Weak Distributive $n$-Semilattices and $n$-Lattices

Seon-Ju Lim  
Department of Mathematics & Statistics, Sookmyung Women’s University, Seoul 140-742, Korea  
e-mail: sjlim@sookmyung.ac.kr

Abstract. We define weak distributive $n$-semilattices and $n$-lattices, using variants of the absorption law and those of the distributive law. From a weak distributive $n$-semilattice, we construct direct system of subalgebras which are weak distributive $n$-lattices and show that its direct limit is a reflection of the category $\text{wD}_n\text{-SLatt}$ of the weak distributive $n$-semilattices.

1. Introduction

A semilattice is an algebra, $S = (S, \lor)$, with one binary operation $\lor$ that is idempotent, commutative and associative, that is, the following identities hold in $S$:

\[
\begin{align*}
x \lor x &= x \quad \text{(idempotence)}, \\
x \lor y &= y \lor x \quad \text{(commutativity)}, \\
(x \lor y) \lor z &= x \lor (y \lor z) \quad \text{(associativity)}.
\end{align*}
\]

An algebra $(B, \lor, \land)$ with two binary operations $\lor$ and $\land$ is called a bisemilattice if both of its reducts $(B, \lor)$ and $(B, \land)$ are semilattices. This notion was introduced by J. Plonka in [8] under the name quasilattice. However, it is called bisemilattice by other author ([3], [6], [7]). In particular, a bisemilattice is distributive if it satisfies the following two distributivity:

\[
\begin{align*}
x \land (y \lor z) &= (x \land y) \lor (x \land z), \\
x \lor (y \land z) &= (x \lor y) \land (x \lor z).
\end{align*}
\]

Plonka has generalized distributive bisemilattice to distributive $n$-semilattice and distributive $n$-lattice([9]). A distributive $n$-semilattice $(S, F)$ which is an algebra with a family $F = \{a_i \mid i \in [n]\}$ of $n$ binary semilattice operations on a common set $S$ in which each pair of semilattice operations satisfy both distributive laws. A distributive $n$-semilattice $(S, F)$ is called a distributive $n$-lattice if it satisfies

Received January 16, 2006.  
2000 Mathematics Subject Classification: 06A12, 06B99, 18A30, 18A40.  
Key words and phrases: $n$-semilattice, $n$-lattice, weak distributive, generalized absorption law, reflection, direct limit.
moreover the following generalized absorption law for the sequence \( I = (1, 2, \cdots, n) \) of indices of \( F = \{\circ_i \mid i \in [n]\} \)

\[
a \circ_1 (a \circ_2 (\cdots (a \circ_{n-1} (a \circ_n b)) \cdots)) = a.
\]

In 1971, R. Padmanbhan define a weak distributive bisemilattice, which is a bisemilattice satisfying the weak distributivity (it was studied under the name quasilattice in [7]):

\[
((a \land b) \lor c) \land (b \lor c) = (a \land b) \lor c \text{ and } ((a \lor b) \land c) \lor (b \land c) = (a \lor b) \land c.
\]

In this paper, we are concerned with categorical properties of certain algebras which we call weak distributive \( n \)-semilattices. These algebras generalize weak distributive bisemilattices. A weak distributive \( n \)-semilattice is an algebra with a family of \( n \) binary semilattice operations on a common underlying set which are mutually weak distributive. A weak distributive \( n \)-semilattice will be called a weak distributive \( n \)-lattice, if it satisfies the generalized absorption law, which generalizes the absorption law for lattices. Furthermore, weak distributive \( n \)-semilattices (or weak distributive \( n \)-lattices) generalize distributive \( n \)-semilattices (or distributive \( n \)-lattices, respectively). We show that every weak distributive \( n \)-semilattices has a partition consisting of weak distributive \( n \)-lattices and then the family of weak distributive \( n \)-lattices in the partition forms a direct system in the category \( \text{wD}_n-\text{Latt} \) of weak distributive \( n \)-lattices and homomorphisms. Furthermore, we prove that its direct limit gives rise to the reflection. For the terminology not introduced in the paper, we refer to [1] for the category theory, [2] for the ordered sets and [4], [5] for the abstract algebra.

2. Weak distributive \( n \)-semilattices

Let us start with a definition of weak distributive \( n \)-semilattice which is a generalization of both weak distributive bisemilattice and distributive \( n \)-semilattice.

**Definition 2.1.** An algebra \( W = (W, F) \) is called a weak distributive \( n \)-semilattice if it has a family \( F = \{\circ_i \mid i \in [n]\} \) consisting of \( n \) binary operations which satisfy the following equations for any \( i, j \in I \):

\[
a \circ_i a = a \quad \text{(idempotence)},
\]

\[
a \circ_i b = b \circ_i a \quad \text{(commutativity)},
\]

\[
(a \circ_i b) \circ_i c = a \circ_i (b \circ_i c) \quad \text{(associativity)},
\]

\[
((a \circ_i b) \circ_j c) \circ_i (b \circ_j c) = (a \circ_i b) \circ_j c \quad \text{(weak distributivity)}.
\]

A weak distributive \( n \)-semilattice is called a weak distributive \( n \)-lattice if it satisfies the generalized absorption law:

\[
(*) 
\quad a \circ_{\sigma(1)} (a \circ_{\sigma(2)} (\cdots (a \circ_{\sigma(n-1)} (a \circ_{\sigma(n)} b)) \cdots)) = a
\]
for any permutation \( \sigma \in \text{Sym}(n) \).

In the case \( n = 2 \), it is clear that a weak distributive \( n \)-semilattice is a weak distributive bisemilattice and a weak distributive \( n \)-lattice is a lattice. In a weak distributive \( n \)-lattice, the condition \((\ast)\) can be reduced to the condition

\[
a \circ_1 (a \circ_2 (\cdots (a \circ_{n-1} (a \circ_n b)) \cdots)) = a,
\]

because it can be easily shown by the weak distributivity. A distributive \( n \)-semilattice (or \( n \)-lattice) is a weak distributive \( n \)-semilattice (or \( n \)-lattice, respectively). But a weak distributive \( n \)-semilattice (or \( n \)-lattice) need not be a distributive \( n \)-semilattice (or \( n \)-lattices, respectively).

From now on, an \( n \)-semilattice \( W = (W, F) \) with a family \( F = \{ \circ_i \mid i \in [n] \} \) of \( n \) semilattice operations will be denoted by \( W = (W, F) \) or \( W \), simply.

**Remark 2.2.**

1. It is easy to see that an \( n \)-semilattice \( W = (W, F) \) is weak distributive if and only if \( a \circ_i b = b \) implies \((a \circ_j c) \circ_i (b \circ_j c) = b \circ_j c\) for any \( j \in [n] \) and any \( c \in W \).

2. Let \((W, F)\) be a weak distributive \( n \)-semilattice. If \( a \circ_i b = a \) and \( c \circ_i d = c \), then for any \( j \in [n] \), we have by (1),

\[
(a \circ_j c) \circ_i (b \circ_j d) = a \circ_j c.
\]

Now we obtain some properties of weak distributive \( n \)-semilattices and \( n \)-lattices, which will be needed in the formation of the direct system in the category \textbf{wDn-Latt} of weak distributive \( n \)-lattices and homomorphisms.

**Lemma 2.3.** Let \( W = (W, F) \) be a weak distributive \( n \)-semilattice. Then for any \( i, j \in [n] \), the following equations hold.

\[
\begin{align*}
(1) \quad a \circ_i (b \circ_j a) & = (a \circ_i b) \circ_j a, \\
(2) \quad a \circ_i (a \circ_j b) \circ_i (c \circ_j b) & = a \circ_i (c \circ_j b), \\
(3) \quad a \circ_i (a \circ_j b) \circ_i (a \circ_j b \circ_j c) & = a \circ_i (a \circ_j b \circ_j c), \\
(4) \quad a \circ_i b \circ_i (a \circ_j c) & = a \circ_i b \circ_i (b \circ_j c), \\
(5) \quad a \circ_i (a \circ_j (b \circ_i c)) & = a \circ_i (a \circ_j b) \circ_i (a \circ_j c), \\
(6) \quad a \circ_i (a \circ_j (b \circ_i (b \circ_j c))) & = a \circ_i (a \circ_j (c \circ_i (c \circ_j b))), \\
(7) \quad a \circ_i (a \circ_j (b \circ_i (b \circ_j c))) & = a \circ_i (a \circ_j b) \circ_i ((a \circ_i (a \circ_j b)) \circ_j c), \\
(8) \quad a \circ_i b \circ_i ((a \circ_i b) \circ_j c) & = a \circ_i (a \circ_j c) \circ_i b = a \circ_i b \circ_i (b \circ_j c), \\
(9) \quad a \circ_i (a \circ_j (b \circ_i (b \circ_j (a \circ_i b)))) & = a \circ_i (a \circ_j b).
\end{align*}
\]
Proof. (1) It follows from the definition of weak distributive \( n \)-semilattice. (2) From the associativity, weak distributivity and (2) of Remark 2.2, we have

\[
a \circ_i (a \circ_j b) \circ_i (c \circ_j b) = ((c \circ_j b) \circ_i a) \circ_i (a \circ_j b) = (((c \circ_j b) \circ_i a) \circ_j (a \circ_j b)) \circ_i (a \circ_j b) = ((c \circ_j b) \circ_i a) \circ_j (b \circ_i a) = (c \circ_j b) \circ_i a.
\]

(3) Equation (3) follows from (2) by the substitution \( b = a \circ_j b \).

(4) From (2),

\[
a \circ_i b \circ_i (a \circ_j c) = a \circ_i (b \circ_i (a \circ_j c)) = a \circ_i (b \circ_i (b \circ_j c)) \circ_i (a \circ_j c) = a \circ_i (b \circ_i (a \circ_j c)) \circ_i (b \circ_j c) = b \circ_i (a \circ_i (b \circ_j c)) = b \circ_i (a \circ_i (b \circ_j c)).
\]

(5) Using (4) and the weak distributivity, we obtain

\[
a \circ_i (a \circ_j b) \circ_i (a \circ_j c) = a \circ_i (a \circ_j b) \circ_i (a \circ_j (b \circ_i c)) = a \circ_i ((a \circ_j b) \circ_j ((a \circ_j b) \circ_i c)) = a \circ_i ((a \circ_j b) \circ_j (b \circ_i c)) = a \circ_i ((a \circ_j b) \circ_i (b \circ_i c)) = a \circ_i (a \circ_j (b \circ_i c)) \circ_i (a \circ_j (b \circ_i c)) = a \circ_i (a \circ_j (b \circ_i c)).
\]

(6) Using (1) and (2), we have

\[
a \circ_i (a \circ_j (b \circ_i (b \circ_j c))) = a \circ_i (a \circ_j b \circ_j (b \circ_i c)) = a \circ_i (a \circ_j (b \circ_i c) \circ_i (a \circ_j b \circ_j (b \circ_i c)) = ((a \circ_i b \circ_i c) \circ_j a) \circ_i (a \circ_j b \circ_j (b \circ_i c)) = a \circ_j (a \circ_i b \circ_i c),
\]

and similarly, \( a \circ_i (a \circ_j (c \circ_i (c \circ_j b))) = a \circ_j (a \circ_i b \circ_i c) \). It is easy to show that equation (7) hold using (1) and (5). Equations (8) and (9) follow from (4) by the substitution \( b = a \circ_i b \) and (1), respectively. This completes the proof. □

Note that a weak distributive \( n \)-semilattice \( W = (W, F) \) is an algebra of type \( n \). Then we may denote the operations of \( W \) by \( \circ_1, \circ_2, \cdots, \circ_n \). We observe that for any \( k \in [n] \), there is a subsequence \( K = (i_1, i_2, \cdots, i_k) \) of the sequence \( I = (1, 2, \cdots, n) \).
In the following, we denote $a \circ_i (a \circ_{i_2} \cdots (a \circ_{i_k} (a \circ_{i_1} b) \cdots ))$ by $f_{i_1,i_2,\ldots,i_k} (a,b)$ or $f_K (a,b)$ for the convenience.

**Lemma 2.4.** If $W = (W,F)$ is a weak distributive $n$-semilattice, then for any $i \in [n]$, we have the following equations:

1. $f_I (a, b \circ_i c) = f_I (a, b) \circ_i f_I (a, c)$ and $f_I (a \circ_i b, c) = f_I (a, c) \circ_i f_I (b, c)$,
2. $f_I (a \circ_i b, a) = a \circ_i b = f_I (a, b, b)$,
3. $f_K (f_K (a,b), c) = f_K (a, f_K (b,c)) = f_K (a, f_K (c,b))$ for any nonempty subsequence $K$ of $I$.

**Proof.** (1) For any $i, k \in [n]$, we denote the subsequences $(1, 2, \cdots, i)$ and $(1, 2, \cdots, k-1, k+1, \cdots, i)$ of the sequence $I = (1, 2, \cdots, n)$ by $I_i$ and $I_i - \{k\}$, respectively. Using (5) and (8) of Lemma 2.3., we have

$$f_I (a, b \circ_i c) = f_{I_{n-1}} (a, a \circ_n (b \circ_i c)) = f_{I_{n-1} - \{i\}} (a, a \circ_i (a \circ_n (b \circ_i c))) = f_{I_{n-2}} (a, f_{I_{n-1},n} (a, b) \circ_i (a \circ_n (b \circ_i c))) = f_{I_{n-2}} (a, f_{I_{n-1},n} (a, b) \circ_i f_{I_n, n} (a, c)) = f_{I_{n-3}} (a, f_{I_{n-2},n-1, n} (a, b) \circ_i f_{I_{n-1}, n} (a, c))$$

and the second part is proved from (8) of Lemma 2.3 and idempotence:

$$f_I (a \circ_i b, c) = f_{I_{n-1} - \{i\}} (a \circ_i (a \circ_i b) \circ_i (a \circ_i b \circ_n c)) = f_{I_{n-1} - \{i\}} (a \circ_i (a \circ_i b) \circ_i (a \circ_i b \circ_n c)) = f_{I_{n-2}} (a, f_{I_{n-1}, n} (a, b) \circ_i f_{I_{n-2}, n} (a, b \circ_n c)) = f_{I_{n-2} - \{i\}} (a \circ_i b, a \circ_i b \circ_i ((a \circ_i (b \circ_n c)))) = f_{I_{n-2} - \{i\}} (a \circ_i b, a \circ_i b \circ_i ((a \circ_i (b \circ_n c)))) = f_{I_{n-3}} (a, f_{I_{n-2}, n-1, n} (a, b) \circ_i f_{I_{n-3}, n} (a, b \circ_n c)) = f_{I_{n-3}} (a, f_{I_{n-2}, n-1, n} (a, b) \circ_i f_{I_{n-3}, n} (a, b \circ_n c)) = f_{I_{n-3}} (a, f_{I_{n-2}, n-1, n} (a, b) \circ_i f_{I_{n-3}, n} (a, b \circ_n c)) = f_{I_{n-3}} (a, f_{I_{n-2}, n-1, n} (a, b) \circ_i f_{I_{n-3}, n} (a, b \circ_n c)) = f_{I_{n-3}} (a, f_{I_{n-2}, n-1, n} (a, b) \circ_i f_{I_{n-3}, n} (a, b \circ_n c)) = f_{I_{n-3}} (a, f_{I_{n-2}, n-1, n} (a, b) \circ_i f_{I_{n-3}, n} (a, b \circ_n c)).$$
\[ f_{I_{n-3}}(a \circ_i b, (a \circ_i b) \circ_i (a \circ_{n-2} (a \circ_{n-1} (a \circ_n c)))) \\
\circ_i f_{I_{n-3}}(a \circ_i b, (a \circ_i b) \circ_i (b \circ_{n-2} (b \circ_{n-1} (b \circ_n c)))) \\
\vdots \\
= f_{1,i}(a \circ_i b, f_{I_{1,i-1}}(a, c)) \circ_i f_{1,i}(a \circ_i b, f_{I_{1,i-1}}(b, c)) \\
= (a \circ_i b) \circ_i ((a \circ_i b) \circ_i f_{I_{1,i-1}}(a, c)) \\
\circ_i (a \circ_i b) \circ_i ((a \circ_i b) \circ_i f_{I_{1,i-1}}(b, c)) \\
= (a \circ_i b \circ_i a \circ_i f_{I_{1,i-1}}(a, c)) \circ_i (a \circ_i b \circ_i b \circ_i f_{I_{1,i-1}}(b, c)) \\
= a \circ_i (a \circ_i f_{I_{1,i-1}}(a, c)) \circ_i b \circ_i (b \circ_i f_{I_{1,i-1}}(b, c)) \\
= f_I(a, c) \circ_i f_I(b, c).
\]

(2) From the weak distributivity and idempotence, we have

\[ f_I(a \circ_i b, b) = f_{1,2,\ldots,n-1}(a \circ_i b, (a \circ_i b) \circ_i b) = f_{1,2,\ldots,i-1,i+1,\ldots,n}(a \circ_i b, (a \circ_i b) \circ_i b) = a \circ_i b. \]

Interchange roles of \(a\) and \(b\), \(f_I(a \circ_i b, a) = a \circ_i b\) holds.

(3) First, we show that for any nonempty subsequence \(K = (i_1, i_2, \ldots, i_k)\) of \((1, 2, \ldots, n)\),

\[ f_{i_1,i_2,\ldots,i_k}(a, f_{i_1,i_2,\ldots,i_k}(b, c)) = a \circ_{i_1}(a \circ_{i_2}(\cdots(a \circ_{i_{k-1}}(a \circ_{i_k} b \circ_{i_k} c))\cdots)). \]

\[ = f_{i_1,i_2,\ldots,i_k}(a, b \circ_{i_k} c). \]

We use the induction on \(k\). If \(k = 2\), then by (5), (2) of Lemma 2.3.,

\[ f_{i_1,i_2}(a, f_{i_1,i_2}(b, c)) = a \circ_{i_1}(a \circ_{i_2}(b \circ_{i_1} (b \circ_{i_2} c))) \\
= a \circ_{i_1}(a \circ_{i_2} b \circ_{i_1} (a \circ_{i_2} b \circ_{i_2} c)) = a \circ_{i_1}(a \circ_{i_2} b \circ_{i_2} c) \\
= f_{i_1,i_2}(a, b \circ_{i_2} c). \]

Assume that the above statement is true for all sequences of indices with the length \(\leq k - 1\). Let \(K = (i_1, i_2, \ldots, i_k)\) and \(J = (i_1, i_2, \ldots, i_{k-1})\). Then by induction hypothesis, (5) and (3) of Lemma 2.3, we have

\[ f_K(a, f_K(b, c)) = f_K(a, f_J(b, b \circ_{i_k} c)) \\
= a \circ_{i_k} f_J(a, b \circ_{i_k} c) \\
= a \circ_{i_k} f_J(a, b \circ_{i_k-1} (b \circ_{i_k} c)) \\
= f_J(a, a \circ_{i_k} (b \circ_{i_k-1} (b \circ_{i_k} c))) \\
= f_J(a, a \circ_{i_k} b \circ_{i_k-1} (a \circ_{i_k} b \circ_{i_k} c)) \\
= f_J(a, a \circ_{i_k} b \circ_{i_k} c) \\
= f_K(a, b \circ_{i_k} c). \]
Hence \( f_K (a, f_K (b, c)) = f_K (a, b \circ_{i_k} c) = f_K (a, f_K (c, b)) \). Also, we show that
\[
f_K (f_K (a, b), c) = a \circ_{i_1} \left( a \circ_{i_2} \left( \cdots (a \circ_{i_{k-1}} (a \circ_{i_k} b \circ_{i_k} c)) \cdots \right) \right)
= f_K (a, b \circ_{i_k} c).
\]

First, we claim that for index \( J = (i_1, i_2, \cdots, i_n) \) \((2 \leq n \leq k - 1)\)
\[
f_J (f_K (a, b), c) = a \circ_{i_1} \left( a \circ_{i_2} \left( \cdots (a \circ_{i_n} f_K (a, b) \circ_{i_n} c) \cdots \right) \right)
= f_J (a, f_K (a, b) \circ_{i_n} c).
\]

We use the induction on \( n \). Let \( S = (i_2, i_3, \cdots, i_k) \) and \( T = (i_3, \cdots, i_k) \). If \( n = 2 \),
then by (7), (5) and (3) of Lemma 2.3,
\[
f_{i_1, i_2} (f_K (a, b), c) = f_K (a, b \circ_{i_1} (f_K (a, b) \circ_{i_2} c)
= \circ_{i_1} f_S (a, b) \circ_{i_1} \left( (a \circ_{i_1} f_S (a, b)) \circ_{i_2} c \right)
= a \circ_{i_1} \left( a \circ_{i_2} f_T (a, b) \circ_{i_1} ((a \circ_{i_1} (a \circ_{i_2} f_T (a, b))) \circ_{i_2} c) \right)
= a \circ_{i_1} \left( a \circ_{i_2} f_T (a, b) \circ_{i_1} ((a \circ_{i_2} f_T (a, b) \circ_{i_2} c) \right)
= a \circ_{i_1} \left( a \circ_{i_2} f_T (a, b) \circ_{i_2} c \right)
= a \circ_{i_1} \left( a \circ_{i_2} f_K (T, i_{i_2}) (a, b) \circ_{i_2} c \right)
= f_{i_1, i_2} (a, f_K (a, b) \circ_{i_2} c).
\]

Assume that
\[
f_J (f_K (a, b), c) = a \circ_{i_1} \left( a \circ_{i_2} \left( \cdots (a \circ_{i_n} f_K (a, b) \circ_{i_n} c) \cdots \right) \right)
= f_J (a, f_K (a, b) \circ_{i_n} c).
\]
holds for all \( n \leq k - 2 \). Then by the induction hypothesis, (1) and (3) of Lemma 2.3, we have
\[
f_{i_1, \cdots, i_{k-1}} (f_K (a, b), c) = f_{i_1, \cdots, i_{k-2}} (f_K (a, b), c) \circ_{i_{k-1}} f_K (a, b)
= f_K (a, b) \circ_{i_{k-1}} f_{i_1, \cdots, i_{k-2}} (a, f_K (a, b) \circ_{i_{k-2}} c)
= a \circ_{i_{k-1}} f_{i_1, \cdots, i_{k-2}} (a, a \circ_{i_k} b) \circ_{i_{k-1}} f_{i_1, \cdots, i_{k-2}} (a, f_K (a, b) \circ_{i_{k-2}} c)
= a \circ_{i_{k-1}} f_{i_1, \cdots, i_{k-2}} (a, a \circ_{i_k} b) \circ_{i_{k-1}} (f_{K} (a, b) \circ_{i_{k-2}} c)
= f_{i_1, \cdots, i_{k-2}} (a, a \circ_{i_k} b) \circ_{i_{k-1}} ((a \circ_{i_k} (a \circ_{i_k} b)) \circ_{i_{k-2}} c)
= f_{i_1, \cdots, i_{k-2}} (a, a \circ_{i_k} (a \circ_{i_k} b)) \circ_{i_{k-2}} (a \circ_{i_k} (a \circ_{i_k} b)) \circ_{i_{k-2}} c)
= f_{i_1, \cdots, i_{k-2}} (a, a \circ_{i_k} (a \circ_{i_k} b)) \circ_{i_{k-1}} c)
= f_{i_1, \cdots, i_{k-1}} (a, f_K (a, b) \circ_{i_{k-1}} c).
\]
Using the above claim, we have
\[
f_{i_1, i_2, \cdots, i_k} (f_K (a, b), c) = f_K (a, b \circ_{i_k} c).
\]
This completes the proof.
3. \textit{wD}_n\text{-SLatt} and \textit{wD}_n\text{-Latt}

In this section, we prove that a weak distributive \(n\)-semilattice has a partition consisting of weak distributive \(n\)-lattices and which form a direct system in the category \textit{wD}_n\text{-Latt} of weak distributive \(n\)-lattices and homomorphisms. Furthermore, we show that the direct limit of this direct system gives to the reflection. Firstly, for a weak distributive \(n\)-semilattice \(W\), we have a partition of weak distributive \(n\)-lattices of \(W\) by the following equivalence relation.

**Proposition 3.1.** Let \(W = (W, F)\) be a weak distributive \(n\)-semilattice. Define a binary relation \(\theta\) on \(W\) as follows:

\[(a, b) \in \theta \text{ if and only if } f_I(a, b) = a \text{ and } f_I(b, a) = b,\]

where \(I = (1, 2, \cdots, n)\). Then \(\theta\) is an equivalence relation and each equivalence class \(\theta(x)\) of \(x\) is a subalgebra of \(W\). Moreover, each \(\theta(x)\) is a weak distributive \(n\)-lattice.

**Proof.** Clearly, \(\theta\) is reflexive and symmetric. Let \((a, b), (b, c) \in \theta\). Then

\[f_I(a, b) = a, f_I(b, a) = b = f_I(b, c) \text{ and } f_I(c, b) = c.\]

Using (3) of Lemma 2.4, we have \((a, c) \in \theta\); \(\theta\) is transitive. Then \(\theta\) is an equivalence relation. It remains to show that each \(\theta(x)\) is a subalgebra which is a weak distributive \(n\)-lattice. Take any \(a, b \in \theta(x)\). Then

\[f_I(a, x) = a, f_I(x, a) = x = f_I(x, b) \text{ and } f_I(b, x) = b.\]

Thus for any \(j \in [n]\),

\[f_I(a \circ_j b, x) = f_I(a, x) \circ_j f_I(b, x) = a \circ_j b, \quad f_I(x, a \circ_j b) = f_I(x, a) \circ_j f_I(x, b) = x \circ_j x = x;\]

\(a \circ_j b \in \theta(x)\). So \(\theta(x)\) is a subalgebra of \(W\). By the definition of \(\theta\) and Lemma 2.4., \(\theta(x)\) satisfies the generalized absorption law and thus each \(\theta(x)\) is a weak distributive \(n\)-lattice. \(\square\)

Proposition 3.1 amounts to saying that for a weak distributive \(n\)-semilattice \(W = (W, F)\) we have a partition \(\{W_\alpha \mid \alpha \in S\}\) of subalgebras of \(W\) which are weak distributive \(n\)-lattices. Here we consider a binary relation \(\leq\) on the set \(S\) of indices of the set \(\{W_\alpha \mid \alpha \in S\}\) defined as follows:

\[\alpha \leq \beta \text{ if and only if there are } a \in W_\alpha, b \in W_\beta \text{ such that } f_I(b, a) = b.\]

Then \((S, \leq)\) is a join semilattice.

For \(\alpha \leq \beta\) let \(\varphi_{\alpha, \beta} : W_\alpha \rightarrow W_\beta\) be the map defined by \(\varphi_{\alpha, \beta}(a) = f_I(a, b)\), where \(b\) is an arbitrary element of \(W_\beta\). Thus we have a family of homomorphisms
\{\varphi_{\alpha,\beta} \mid \alpha \leq \beta\}. Moreover, for \alpha \leq \beta and \beta \leq \gamma, \varphi_{\alpha,\beta}(a) = f_I(a, b) and \varphi_{\beta,\gamma}(b) = f_I(b, c), where b \in W_\beta and c \in W_\gamma, and thus

\[
\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta}(a) = \varphi_{\beta,\gamma}(f_I(a, b)) = f_I(f_I(a, b), c) = f_I(a, f_I(b, c)) = f_I(a, f_I(c, b)) = f_I(a, c) = \varphi_{\alpha,\gamma}(a)
\]

and

\[
\varphi_{\alpha,\alpha}(a) = f_I(a, a) = a = 1_{W_n}(a).
\]

Then we obtain a direct system (see [4]) \(((S, \leq), \{W_\alpha \mid \alpha \in S\}, \{\varphi_{\alpha,\beta} \mid \alpha \leq \beta\})\) of weak distributive \(n\)-lattices, where \(\{W_\alpha \mid \alpha \in S\}\) is the partition of the given weak distributive \(n\)-semilattice \(W\), given by Proposition 3.1.

Let \(S(W) = (\bigcup_{\alpha \in S} W_\alpha, *,_1, *_2, \ldots, *_n)\) be an algebra with \(n\) binary operations such that for \(x \in W_\alpha, y \in W_\beta, x *_i y = \varphi_{\alpha,\gamma}(x) *_i \varphi_{\beta,\gamma}(y)\), where \(\gamma = \alpha \lor \beta\) in the join semilattice \((S, \leq)\). Then one has the following Proposition:

**Proposition 3.2.** For any weak distributive \(n\)-semilattice \(W = (W, o_1, o_2, \ldots, o_n)\), \(W\) and \(S(W)\) are identical.

**Proof.** For any \(x, y \in W\), assume that \(x \in W_\alpha, y \in W_\beta\) and let \(\gamma = \alpha \lor \beta\), then \(z = x o_i y \in W_\gamma\), by the above argument. Then \(x *_i y = \varphi_{\alpha,\gamma}(x) *_i \varphi_{\beta,\gamma}(y) = f_I(x, z) *_i f_I(y, z) = f_I(x o_i y, z) = f_I(z, z) = z = x o_i y\) for all \(i \in [n]\). 

From the definition of homomorphism \(\varphi_{\alpha,\beta}\) and Proposition 3.2, we have the following theorem:

**Theorem 3.3.** For a weak distributive \(n\)-semilattice \(W = (W, o_1, o_2, \ldots, o_n)\), let \(S(W) = (\bigcup_{\alpha \in S} W_\alpha, *,_1, *_2, \ldots, *_n) = (W, o_1, o_2, \ldots, o_n)\) in the above Proposition 3.2. Define a binary relation \(\Lambda\) on \(S(W)\) by \((x, y) \in \Lambda\) if and only if \(x \in W_\alpha, y \in W_\beta\) for some \(\alpha, \beta \in S\) and there exists \(\gamma \in S\) such that \(\alpha \leq \gamma, \beta \leq \gamma, \varphi_{\alpha,\gamma}(x) = \varphi_{\beta,\gamma}(y)\), i.e., \(f_I(x, z) = f_I(y, z)\), where \(z\) is an arbitrary element of \(W_\gamma\). Then the relation \(\Lambda\) is a congruence on \(S(W)\).

The following theorem follows from Theorem 3.3.

**Theorem 3.4.** The quotient algebra \((S(W) / \Lambda, *,_1, *_2, \ldots, *_n)\) of \(S(W)\) is a weak distributive \(n\)-lattice.

**Proof.** It is enough to show that \(S(W) / \Lambda\) satisfies the generalized absorption law. Let \(x \in W_\alpha\) and \(y \in W_\beta\) for some \(\alpha, \beta \in S\), then there exists \(\gamma \in S\) such that \(\alpha, \beta \leq \gamma\). So

\[
[x]_{\Lambda} *_1 ([x]_{\Lambda} *_2 (\cdots ([x]_{\Lambda} *_{n-1} ([x]_{\Lambda} *_{n} [y]_{\Lambda})))) \cdots) \\
= [\varphi_{\alpha,\gamma}(x) o_1 (\varphi_{\alpha,\gamma}(x) o_2 (\cdots (\varphi_{\alpha,\gamma}(x) o_{n-1} (\varphi_{\alpha,\gamma}(x) o_{n} \varphi_{\beta,\gamma}(y)))))) \cdots)]_{\Lambda} \\
= [\varphi_{\alpha,\gamma}(x)]_{\Lambda} = [x]_{\Lambda}.
\]
Hence $S(W)/\Lambda$ is a weak distributive $n$-lattice.

As the following terminologies are refer to [4], we obtain the following facts:

**Remark 3.5.** For a weak distributive $n$-semilattice $W = (W, o_1, o_2, \ldots, o_n)$, $W$ and $S(W)$ are identical. Thus, $S(W)/\Lambda$ may be viewed as a quotient algebra of $W$. In fact, $W/\Lambda$ is the direct limit of the direct system

$$((S, \leq), \{W_\alpha | \alpha \in S\}, \{\varphi_{\alpha,\beta} | \alpha \leq \beta\}) .$$

The class of weak distributive $n$-semilattices and homomorphisms between them forms a category, which will be denoted by $\textbf{wDn-SLatt}$, and the class of weak distributive $n$-lattices forms a full subcategory of $\textbf{wDn-SLatt}$, which will be denoted by $\textbf{wDn-Latt}$.

**Theorem 3.6.** The category $\textbf{wDn-Latt}$ is a reflective subcategory of the category $\textbf{wDn-SLatt}$.

**Proof.** For a weak distributive $n$-semilattice $W = (W, o_1, o_2, \ldots, o_n)$, let $q : W \longrightarrow W/\Lambda$ be the quotient homomorphism, where $\Lambda$ is the congruence given in Theorem 3.3. Then $(q, W/\Lambda)$ is the $\textbf{wDn-Latt}$-reflection of $W \in \textbf{wDn-SLatt}$. In fact, take any $L = (L, \ast_1, \ast_2, \ldots, \ast_n) \in \textbf{wDn-Latt}$ and any homomorphism $f : W \longrightarrow L$, then $\ker(q) \subseteq \ker(f)$. For any $(x, y) \in \ker(q)$, there are $\alpha, \beta \in S$ such that $x \in W_\alpha$, $y \in W_\beta$. Then there is $\gamma \in S$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$, $q(x) = [\varphi_{\alpha, \gamma}(x)]_\Lambda = [\varphi_{\beta, \gamma}(y)]_\Lambda = q(y)$ so, $x \circ_1 (x \circ_2 \cdots (x \circ_{n-1} (x \circ_n z)) \cdots) = y \circ_1 (y \circ_2 \cdots (y \circ_{n-1} (y \circ_n z)) \cdots)$. Since $f$ is a homomorphism and each element of $L$ satisfies the generalized absorption law,

$$f(x) = (f(x) \ast_1 (f(x) \ast_2 \cdots (f(x) \ast_{n-1} (f(x) \ast_n f(z))) \cdots)) = f(y) \ast_1 (f(y) \ast_2 \cdots (f(y) \ast_{n-1} (f(y) \ast_n f(z))) \cdots) = f(y);$$

therefore $(x, y) \in \ker(f)$. So by the Fundamental Theorem of Factorization, there is a unique homomorphism $f : W/\Lambda \longrightarrow L$ with $f \circ q = f$. Hence $\textbf{wDn-Latt}$ is a reflective subcategory of $\textbf{wDn-SLatt}$. 

**Corollary 3.7.** The category $\textbf{wDn-Latt}$ is closed under the formation of limits in the category $\textbf{wDn-SLatt}$.

Note that $W/\Lambda$ is the direct limit of the following direct system

$$((S, \leq), \{W_\alpha | \alpha \in S\}, \{\varphi_{\alpha,\beta} | \alpha \leq \beta\}) .$$

Then we have the following corollary, directly.

**Corollary 3.8.** If $W = (W, F)$ is a finite weak distributive $n$-semilattice, then $W_p \cong W/\Lambda$, where $p$ is the largest element of $(S, \leq)$.

**References**


