Two More Radicals for Right Near-Rings: The Right Jacobson Radicals of Type-1 and 2

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Abstract. Near-rings considered are right near-rings and $R$ is a near-ring. $J^r_0(R)$, the right Jacobson radical of $R$ of type-0, was introduced and studied by the present authors. In this paper $J^r_1(R)$ and $J^r_2(R)$, the right Jacobson radicals of $R$ of type-1 and type-2 are introduced. It is proved that both $J^r_1$ and $J^r_2$ are radicals for near-rings and $J^r_0(R) \subseteq J^r_1(R) \subseteq J^r_2(R)$. Unlike the left Jacobson radical classes, the right Jacobson radical class of type-2 contains $M_0(G)$ for many of the finite groups $G$. Depending on the structure of $G$, $M_0(G)$ belongs to different right Jacobson radical classes of near-rings. Also unlike left Jacobson-type radicals, the constant part of $R$ is contained in every right 1-modular (2-modular) right ideal of $R$. For any family of near-rings $R_i$, $i \in I$, $J^r_{\nu}(\bigoplus_{i \in I} R_i) = \bigoplus_{i \in I} J^r_{\nu}(R_i)$, $\nu \in \{1, 2\}$. Moreover, under certain conditions, for an invariant subnear-ring $S$ of a d.g. near-ring $R$ it is shown that $J^r_2(S) = S \cap J^r_2(R)$.

1. Introduction

Throughout this paper we consider only right near-rings. Many radicals of near-rings and the corresponding structure theories have been developed. But almost all of them give structures of near-rings in term of ideals and left ideals but not in terms of right ideals (which are not ideals). In [2] and [3] the first author has established that only right ideals are relevant for the extension of the Wedderburn-Artin theorem to near-rings. This motivated the authors to develop and study the right Jacobson radicals for near-rings. In [4] the right Jacobson radical of type-0 was introduced and studied. In subsequent papers the authors will present the structure theorems of near-rings given by the right Jacobson radicals. In this paper, the right Jacobson radicals of type-1 and 2 for near-rings are intro-
duced and studied. It is proved that both of them are radicals for near-rings. Let 
\((G, +)\) be a finite group. We know that the left ideals of \(M_0(G)\) do not depend on 
the structure of the group \(G\) and that left Jacobson radicals also do not depend on 
the structure of \(G\). We know that \(J_2(M_0(G)) = \{0\}\). But, we see in this paper 
that the right Jacobson radicals of \(M_0(G)\) depend on the nature of \(G\) and many of 
them are right Jacobson radical near-rings of type-2, in contrast to the left Jacobson 
radicals. Unlike left Jacobson-type radicals, the constant part of \(R\) is contained in 
every right 1-modular (2-modular) right ideal of \(R\). For any family of near-rings \(R_i, \ i \in I\), \(J_\nu^r(\oplus_{i \in I} R_i) = \oplus_{i \in I} J_\nu^r(R_i)\), where \(J_\nu^r\) is the right Jacobson radical of type-\(\nu\), 
\(\nu \in \{1, 2\}\). Moreover, under certain conditions, for an invariant subnear-ring \(S\) of a 
d.g. near-ring \(R\), it is shown that \(J_2^r(S) = S \cap J_2^r(R)\).

2. Preliminaries

Throughout this paper \(R\) is a right near-ring and all notations and definitions 
will be as in [1].

Definition 2.1. A group \((G, +)\) is called a right \(R\)-group if there is a mapping 
\(((g, r) \to gr)\) of \(G \times R\) into \(G\) such that

1. \((g + h)r = gr + hr\), and

2. \(g(rs) = (gr)s\), for all \(g, h \in G\) and \(r, s \in R\).

Definition 2.2. Let \(G\) be a right \(R\)-group. An element \(g \in G\) is called a generator 
of \(G\), if \(gR = G\) and \(g(r + s) = gr + gs\) for all \(r, s \in R\). \(G\) is said to be monogenic, 
if \(G\) has a generator.

Definition 2.3. Let \(G\) be a right \(R\)-group. A normal subgroup \(B\) of \(G\) is called an 
ideal of \(G\) if \(BR \subseteq B\). A subgroup \(B\) of \(G\) is called an \(R\)-subgroup of \(G\) if \(BR \subseteq B\). 
\(G\) is said to be simple if \(GR \neq \{0\}\) and \(\{0\}\) and \(G\) are the only ideals of \(G\).

Definition 2.4. A monogenic right \(R\)-group \(G\) is said to be of type-0 if \(G\) is simple.

Definition 2.5. Let \(G, H\) be right \(R\)-groups. A mapping \(f : G \to H\) is called an 
\(R\)-homomorphism if

1. \(f(x + y) = f(x) + f(y)\) and

2. \(f(xr) = f(x)r\), for all \(x, y \in G\) and for all \(r \in R\).

We say that \(G\) is \(R\)-isomorphic to \(H\) if there is a one-to-one \(R\)-homomorphism 
of \(G\) onto \(H\).

Definition 2.6. A right ideal \(K\) of \(R\) is called right modular if there is an element 
\(e \in R\) such that \(xe - ex \in K\) for all \(x \in R\). In this case we say that \(K\) is right 
modular by \(e\).
Definition 2.7. An element $a \in R$ is called right quasi-regular if and only if the right ideal of $R$ generated by the set $\{x - ax \mid x \in R\}$ is $R$.

We need the following results of [4].

Proposition 2.8. Let $R_c$ be the constant part of $R$. Then each element of $R_c$ is right quasi-regular.

Proposition 2.9. Let $G$ be a right $R$-group. Then $G$ is monogenic if and only if there is a right modular right ideal $K$ of $R$ such that $G$ is $R$-isomorphic to $R/K$.

Proposition 2.10. Let $G$ be a right $R$-group. $G$ is a right $R$-group of type-0 if and only if there is a maximal right modular right ideal $K$ of $R$ such that $G$ is $R$-isomorphic to $R/K$.

Proposition 2.11. Let $R$ be a zero-symmetric near-ring and $K$ be a right ideal of $R$ right modular by $e$. Then $(K: R) = (K: eR)$ and the largest ideal of $R$ contained in $K$ is the largest ideal of $R$ contained in $(K: R)$.

Theorem 2.12. A right 0-primitive ideal of $R$ is a prime ideal of $R$.

Proposition 2.13. Let $G$ be a monogenic right $R$-group. If $R$ is a distributively generated (d.g.) near-ring then there is a subset $T$ of $G$ such that $h(a + b) = ha + hb$ for all $h \in T$ and $a, b \in R$ and $T$ generates $(G, +)$.

3. Right Jacobson radicals of type-1 and 2

Definition 3.1. A right $R$-group $G$ of type-0 is said to be of type-1 if $G$ has exactly two $R$-subgroups, namely $\{0\}$ and $G$.

Remark 3.2. Let $G$ be a right $R$-group. Then $\{g \in G \mid gR = \{0\}\}$ is an ideal of $G$.

Definition 3.3. A right $R$-group $G$ of type-0 is said to be of type-2, if $gR = G$ for all $0 \neq g \in G$.

Remark 3.4. Clearly a right $R$-group of type-2 is of type-1.

Now we give some examples of right $R$-groups of type-0, 1 and 2.

Example 3.5. Let $(G, +)$ be a finite non-abelian simple group. Since $\{0\}$ is the maximal normal subgroup of $(G, +)$, $\{0\}$ is the maximal right ideal of $M_0(G)$ and hence $M_0(G)$ is a right $M_0(G)$-group of type-0. But $M_0(G)$ is not a $M_0(G)$-group of type-1. Let $0 \neq a \in G$ and let $H$ be the cyclic subgroup of $G$ generated $a$. Now $H \neq \{0\}$ and $H \neq G$. So $(H : G) = \{f \in M_0(G) \mid f(x) \in H, \text{ for all } x \in G\}$ is a right $M_0(G)$-subgroup of $M_0(G)$ and $(H : G) \neq \{0\}$, $(H : G) \neq M_0(G)$. Therefore, $M_0(G)$ is not a right $M_0(G)$-group of type-1.

Example 3.6. Let $(G, +)$ be a finite cyclic group of prime order greater than
2. Then $M_0(G)$ is a right $M_0(G)$-group of type-1. Since $\{0\}$ is the only proper subgroup of $G$, $\{0\}$ is the only proper right $M_0(G)$-subgroup of $M_0(G)$ and is right modular by the identity element of $M_0(G)$. Therefore, $M_0(G)$ is a right $M_0(G)$-group of type-1. Clearly $M_0(G)$ is not a right $M_0(G)$-group of type-2, as $M_0(G)$ is not a near-field.

**Example 3.7.** A near-field $R$ is a right $R$-group of type-2.

**Definition 3.8.** Let $\nu \in \{0,1,2\}$. A right modular right ideal $K$ of $R$ is called right $\nu$-modular if $R/K$ is a right $R$-group of type-$\nu$.

**Remark 3.9.** Let $K$ be a right ideal of $R$. $K$ is right 0-modular if and only if $K$ is a maximal right modular right ideal of $R$ and $K$ is a right 1-modular if and only if $K$ is a maximal right modular right $R$-subgroup of $R$.

**Definition 3.10.** Let $\nu \in \{0,1,2\}$. An ideal $P$ of $R$ is called right $\nu$-primitive if $P$ is the largest ideal of $R$ contained in a right $\nu$-modular right ideal of $R$.

**Definition 3.11.** Let $\nu \in \{0,1,2\}$. $D_\nu'(R)$ denotes the intersection of all right $\nu$-modular right ideals of $R$. If $R$ has no right $\nu$-modular right ideals then $D_\nu'(R)$ is defined as $R$.

**Definition 3.12.** Let $\nu \in \{0,1,2\}$. $J_\nu'(R)$ denotes the intersection of all right $\nu$-primitive ideals of $R$. If $R$ has no right $\nu$-primitive ideals then $J_\nu'(R)$ is defined as $R$. $J_\nu'$ is called the right Jacobson radical of type-$\nu$.

Let $G$ be a finite group and $R = M_0(G)$. Then, we have that $\{0\} = P(R) = N(R) = J_0(R) = J_2(R) = J_3(R)$.

**Remark 3.13.** Let $R$ be the near-ring given in Example 3.5. We get that $J_0'(R) = \{0\}$ and $J_1'(R) = R$.

**Remark 3.14.** Let $R$ be the near-ring given in Example 3.6. It is clear that $J_1'(R) = \{0\}$ and $J_2'(R) = R$.

The following result shows that the abelian property of $(R, +)$ depends on the abelian property of a monogenic faithful right $R$-group, if $R$ is d.g..

**Proposition 3.15.** Let $G$ be a monogenic right $R$-group and $R$ be a d.g. near-ring. If $(0 : G) = \{0\}$ and $G$ is abelian then $(R, +)$ is also an abelian group.

**Proof.** Suppose that $(0 : G) = \{0\}$ and $G$ is abelian. Since $R$ is d.g., by Proposition 2.13 we get a subset $H$ of $G$ such that $H$ generates $(G, +)$ and $h(r+s) = hr + hs$, for all $h \in H$ and $r, s \in R$. Let $r, s \in R$. Now $h(r+s-r-s) = hr + hs - hr - hs = 0$ as $G$ is abelian. So, $r+s-r-s \in (0 : H) = (0 : G) = \{0\}$. Therefore, $r+s = s+r$ and hence $(R, +)$ is an abelian group.

The following propositions follow easily from the definitions.
Proposition 3.16. Let \( \nu \in \{1, 2\} \). Let \( K \) be a right \( \nu \)-modular right ideal of \( R \). Let \( I \) be an ideal of \( R \) contained in \( K \). Then \( K/I \) is also a right \( \nu \)-modular right ideal of \( R/I \).

Proposition 3.17. Let \( \nu \in \{1, 2\} \). Let \( I \) be an ideal of \( R \). If \( K/I \) is a right \( \nu \)-modular right ideal of \( R/I \) then \( K \) is also a right \( \nu \)-modular right ideal of \( R \).

We see now that for \( \nu \in \{1, 2\} \), \( J_\nu^r \) is a radical map.

Theorem 3.18. \( R \to J_\nu^r(R) \) is a radical map for \( \nu \in \{1, 2\} \).

Proof. Let \( \nu \in \{1, 2\} \). First suppose that \( R \) has no right \( \nu \)-modular right ideal. Now \( R = J_\nu^r(R) \) and hence \( R/J_\nu^r(R) = \{0\} \). So \( J_\nu^r(R/J_\nu^r(R)) = \{0\} \). Suppose now that \( R \) has a right \( \nu \)-modular right ideal. Let \( \{K_\alpha \mid \alpha \in \Delta\} \) be the collection of all right \( \nu \)-modular right ideals of \( R \). Since \( K_\alpha \) is a right \( \nu \)-modular right ideal of \( R \) and \( J_\nu^r(R) \subseteq K_\alpha \), \( K_\alpha /J_\nu^r(R) \) is a right \( \nu \)-modular right ideal of \( R/J_\nu^r(R) \) for all \( \alpha \in \Delta \).

So, \( J_\nu^r(R/J_\nu^r(R)) \subseteq \cap_{\alpha \in \Delta} (K_\alpha /J_\nu^r(R)) = (\cap_{\alpha \in \Delta} K_\alpha /J_\nu^r(R)). \) Since \( J_\nu^r(R) \) is the largest ideal of \( R \) contained in \( \cap_{\alpha \in \Delta} K_\alpha \), we get that the largest ideal of \( R/J_\nu^r(R) \) contained in \( \cap_{\alpha \in \Delta} (K_\alpha /J_\nu^r(R)) \) is the zero ideal. Therefore \( J_\nu^r(R/J_\nu^r(R)) = \{0\} \).

Let \( h \) be a homomorphism of the the near-ring \( R \) onto a near-ring \( S \). If \( S \) has no right \( \nu \)-modular right ideal then \( J_\nu^r(S) = S \). Then clearly \( h(J_\nu^r(R)) \subseteq S = J_\nu^r(S) \).

Suppose that \( S \) has a right \( \nu \)-modular right ideal. Let \( \{L_\alpha \mid \alpha \in \Delta\} \) be the collection of all right \( \nu \)-modular right ideals of \( S \).

Now \( h^{-1}(L_\alpha) \) is a right \( \nu \)-modular right ideal of \( R \) for each \( \alpha \in \Delta \). Let \( K_\alpha = h^{-1}(L_\alpha), \alpha \in \Delta \). We have that \( h(h^{-1}(L_\alpha)) = L_\alpha, \) for all \( \alpha \in \Delta \) and also \( J_\nu^r(R) \subseteq \cap_{\alpha \in \Delta} K_\alpha \). So \( h(J_\nu^r(R)) \subseteq h(\cap_{\alpha \in \Delta} K_\alpha) \subseteq \cap_{\alpha \in \Delta} h(K_\alpha) = \cap_{\alpha \in \Delta} L_\alpha \). Since \( h(J_\nu^r(R)) \) is an ideal of \( S \) and \( J_\nu^r(S) \) is the largest ideal of \( S \) contained in \( \cap_{\alpha \in \Delta} L_\alpha \), \( h(J_\nu^r(R)) \subseteq J_\nu^r(S) \).

Therefore \( R \to J_\nu^r(R) \) is a radical map.

\( \square \)

Theorem 3.19. \( D_1^r(R) \) contains all right quasi-regular right \( R \)-subgroups of \( R \).

Proof. If \( D_1^r(R) = R \), then we get the result. Suppose that \( D_1^r(R) \neq R \). Let \( K \) be a right quasi-regular right \( R \)-subgroup of \( R \). Assume that \( K \not\subseteq D_1^r(R) \). We get a right 1-modular right ideal \( M \) of \( R \) such that \( K \not\subseteq M \). So, \( M + K = R \). Let \( M \) be right modular by \( e \). Now \( m + k = e, m \in M, k \in K \). It is clear that \( x - ex \in M \), for all \( x \in R \). Since \( e - k = m \in M \), \( ex - kx \in M \), for all \( x \in R \). Now \( x - kx = (x - kx) \), \( kx \in M \), for all \( x \in R \). This is a contradiction to the fact that \( k \) is right quasi-regular. Therefore, \( K \subseteq D_1^r(R) \). Hence, \( D_1^r(R) \) contains all right quasi-regular right \( R \)-subgroups of \( R \). \( \square \)

Corollary 3.20. \( R_c \) is a subset of \( D_1^r(R) \), where \( R_c \) is the constant part of \( R \).

Proof. By Proposition 2.8, \( R_c \) is right quasi-regular. Since \( R_c \) is right \( R \)-subgroup of \( R \), by Theorem 3.19, \( R_c \) is contained in \( D_1^r(R) \).

\( \square \)

Corollary 3.21. If \( D_1^r(R) \) is an ideal of \( R \), then \( R/D_1^r(R) \) is zero-symmetric.

Corollary 3.22. If \( D_1^r(R) = \{0\} \), then \( R \) is zero-symmetric.
Theorem 3.23. $D^2_\nu(R)$ contains all right quasi-regular subsets of $R$ of the form $aR = \{ax \mid x \in R\}$, $a \in R$.

Proof. If $D^2_\nu(R) = R$, then we get the result. Suppose that $D^2_\nu(R) \neq R$. Let $aR$ be a right quasi-regular subset of $R$, $a \in R$. Assume that $aR \not\subseteq D^2_\nu(R)$. We get a right 2-modular right ideal $M$ of $R$ such that $aR \not\subseteq M$. Therefore, $M + aR = R$. Let $M$ be right modular by $e$. Now $m + ab = e$, $m \in M$, $b \in R$. We have that $x - ex \in M$, for all $x \in R$. Since $e - ab = m \in M$, $ex - abx \in M$, for all $x \in R$. Now $x - abx = (x - ex) + (ex - abx) \in M$, for all $x \in R$. This is a contradiction to the fact that $ab$ is right quasi-regular. Therefore, $aR \subseteq D^2_\nu(R)$. Hence, $D^2_\nu(R)$ contains all right quasi-regular subsets of $R$ of the form $aR$, $a \in R$.

Corollary 3.24. If $D^2_\nu(R)$ is an ideal of $R$, then $R/D^2_\nu(R)$ is zero-symmetric.

Proof. By Corollary 3.20, $R_\nu \subseteq D^1_\nu(R) \subseteq D^2_\nu(R)$. Therefore, $R/D^2_\nu(R)$ is zero symmetric.

Corollary 3.25. If $D^2_\nu(R) = \{0\}$, then $R$ is zero-symmetric.

Definition 3.26. $R$ is called a right $\nu$-primitive near-ring if $\{0\}$ is a right $\nu$-primitive ideal of $R$, $\nu \in \{1, 2\}$.

Theorem 3.27. An ideal $P$ of $R$ is right $\nu$-primitive if and only if $R/P$ is a right $\nu$-primitive near-ring, $\nu \in \{1, 2\}$.

Proof. Let $\nu \in \{1, 2\}$. Let $P$ be a right $\nu$-primitive ideal of $R$. We get a right $\nu$-modular right ideal $M$ of $R$ such that $P$ is the largest ideal of $R$ contained in $M$. $M/P$ is a right $\nu$-modular right ideal of $R/P$. Since $P$ is the largest ideal of $R$ contained in $M$, the zero ideal of $R/P$ is the largest ideal of $R/P$ contained in $M/P$. Therefore $R/P$ is a right $\nu$-primitive near-ring. On the other hand, suppose that $R/P$ is a right $\nu$-primitive near-ring. We get a right $\nu$-modular right ideal $M/P$ of $R/P$ such that the largest ideal of $R/P$ contained in $M/P$ is the zero ideal of $R/P$. Clearly $M$ is a right $\nu$-modular right ideal of $R$ and $P$ is the largest ideal of $R$ contained in $M$. Therefore, $P$ is a right $\nu$-primitive ideal of $R$.

Theorem 3.28. A right $\nu$-primitive ideal of $R$ is prime, $\nu \in \{1, 2\}$.

Proof. Let $\nu \in \{1, 2\}$. Let $P$ be a right $\nu$-primitive ideal of $R$. We have that a right 0-primitive ideal of $R$ is prime. Since a right $\nu$-primitive ideal of $R$ is right 0-primitive, from Theorem 2.12, we get that $P$ is prime.

4. Properties of the right Jacobson radicals of type-1 and 2

We now see the relation between right $\nu$-primitive ideals of a zero-symmetric near-ring $R$ and the annihilators of the right $R$-groups of type-$\nu$.

Proposition 4.1. Let $P$ be an ideal of a zero-symmetric near-ring $R$. Then $P$ is right $\nu$-primitive if and only if $P$ is the largest ideal of $R$ contained in $(0 : G)$ for some right $R$-group $G$ of type-$\nu$, $\nu \in \{1, 2\}$. 

Proof. Let $P$ be an ideal of a zero-symmetric near-ring $R$. Suppose that $P$ is a right $\nu$-primitive ideal of $R$. So, we get a right $\nu$-modular right ideal $K$ of $R$ such that $P$ is the largest ideal of $R$ contained in $K$. Now $R/K$ is a right $R$-group of type-$\nu$.

By Proposition 2.11, $P$ is the largest ideal of $R$ contained in $(0 : R/K) = (K : R)$. Conversely, suppose that $P$ is the largest ideal of $R$ contained in $(0 : G)$, where $G$ is a right $R$-group of type-$\nu$. Now $G$ is $R$-isomorphic to $R/K$ for some right $\nu$-modular right ideal $K$ of $R$. So, $(0 : G) = (0 : R/K) = (K : R)$. Since $P$ is the largest ideal of $R$ contained in $(0 : G) = (K : R)$, by Proposition 2.11, $P$ is the largest ideal of $R$ contained in $K$. Hence $P$ is a right $\nu$-primitive ideal of $R$. □

**Proposition 4.2.** Let $R$ be a d.g. near-ring. Then $J_\nu^r(R) = D_\nu^r(R), \nu \in \{1, 2\}$.

Proof. If $J_\nu^r(R) = R$, then the result is obvious. Suppose that $J_\nu^r(R) \neq R$. Let $P$ be a right $\nu$-primitive ideal of $R$, $\nu \in \{1, 2\}$. Since $R$ is d.g., we get a right $R$-group $G$ of type-$\nu$, such that $P = (0 : G)$. By Proposition 2.13, we get a subset $T$ of $G$ such that $T$ generates $(G, +)$ and $t(r+s) = tr+ts$ for all $t \in T$ and $r, s \in R$. If $Tr = \{0\}$ then $Gr = \{0\}$. Therefore, $(0 : T) = (0 : G)$. Let $0 \neq t \in T$. Since $tr$ is a right $R$-subgroup of $G$ and $tR \neq \{0\}$, $tR = G$. Define $h : R \rightarrow G$ by $h(r) = tr$, for all $r \in R$. Clearly $h$ is a $R$-homomorphism of $R$ onto $G$ with kernel $(0 : t)$. Therefore, $R/(0 : t)$ is a $R$-isomorphic to $G$. So, $(0 : t)$ is a right $\nu$-modular right ideal of $R$. Therefore, $(0 : G) = (0 : T) = \cap_{0 \neq s \in T} (0 : s)$ and hence $D_\nu^r(R) \subseteq J_\nu^r(R)$. It is clear that $J_\nu^r(R) \subseteq L$ for each right $\nu$-modular right ideal $L$ of $R$. So, $J_\nu^r(R) \subseteq D_\nu^r(R)$. Hence, $J_\nu^r(R) = D_\nu^r(R)$. □

**Corollary 4.3.** If $R$ is a d.g. near-ring, then $J_1^r(R)$ contains all right quasi-regular right $R$-subgroups of $R$.

**Corollary 4.4.** If $R$ is a d.g. near-ring, then $J_2^r(R)$ contains all right quasi-regular subsets of $R$ of the form $aR = \{ax | x \in R\}, a \in R$.

We see some of the properties of the $J_2^r$-radical.

**Theorem 4.5.** Let $G$ be a right $R$-group of type-2. If $I$ is a left invariant ideal of $R$ and $GI \neq \{0\}$, then $G$ is also a right $I$-group of type-2.

Proof. We have that $G$ is a right $R$-group of type-2, $I$ is a left invariant ideal of $R$ and $GI \neq \{0\}$. So for all $0 \neq g \in G$, $gR = G$. Also the right $R$-group $G$ has a generator $I$. Now clearly $G$ is a right $I$-group. Let $0 \neq g \in G$. We claim that $gI = G$. We first show that $gI \neq 0$. We have that $gI \supseteq g(RI) = (gR)I = GI \neq \{0\}$. So, $gI \neq \{0\}$ and hence $(gI)R = G$. Therefore, $G = (gI)R = g(IR) \subseteq gI$. Hence, $gI = G$. This also shows that $h$ is a generator of the right $I$-group $G$. Therefore, $G$ is right $I$-group of type-2. □

Now we want to study the hereditary property of the semisimple class of the radical $J_2^r$.

**Theorem 4.6.** Let $R$ be a d.g. near-ring and $I$ be a non-zero ideal of $R$. If $R$ is right 2-primitive, then $I$ is also right 2-primitive.
Proof. Suppose that $R$ is right 2-primitive. Since $R$ is zero-symmetric, we get a right $R$-group $G$ of type-2 such that $\{0\}$ is the largest ideal of $R$ contained in $(0 : G)$. Since $R$ is d.g., $(0 : G)$ is an ideal of $R$ and hence $(0 : G) = \{0\}$. Now, since $GI \neq \{0\}$, by Theorem 4.5, $G$ is a right $I$-group of type-2. The largest ideal of $I$ contained in $(0 : G)I$ is a right 2-primitive ideal of $I$ as $G$ is right $I$-group of type-2. Since $(0 : G)I = \{0\}$, $\{0\}$ is a right 2-primitive ideal of $I$, that is, $I$ is right 2-primitive. $\Box$

Now we develop a few results to prove that for any family of near-rings $R_i$, $i \in I$, $J'_\nu(\bigoplus_{i \in I} R_i) = \bigoplus_{i \in I} J'_\nu(R_i)$, $\nu \in \{0, 1, 2\}$.

**Proposition 4.7.** Let $I$ be an ideal of $R$ and let $I$ be a direct summand of $R$. If $K$ is a right $\nu$-modular right ideal of $I$, then there is a right $\nu$-modular right ideal $M$ of $R$ such that $K = R \cap M$, $\nu \in \{0, 1, 2\}$.

**Proof.** Let $R = I \oplus J$, $J$ is an ideal of $R$. Suppose that $K$ is a right $\nu$-modular right ideal of $I$, $\nu \in \{0, 1, 2\}$. Let $M = K + J$. Now $M \cap I = K$. Let $K$ be right modular by $e \in I$. Now $M$ is a right ideal of $R$ and right modular in $R$ by $e$. Define $f : R \rightarrow I/K$ by $f(r = i + j) = i + K$, for all $r \in R$, where $i \in I$, $j \in J$. Clearly $f$ is a $R$-homomorphism of $R$ onto $I/K$ and kernel of $f$ is $M$. Therefore, $R/M$ is $R$-isomorphic to $I/K$. So, $R/M$ is isomorphic to $I/K$ as right $I$-groups. Since $I/K$ is a right $I$-group of type-$\nu$, $R/M$ is also a right $I$-group of type-$\nu$, that is, $R/M$ is a right $R/I$-group of type-$\nu$. Hence, $R/M$ is a right $R$-group of type-$\nu$, that is, $M$ is a right $\nu$-modular right ideal of $R$. $\Box$

**Proposition 4.8.** Let $R$ be the direct sum of its ideals $I_1$ and $I_2$ and $M$ be a right ideal of $R$. Let $M_i = M \cap I_i$, $i = 1, 2$. If $M$ is a right $\nu$-modular right ideal of $R$, then for some $i \in \{1, 2\}$, $M_i$ is a right $\nu$-modular right ideal of $I_i$, $\nu \in \{0, 1, 2\}$.

**Proof.** Let $M$ be a right $\nu$-modular right ideal of $R$. Now $M_i$ is a right ideal of $I_i$. If $M_i \neq I_i$, then $M = R$, a contradiction. So, we may assume that $M_i \neq I_i$. Suppose that $M$ is right modular by $e_1 + e_2$, $e_i \in I_i$. If $e_2$ is in the zero symmetric part of $I_2$, then $e_1 - (e_1 + e_2)i_1 = i_1 - e_1i_1 \in M_1$, for all $i_1 \in I_1$, we get that $M_1$ is right modular by $e_1$ in $I_1$. Suppose now that $e_2$ is in the constant part of $I_2$. If $e_1$ is also in the constant part of $I_1$, then $e_1 + e_2$ is in the constant part of $R$ and hence it must be right quasi-regular, which is a contradiction to the fact that $M$ is right modular by $e_1 + e_2$ and $M \neq R$. Therefore, $e_1$ is in the zero-symmetric part of $I_1$. Now $0 - (e_1 + e_2)0 = e_2 \in M$. $i_1 - (e_1 + e_2)i_1 = (i_1 - e_1i_1) - e_2 \in M$ and hence $i_1 - e_1i_1 \in M$, for all $i_1 \in I_1$. Therefore, in this case also $M_1$ is right modular by $e_1$ in $I_1$.

(a) We see now that $M_1$ is a maximal right ideal of $I_1$. Suppose that $N$ is a proper right ideal of $I_1$ properly containing $M_1$. Now, as $N$ is not contained in $M$, we have that $N + M = R$. Let $x \in I_1 - N$. $x = n + m$, $n \in N$, $m \in M$. Clearly $m \in I_1$ and hence $x \in M_1$, a contradiction. So, $M_1$ is a maximal right ideal of $I_1$.

(b) Suppose that $M$ is right 1-modular right ideal of $R$. So, $M$ is a maximal right
Suppose that $M_1$ is a maximal right $I_1$-subgroup of $I_1$, and hence $M_1$ is right 1-modular right ideal of $I_1$.

(c) Suppose that $M$ is a right 2-modular right ideal of $R$. Let $a \in I_1 - M_1$. Since $a \notin M$, $M + aR = R$. Let $b \in I_1$. $b = m + ar$, $m \in M$ and $r \in R$. Since $b$, $ar \in I_1$, $m \in I_1$ we get that $m \in M_1$. If $r = i_1 + i_2$, $i_1 \in I_1$, $i_2 \in I_2$ then as $ar \in I_1$, $ar = ai_1 \in aI_1$. Therefore $M_1 + aI_1 = I_1$ and hence $M_1$ is a right 2-modular right ideal of $I_1$.

□

**Proposition 4.9.** Let $I$ be an ideal of $R$ and $I$ be a direct summand of $R$. A right $
u$-primitive ideal $P$ of $I$, is of the form $P = I \cap Q$, where $Q$ is a right $
u$-primitive ideal of $R$, $\nu \in \{0, 1, 2\}$.

**Proof.** Let $P$ be a right $
u$-primitive ideal of $I$, $\nu \in \{0, 1, 2\}$. We get a right $
u$-modular right ideal $K$ of $I$ such that $P$ is the largest ideal of $I$ contained in $K$. By Proposition 4.7 there is a right $
u$-modular right ideal $M$ of $R$ such that $K = M \cap I$. Let $Q$ be the largest ideal of $R$ contained in $M$. Now $Q$ is a right $
u$-primitive ideal of $R$. Since an ideal $T$ of $I$ is also an ideal of $R$, we have that $P$ is an ideal of $R$ contained in $I \cap M$. So, $P \subseteq I \cap Q$. Since an ideal of $R$ contained in $I$ is also an ideal of $I$, $Q \cap I$ is an ideal of $I$ contained in $K$. So, $Q \cap I \subseteq P$. Therefore $P = I \cap Q$. □

**Proposition 4.10.** Let $I$ be an ideal of $R$ and $I$ be a direct summand of $R$. If $Q$ is a right $
u$-primitive ideal of $R$, then $Q \cap I = I$ or a right $
u$-primitive ideal of $I$, $\nu \in \{0, 1, 2\}$.

**Proof.** Let $Q$ be a right $
u$-primitive ideal of $R$, $\nu \in \{0, 1, 2\}$. We get a right $
u$-modular right ideal $M$ of $R$ such that $Q$ is the largest ideal of $R$ contained in $M$. If $I \subseteq M$, then $Q \cap I = I$. If $I \nsubseteq M$, then as seen in Proposition 4.8, $I \cap M$ is a right $
u$-primitive right ideal of $I$. Since $Q \cap I$ is the largest ideal of $I$ contained in $I \cap M$, it is a right $\nu$-primitive ideal of $I$. □

**Theorem 4.11.** Let $I$ be an ideal of $R$ and $I$ be a direct summand of $R$. Then, $J_{\nu}^{c}(I) = I \cap J_{\nu}^{c}(R)$, $\nu \in \{0, 1, 2\}$.

**Proof.** Let $\nu \in \{0, 1, 2\}$. If $I$ has no right $\nu$-primitive ideal, then by Proposition 4.10, it follows that $I$ is contained in every right $\nu$-primitive ideal of $R$ and hence $J_{\nu}^{c}(I) = I = I \cap J_{\nu}^{c}(R)$. Suppose now that $I$ has right $\nu$-primitive ideals. Now by Propositions 4.9 and 4.10, $J_{\nu}^{c}(I) = \cap\{P \mid P$ is a right $\nu$-primitive ideal of $I\} = \cap\{Q \cap I \mid Q$ is a right $\nu$-primitive ideal of $R\} = I \cap J_{\nu}^{c}(R)$. □

**Theorem 4.12.** Let $R_i$, $i \in I$, be a family of near-rings. Then $J_{\nu}^{c}(\oplus_{i \in I} R_i) = \oplus_{i \in I} J_{\nu}^{c}(R_i)$, $\nu \in \{0, 1, 2\}$.

**Proof.** Let $\nu \in \{0, 1, 2\}$. Since $J_{\nu}^{c}$ is a radical map, we have that $J_{\nu}^{c}(\oplus_{i \in I} R_i) \subseteq \oplus_{i \in I} J_{\nu}^{c}(R_i)$. We show that $J_{\nu}^{c}(\oplus_{i \in I} R_i) \supseteq \oplus_{i \in I} J_{\nu}^{c}(R_i)$. Since $R_i$ is a direct sum-
mand of \(\oplus_{i \in I} R_i\), by Theorem 4.11, we have \(J'_\nu(R_i) \subseteq R_i \cap J'_\nu(\oplus_{i \in I} R_i)\). Therefore, \(J'_\nu(\oplus_{i \in I} R_i) \supseteq \oplus_{i \in I} J'_\nu(R_i)\). Hence, \(J'_\nu(\oplus_{i \in I} R_i) = \oplus_{i \in I} J'_\nu(R_i)\). \(\square\)

Now we develop a more general result related to the hereditariness of the \(J'_\nu\)-radical.

**Theorem 4.13.** Let \(S\) be an invariant subnear-ring of \(R\) and let \(K\) be a right 2-modular right ideal of \(S\). Then \(K\) is an ideal of the right \(R\)-group \(S\).

**Proof.** Let \(s \in S\). Suppose that \(sS \subseteq K\). We claim now that \(sR \subseteq K\). On the contrary suppose that \(sR \not\subseteq K\). We have \(sR \subseteq S\). Let \(t \in sR - K\). Now \(tS + K = S\), as \(K\) is a right 2-modular right ideal of \(S\); \(t \not\in K\) and \(t \in S\). Now \(t = sx, x \in R\).

Since \(tS = sxS \subseteq sS \subseteq K\), we have that \(K = S\), a contradiction. So, \(sR \subseteq K\). Therefore, as \(K \subseteq S\) and \(K S \subseteq K\), we have \(K R \subseteq K\). Hence \(K\) is an ideal of the right \(R\)-group \(S\). \(\square\)

**Theorem 4.14.** Let \(S\) be an ideal of a d.g. near-ring \(R\). Suppose that \((S,+)\) is generated by \(S \cap D\), where \(D\) is the set of all distributive elements in \(R\). If \(T\) is a right 2-primitive ideal of \(S\) then \(T\) is an ideal of \(R\).

**Proof.** Let \(T\) be a right 2-primitive ideal of \(S\). Since \(S\) is d.g., we have \(T = (0 : G)\) for a right \(S\)-group \(G\) of type-2. Now \(G\) is \(S\)-isomorphic to \(S/K\) for some right 2-modular right ideal \(K\) of \(S\). So, \(T = (0 : (S/K)_S) = (K : S)_S = S \cap (K : S)\).

By Theorem 4.13, \(K\) is an ideal of the right \(R\)-group \(S\). Therefore, \(S/K\) is a right \(R\)-group. Since \(S \cap D\), generates \((S,+)\), \((0 : S/K) = (K : S)\) is an ideal of \(R\).

Hence, \(T = S \cap (K : S)\) is an ideal of \(R\). \(\square\)

**Theorem 4.15.** Let \(S\) be an invariant subnear-ring of the d.g. near-ring \(R\). Suppose that \((S,+)\) is generated by \(S \cap D\), where \(D\) is the set of all distributive elements in \(R\). Then, \(J'_2(S) = S \cap J'_2(R)\).

**Proof.** Let \(T\) be a right 2-primitive ideal of \(S\). As seen in the Theorem 4.14, there is a right 2-modular right ideal \(K\) of \(S\) such that \(T = S \cap (K : S)\), where \(K\) is an ideal of the right \(R\)-group \(S\) and \(S/K\) is a right \(R\)-group. Choose \(d \in (S \cap D) - K\). Now \((d + K)S = S/K\) and hence \((d + K)R = S/K\). Since \(d \in D\), \(d + K\) is a generator of the right \(R\)-group, \(S/K\). If \(s \in S\) and \(s + K \not\in K\) then \((s + K)S = S/K\) and hence \((s + K)R = S/K\). Therefore, \(S/K\) is a right \(R\)-group of type-2. So, \((K : S)\) is a right 2-primitive ideal of \(R\), as \(R\) is d.g..

Let \(A\) be the collection of all right 2-primitive ideals \(\{T_\alpha \mid \alpha \in \Delta\}\) of \(S\). If \(A\) is empty then clearly \(J'_2(S) = S \supseteq J'_2(R)\). Suppose that \(A\) is not empty. For each \(\alpha \in \Delta\) we get a right 2-primitive ideal \(I_\alpha\) of \(R\) such that \(T_\alpha = I_\alpha \cap S\). Now \(J'_2(S) = \cap_{\alpha \in \Delta} I_\alpha = \cap_{\alpha \in \Delta} (I_\alpha \cap S) = S \cap (\cap_{\alpha \in \Delta} I_\alpha) \supseteq S \cap J'_2(R)\). Therefore, we have \(J'_2(S) \supseteq J'_2(R) \cap S\).

We prove now that \(J'_2(S) \subseteq J'_2(R)\). Let \(J\) be a right 2-primitive ideal of \(R\). Since \(R\) is d.g., we get a right 2-modular right ideal \(L\) of \(R\) such that \(J = (L : R)\), where \(R/L\) is a right \(R\)-group of type-2. If \(S \subseteq L\) then \(RS \subseteq S \subseteq L\) and hence \(S \subseteq J\). So, we get that \(J'_2(S) \subseteq S \subseteq J \cap S\). Now Suppose that \(S \not\subseteq L\). Since \(L\) is a right ideal of \(R\), \(S \cap L\) is a right ideal of \(S\). So, \(S/(S \cap L)\) is a right \(S\)-group. We show
that \( S \cap L \) is a right 2-modular right ideal of \( S \), that is, \( S/(S \cap L) \) is a right \( S \)-group of type-2. We have that \( S \neq S \cap L \) and hence \( S/(S \cap L) \) is a non zero right \( S \)-group. Let \( s \in S - L \). We see that \((s + (S \cap L))S = S/(S \cap L)\), that is, \( sS + (S \cap L) = S \).

Since \( L \) is a right 2-modular right ideal of \( R \), we get \( ae \in R \) such that \( r - er \in L \) for all \( r \in R \). Now, as \( L \) is right 2-modular, \( sR + L = R \). Now \( sr + l = e \), for some \( r \in R \) and \( l \in L \). Let \( t \in S \). Now \( et = srt + lt \). Since \( et - t \in L \) we have that \( srt + lt - t = l_1 \), for some \( l_1 \in L \). Now \( srt - l_2 \), for some \( l_2 \in L \). So, \( t \in sS + L \). Therefore, \( S \subseteq sS + L \) and hence \( S = sS + (L \cap S) \). We get a distributive element \( d \) of \( R \) in \( S - L \). Now \( d + (L \cap S) \) is a generator of the right \( S \)-group \( S/(L \cap S) \). Hence, \( S/(L \cap S) \) is a right \( S \)-group of type-2. Since \( R/L = S + L/L \) is \( R \)-isomorphic to \( S/(L \cap S) \), we have that \( R/L \) is also \( S \)-isomorphic to \( S/(L \cap S) \). Therefore, as \((S, +)\) is generated by \( S \cap D \), we have that \(((L \cap S) : S) \cap S \) is a right 2-primitive ideal of \( S \) and \(((L \cap S) : S) \cap S = (L : R) \cap S = (L : R) \cap S = J \cap S \). So, \( J \cap S \) is a right 2-primitive ideal of \( S \). Hence, \( J_2^2(S) \subseteq J \cap S \). So, \( J_2^2(S) \subseteq J_2^2(R) \cap S \). Therefore, \( J_2^2(S) = S \cap J_2^2(R) \).

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**References**


