Note on $\sigma$-derivations in Near-rings and Reduced Near-rings

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Abstract. We study $\sigma$-derivations on reduced near-rings and extend certain results on derivations to $\sigma$-derivations with some conditions.

1. Introduction

A (left) near-ring is a set with two binary operations $+$ and $\cdot$ such that

(i) $(N, +)$ is a group (not necessarily abelian) with identity $0$,

(ii) $(N, \cdot)$ is a semigroup and

(iii) for all $x, y, z \in N$, $x(y + z) = xy + xz$.

Throughout this paper, $N$ will denote a zero symmetric near-ring ($a0 = 0a = 0$ for all $a \in N$). A derivation $d$ on a near-ring $N$ is an additive endomorphism having $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ and a $\sigma$-derivation $d$ on $N$ is defined to be an additive endomorphism satisfying the product rule $d(xy) = \sigma(x)d(y) + d(x)y$ for all $x, y \in N$, where $\sigma$ is an automorphism on $N$. Posner [7] defined derivations on prime rings. Herstein [3] derived commutativity property of prime rings with derivations. Recently in near-ring theory, Bell, Mason [1], Kamal [4] and Cho [2] researched in prime and semi prime near-rings. A near-ring $N$ is said to be reduced if it has no nilpotent elements. $N$ is said to have IFP (Insertion Factor Property) if whenever $ab = 0$ with $a, b \in N$ implies $anb = 0$ for all $n \in N$. A $\sigma$-derivation $d$ on $N$ is said to be nilpotent if there exist a positive integer $k$ such that $d^k = 0$. The smallest $k$ having the property is called the index of nilpotency and denoted by $\text{nil}(d)$. A $\sigma$-derivation is called nil, if for every $a \in N$, there exists a natural number $k$ such that $d^k(a) = 0$. The smallest such number is denoted by $\text{nil}(d,a)$. Clearly, nilpotent derivation is nil but not the converse [2]. A non-empty subset $U$ of $N$ is called a left (resp. right) $N$-subset if $NU \subset U$ (resp. $UN \subset U$). If $U$ is both right and left $N$-subset, it is called an $N$-subset of $N$. An $N$-subset $U$ of $N$, which is also a group with respective to $+$, is called an $N$-subgroup of $N$. Basic concepts on near-rings can be found in Meldrum [5] and Pilz [6].

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2. \(\sigma\)-derivations and annihilators

In this section, we see the role of \(\sigma\)-derivations on annihilator ideals of a reduced near-ring. The following lemmas are useful in this sequel.

**Lemma 2.1 ([6]).** Let \(N\) be a reduced near-ring. If \(ab = 0\) for all \(a, b \in N\), then \(ba = 0\). Moreover, for \(s \in N\), \(\text{Ann}(s) = \{x \in N \mid sx = 0\}\) is an ideal of \(N\).

**Lemma 2.2 ([6]).** Let \(N\) be a reduced near-ring. Then \(N\) has IFP and for any subset \(S\) of \(N\), \(\text{Ann}(S)\) is an ideal of \(N\).

**Lemma 2.3.** Let \(d\) be an additive endomorphism on a near-ring \(N\). Then \(d(ab) = \sigma(a)d(b) + d(a)b\) and \(d(ab) = d(a)b + \sigma(a)d(b)\) are equivalent, for all \(a, b \in N\) and any automorphism \(\sigma\) on \(N\).

**Proof.** Follows from the equality \(d(a(b + b)) = d(ab + ab)\). \(\square\)

**Lemma 2.4.** Let \(N\) be a near-ring and \(S_1\) and \(S_2\) be subsets of \(N\) such that \(S_1 \subset S_2\). Then \(\text{Ann}(S_2) \subset \text{Ann}(S_1)\).

**Proof.** Let \(x \in \text{Ann}(S_2)\). Then \(S_2x = 0\). That is, \(S_2x = 0\) for all \(s_2 \in S_2\). This means that \(x\) annihilates all elements of \(S_2\). In particular, \(x\) annihilates all elements of \(S_1\). Therefore, \(x \in \text{Ann}(S_1)\). This completes the proof. \(\square\)

**Theorem 2.5.** Let \(d\) be a \(\sigma\)-derivation on a reduced near-ring \(N\). Then for any subset \(S\) of \(N\), \(d(\text{Ann}(S)) \subset \text{Ann}(\sigma(S))\).

**Proof.** Let \(x \in \text{Ann}(S \cap \sigma(S))\). Then \(sx = 0\) and \(\sigma(s)x = 0\) for all \(s \in S\). Since \(sx = 0\), we get \(d(sx) = 0\) and so by Lemma 2.3, \(d(s)x + \sigma(s)d(x) = 0\). Multiplying by \(\sigma(s)\) from the right, we get \(d(s)x\sigma(s) + \sigma(s)d(x)\sigma(s) = 0\). Since \(\sigma(s)x = 0\), we have \(x\sigma(s) = 0\). Thus \(d(s)x\sigma(s) = 0\). Therefore, from the above equation \(d(s)d(x)\sigma(s) = 0\) and so \(d(x)\sigma(s)d(x)\sigma(s) = 0\). That is, \((d(x)\sigma(s))^2 = 0\). Since \(N\) is reduced, \(d(x)\sigma(s) = 0\). Thus, we obtain \(\sigma(s)d(x) = 0\), which means that \(d(x) \in \text{Ann}(\sigma(S))\). Thus, \(d(\text{Ann}(S \cap \sigma(S))) \subset \text{Ann}(\sigma(S))\). Clearly, \(S \cap \sigma(S) \subset S\). By Lemma 2.4, \(\text{Ann}(S) \subset \text{Ann}(S \cap \sigma(S))\). Since derivation \(d\) is a function from \(N\) to \(N\), we obtain \(d(\text{Ann}(S)) \subset d(\text{Ann}(S \cap \sigma(S)))\). Combining these two inclusions, we have \(d(\text{Ann}(S)) \subset \text{Ann}(\sigma(S))\). \(\square\)

The above Theorem 2.5 is valid, even if \(\sigma\) is an endomorphism. For the subsequent discussions, we require \(\sigma\) to be an automorphism. When \(\sigma\) is an identity automorphism, we have the following corollary.

**Corollary 2.6** [2, Theorem 2.2]. Let \(d\) be a derivation on a reduced near-ring. Then every annihilator ideal is invariant under \(d\).

**Theorem 2.7.** Let \(N\) be a reduced near-ring and \(d\) be a \(\sigma\)-derivation on \(N\). Then \(\text{Ann}(S) \subset \text{Ann}(d(S))\) for any subset \(S\) of \(N\).

**Proof.** Let \(S\) be a subset of \(N\) and \(d\) be a \(\sigma\)-derivation on \(N\). Suppose \(x \in \text{Ann}(S)\). Then \(sx = 0\) for all \(s \in S\). Since \(sx = 0\), we have \(d(sx) = 0\).
Therefore, \( d(sx) = d(s)x + \sigma(s)d(x) = 0 \). Multiplying by \( \sigma(x) \) from the right,
\( d(s)x\sigma(x) + \sigma(s)d(x)(\sigma(x)) = 0 \). Since \( sx = 0 \), \( \sigma(s)\sigma(x) = 0 \). By IFP property,
\( \sigma(s)d(x)\sigma(x) = 0 \). Since \( \sigma \) is an automorphism, \( d(s)x\sigma^{-1}(x) = 0 \) and so \( d(s)x = 0 \). That is, \( (d(s)x)(x) = 0 \). By IFP, \( d(s)x\sigma(x) = 0 \). So, we obtain
\( (d(s)x)^2 = 0 \). Since \( N \) is reduced, \( d(s)x = 0 \). Hence \( x \in \text{Ann}(d(S)) \). Therefore,
\( \text{Ann}(S) \subseteq \text{Ann}(d(S)) \). \( \square \)

**Remark 2.8.** If we repeat the procedure mentioned in Theorem 2.7 continuously,
we have the following ascending chain of annihilator ideals of \( N \) namely,
\( \text{Ann}(S) \subseteq \text{Ann}(d(S)) \subseteq \text{Ann}(d^2(S)) \subseteq \cdots \). In particular, for any \( a \in N \), we get \( \text{Ann}(a) \subseteq \text{Ann}(d(a)) \subseteq \text{Ann}(d^2(a)) \cdots \). Moreover, if the \( \sigma \)-derivation \( d \) is nil, there exists a positive integer \( k \) such that \( d^k(a) = 0 \). Thus we have
\( \text{Ann}(a) \subseteq \text{Ann}(d(a)) \subseteq \cdots \subseteq \text{Ann}(0) = N \). In other words, \( N \) has ascending chain condition on principal annihilator ideals of \( N \) by applying \( d \).

### 3. Existence of \( N \)-subsets and \( N \)-subgroups

We conclude this paper, by showing the existence of \( N \)-subsets and \( N \)-subgroups using the \( \sigma \)-derivation \( d \).

**Theorem 3.1.** Let \( N \) be a near-ring with a \( \sigma \)-derivation \( d \) such that \( d^2\sigma d \neq 0 \). Then every subnear-ring \( B \) generated by \( d(N) \) and satisfies \( \sigma(B) \subseteq B \) has a two sided \( N \)-subset of \( N \). Moreover, if \( \sigma d = d\sigma \), then \( B \) has a two sided \( N \)-subgroup of \( N \).

**Proof.** Clearly, \( d(N) \subseteq B \). So \( d^2\sigma(d(N)) \subseteq d^2\sigma(B) \). Since \( d^2\sigma d \neq 0 \), we get \( d^2\sigma(d(N)) \neq 0 \) and there by \( d^2\sigma(B) \neq 0 \). Select \( y \in B \) such that \( d^2\sigma(y) \neq 0 \). Let \( x \in N \). From the definition of derivation, we get
\[
   d(x\sigma(y)) = d(x)\sigma(y) + \sigma(x)d(\sigma(y)).
\]
Since \( d(x\sigma(y)) \in B \) and \( \sigma(y) \in \sigma(B) \subseteq B \), \( d(x) \in B \) gives \( \sigma(x)d(\sigma(y)) \in B \). Thus, we get \( \sigma(x)d(\sigma(y)) \in N \) which implies \( \sigma(N)d(\sigma(y)) \subseteq B \). Since \( \sigma \) is an automorphism, we have \( \sigma(N) = N \). Therefore, \( N\sigma(y) \subseteq B \). Also, \( \sigma(y)\sigma(\sigma(y)) = d(\sigma(y))\sigma(\sigma(y)) + \sigma(\sigma(y))d(\sigma(y)) \). Since \( \sigma(y) \), \( B \), we have \( \sigma(\sigma(y)) \subseteq B \). Also \( d(\sigma(y)) \in d(N) \subseteq B \). Therefore, \( \sigma(\sigma(y))d(\sigma(x)) \in B \). But \( d(\sigma(y))\sigma(x) \in B \). Hence \( d(\sigma(y))\sigma(N) \subseteq B \). This in turn implies that \( d(\sigma(y))N \subseteq B \). Consider \( a, b \in N \). Now,
\[
   d(ad(\sigma(y)b)) = d(a)d(\sigma(y)b) + \sigma(a)d(d(\sigma(y)b))
   = d(a)d(\sigma(y)b) + \sigma(a)(d^2\sigma(y)b) + \sigma(d(\sigma(y)))d(b)
   = d(a)d(\sigma(y)b) + \sigma(a)d^2(\sigma(y)b) + \sigma(a)d(\sigma(y))d(b).
\]
Clearly, \( d(ad(\sigma(y)b)) \in B \). Since \( d(a) \in B, d(\sigma(y))N \subseteq B \), and so \( d(a)d(\sigma(y)) \in B \). Since \( \sigma \) is a homomorphism, \( \sigma(a)d(\sigma(y))d(b) = \sigma(ad(\sigma(y)))d(b) \). But \( ad(\sigma(y)) \in Nd(\sigma(y)) \subseteq B \). This means that \( \sigma(ad(\sigma(y)) \in \sigma(B) \subseteq B \). Also \( d(b) \in d(N) \subseteq B \).
Since $B$ is a near-ring, $\sigma(a)\sigma(d(\sigma(y)))d(b)$ is in $B$. Therefore, $\sigma(a)d^2(\sigma(y))b \in B$. That is, $\sigma(N)d^2(\sigma(y))N \subset B$. Thus, $Nd^2(\sigma(y))N$ is the required two sided $N$-subset of $N$ contained in $B$. Further, $d(\sigma\sigma(y)) = d(a)d(\sigma(y)) + \sigma(a)d^2(\sigma(y))$. This implies that $\sigma(a)d^2(\sigma(y)) \in B$, which in turn implies that $\sigma(N)d^2(\sigma(y)) \subset B$. Thus, $Nd^2(\sigma(y)) \subset B$. Therefore, $Nd^2(\sigma(y))$ is the required two sided $N$-subset of $N$ contained in $B$. Further, $d(ad(\sigma(y))) = d(a)d(\sigma(y)) + \sigma(a)d^2(\sigma(y))$. Since $\sigma$ and $d$ commute,

$$\sigma(d(\sigma(y)))d(a) = d(\sigma(\sigma(y)))d(a) \in B$$

and hence $d^2(\sigma(y))a \in B$. This in turn implies that, $d^2(\sigma(y))N \subset B$. Thus $d^2(\sigma(y))N$ is a right $N$-subgroup of $N$ contained in $B$. Hence $B$ has a non-zero two sided $N$-subgroup of $N$ generated by $d^2(\sigma(y)) \neq 0$. □

Suppose $N$ is any near-ring with a $\sigma$-derivation $d$ and $\sigma$ and $d$ commutes, then the condition $d^2\sigma d \neq 0$ is equivalent to $d^3 \neq 0$. Since the identity homomorphism commutes with the derivation $d$, if we take $\sigma$ as identity automorphism in the above Theorem 3.1, we get the following corollary.

**Corollary 3.2** [2, Theorem 2.8]. Let $N$ be any near-ring with derivation $d$ such that $d^3 \neq 0$. Then every subnear-ring $B$ generated by $d(N)$ contains an $N$-Subgroup of $N$.

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**References**


