over the ring of integers.

**Proposition 4.1.** Let \( A \) and \( B \) be matrices in \( \mathcal{M}_{m \times n}(\mathbb{Z}) \) with \( \alpha A \neq \beta B \) for all nonzero scalars \( \alpha, \beta \in \mathbb{Z} \). If \( fr(A) = fr(B) = 1 \), then there exists a factor rank-1 matrix \( C \) in \( \mathcal{M}_{m \times n}(\mathbb{Z}) \) such that \( fr(A + C) = 1 \) and \( fr(B + C) = 2 \).

**Proof.** Since \( fr(A) = fr(B) = 1 \), it follows from \( \alpha A \neq \beta B \) that either \( fr(A + B) = 2 \) or \( fr(A + B) = 1 \). For the case of \( fr(A + B) = 2 \), the conclusion is satisfied by letting \( C = A \). So we may assume that \( fr(A + B) = 1 \). By Lemma 2.3, \( A \) and \( B \) have an irreducible common factor. If \( A \) and \( B \) have an irreducible common left factor, then we may write \( A \) and \( B \) as

\[
A = a x^t = [x_1a, \cdots, x_na] \quad \text{and} \quad B = a y^t = [y_1a, \cdots, y_na],
\]

where \( a \) is an irreducible vector. Then we have \( \alpha x \neq \beta y \) for all nonzero integers \( \alpha \) and \( \beta \) because \( \alpha A \neq \beta B \). Since \( a = [a_i] \) is not zero-vector, \( a_i \neq 0 \) for some \( i = 1, \cdots, m \). Let

\[
C = \begin{cases} 
    e_{(m)}^{(i)} x^t & \text{if } a_j = 0 \text{ for some } j \neq i, \\
    e_{(m)}^{(i)} x^t & \text{otherwise.}
\end{cases}
\]

Then \( C \) is a matrix in \( \mathcal{M}_{m \times n}(\mathbb{Z}) \) with \( fr(C) = 1 \). Moreover \( fr(A + C) = 1 \) because \( A \) and \( C \) have a common right factor. But \( B \) and \( C \) have neither a common left factor nor a common right factor. It follows from Lemma 2.3 that \( fr(B + C) = 2 \).

Similarly, a parallel argument holds if \( A \) and \( B \) have an irreducible common right factor. \( \square \)

**Lemma 4.2.** Let \( T \) be a factor rank-1 preserver on \( \mathcal{M}_{m \times n}(\mathbb{Z}) \). If \( T \) is not injective, then \( T \) decreases the factor rank of some factor rank-2 matrix.

**Proof.** By the similar proof to that of Theorem 3.3, we can see that \( T \) is a \((U, V)\)-operator if \( T \) is a factor rank-1 preserver and is injective in the set of all factor rank-1
matrices in $M_{m \times n}(\mathbb{Z})$. If $T$ is not injective, then $T$ is not a $(U, V)$-operator. From above fact we have that $T$ is not injective in the set of all factor rank-1 matrices in $M_{m \times n}(\mathbb{Z})$. Thus there exist distinct factor rank-1 matrices $X$ and $Y$ such that $T(X) = T(Y)$. Suppose that there exist distinct nonzero integers $\alpha$ and $\beta$ such that $\alpha X = \beta Y$. Then we have

$$\alpha T(X) = T(\alpha X) = T(\beta Y) = \beta T(Y) = \beta T(X).$$

Since $\mathbb{Z}$ has no zero divisors and $T(X) \neq O$, we have $\alpha = \beta$, a contradiction. So, we may assume that $\alpha X \neq \beta Y$ for all nonzero scalars $\alpha, \beta \in \mathbb{Z}$. By Proposition 4.1, there exists a factor rank-1 matrix $C$ such that $fr(X + C) = 1$ while $fr(Y + C) = 2$.

But we then have $T(Y + C) = T(X + C)$ so that $fr(T(Y + C)) = fr(T(X + C)) = 1$ because $T$ is a factor rank-1 preserver. Therefore $T$ decreases the factor rank of some factor rank-2 matrix. □

**Theorem 4.3.** Let $T$ be a linear operator on $M_{m \times n}(\mathbb{Z})$. Then the following are equivalent:

(i) $T$ is an injective factor rank-1 preserver;

(ii) $T$ is a $(U, V)$-operator;

(iii) $T$ is a factor rank preserver;

(iv) $T$ is a factor rank-1 and factor rank-2 preserver.

**Proof.** It follows from Theorem 3.3 that (i) and (ii) are equivalent. Statement (ii) implies (iii) by Lemma 3.1. Clearly, (iii) implies (iv). Lemma 4.2 shows if $T$ preserves the factor ranks of all factor rank-1 matrices and factor rank-2 matrices, then $T$ is injective. Thus, (iv) implies (i). □

Thus we have characterized the linear operators that preserve the factor rank of integer matrices.

**References**


