The Seifert Matrices of Periodic Links with Rational Quotients

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Abstract. In this paper, we characterize the Seifert matrices of \( p \)-periodic links whose quotients are 2-bridge links \( C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r) \) and give formulas for the signatures and determinants of the 3-periodic links of these kind in terms of \( n_1, n_2, \ldots, n_r \).

1. Introduction

A link \( L \) in \( S^3 \) is called a \( p \)-periodic link (\( p \geq 2 \) an integer) if there exists an orientation preserving auto-homeomorphism \( h \) of \( S^3 \) such that \( h(L) = L \), \( h \) is of order \( p \) and the set of fixed points of \( h \) is a circle disjoint from \( L \). In this paper, we are interested in a special class of periodic knots and links.

A link in \( S^3 \) is called a \( p \)-periodic link with rational quotient if it is obtained as the preimage of one component of a 2-bridge link in \( S^3 \) by the \( p \)-fold branched cyclic covering branched along the other component. In [5], the authors introduced a special kind of Conway’s normal form \( C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r) \) of a 2-bridge link with two components and studied the excellent component of the character variety of periodic knots in \( S^3 \) with rational quotient. In [10], the authors re-examined this presentation to study the Alexander polynomials of 2-bridge links and periodic links in \( S^3 \) with rational quotients in terms of \( n_1, n_2, \ldots, n_r \). In [7, 11], the authors gave formulas for the Casson knot invariant and the \( \Delta \)-unknotting number of \( p \)-periodic knots with rational quotients via \( n_1, n_2, \ldots, n_r \).

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The purpose of this paper is to give a characterization of the Seifert matrices of periodic links with rational quotients and to study the properties of numerical invariants of the Seifert matrices. In Section 2, we review presentations of 2-bridge links and $p$-periodic links with rational quotients. In Section 3, we show that the Seifert matrices of $(p + 1)$-periodic links with rational quotient $C(2, n_1, -2, n_2, \ldots, n_r, (1)^{r}2)$ is $S$-equivalent to a $p \times p$ block tridiagonal matrix in which each block is also a $r \times r$ tridiagonal matrix whose entries are completely determined by the integers $n_1, n_2, \ldots, n_r$. In Section 4, we give formulas for the signature and determinant of a 3-periodic link with rational quotient $C(2, n_1, -2, n_2, \ldots, n_r, (1)^{r}2)$ in terms of $n_1, n_2, \ldots, n_r$.

2. Periodic links with rational quotients

To each pair $(\alpha, \beta)$ of two co-prime integers subject to the condition that $\beta$ is odd and $0 < |\beta| < \alpha$, Schubert [14] associated an oriented diagram on the 2-sphere $S^2$ of an oriented 2-bridge knot ($\alpha$ odd) or link ($\alpha$ even) $L$ in $S^3$, now called the Schubert normal form of $L$ and denoted by $S(\alpha, \beta)$, and showed that any (oriented) 2-bridge knots and links in $S^3$ can be represented in this way. Two such pairs of integers $(\alpha, \beta)$ and $(\alpha', \beta')$ define an equivalent oriented (resp. unoriented) knot or link if and only if

$$\alpha = \alpha' \text{ and } \beta^{k+1} \equiv \beta' \mod 2\alpha \text{ (resp. mod } \alpha),$$

where $\beta^{-1}$ denotes the integers with the properties $0 < \beta^{-1} < 2\alpha$ and $\beta \beta^{-1} \equiv 1 \mod 2\alpha$.

Let $[a_1, a_2, \ldots, a_n]$ denote the continued fraction of $\alpha/\beta$:

$$[a_1, a_2, \ldots, a_n] \equiv a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_n}}} = \frac{\alpha}{\beta},$$

Then $L = S(\alpha, \beta)$ has also a diagram $C(a_1, a_2, \ldots, a_n)$, called Conway normal form of $L$, as shown in Figure 1, depending on whether $n$ is even or odd [1]. The integral tangles in Figure 1, which are rectangles labeled $a_i$, are the 2-braids with $|a_i|$ crossings as shown in Figure 2. It is well known that $L = S(\alpha, \beta)$ admits a diagram $C(2b_1, 2b_2, \ldots, 2b_n)$, which is equivalent to $C(a_1, a_2, \ldots, a_n)$ [6].

It is known [5], [10] that the 2-bride link $L = S(\alpha, \beta)(\alpha$ even) can also be represented by Conway diagram of the form $C(2, n_1, -2, n_2, \ldots, n_r, (1)^{r}2)$ as shown in Figure 3. We choose an orientation of the 2-bride link $C(2, n_1, -2, n_2, \ldots, n_r, (1)^{r}2)$ as shown in Figure 3. Then it is easy to see that the diagram shown in Figure 3 can be deformed to the diagrams in Figure 4 by using Reidemeister moves. Throughout this paper, an oriented 2-bride link $L$ in $S^3$ represented by the Conway normal form $C(2, n_1, -2, n_2, \ldots, n_r, (1)^{r}2)$ is denoted by $L = C[[n_1, n_2, \ldots, n_r]].$
A link $L$ in $S^3$ is called a $p$-periodic link ($p \geq 2$ an integer) if there exists an orientation preserving auto-homeomorphism $h$ of $S^3$ such that $h(L) = L$, $h$ is of order $p$ and the set $\text{Fix}(h)$ of fixed points of $h$ is a circle disjoint from $L$. In this case, the link $L/\langle h \rangle \cup \text{Fix}(h)$ in the orbit space $S^3/\langle h \rangle \cong S^3$ is called the quotient link of $L$. Let $K$ be an oriented link in $S^3$ and $U$ an oriented trivial knot with $K \cap U = \emptyset$. For any integer $p \geq 2$, let $\phi_p^U : \Sigma^3 \to S^3$ be a $p$-fold branched cyclic covering branched along $U$. Then $\Sigma^3$ is homeomorphic to the 3-sphere $S^3$. Then $(\phi_p^U)^{-1}(K)$ is a $p$-periodic link in $\Sigma^3$ with $L = K \cup U$ as its quotient link. We give an orientation to $(\phi_p^U)^{-1}(K)$ induced by the orientation of $K$. Note that any periodic knot or link in $S^3$ arises in this manner.

**Definition 2.1.** A link $\tilde{L}$ in $S^3$ is called a $p$-periodic link with rational quotient if it is a $p$-periodic link whose quotient link is a 2-bridge link, or equivalently, if there exists a 2-bridge link $L = U_1 \cup U_2$ in $S^3$ such that $\tilde{L}$ is equivalent to the preimage $(\phi_p^{U_2})^{-1}(U_1)$ of the component $U_1$ of $L$ by a $p$-fold cyclic covering $\phi_p^{U_2} : \Sigma^3 \to S^3$ branched along the component $U_2$ of $L$.

Note that each component $U_1$ and $U_2$ of $L$ is a trivial knot and they can be interchanged each other by an orientation preserving homeomorphism of
This implies that \((\phi_{U_1}^p)^{-1}(U_1)\) is equivalent to \((\phi_{U_2}^p)^{-1}(U_2)\). Now let
\[
L = \overrightarrow{C}[n_1, n_2, \ldots, n_r] = U_1 \cup U_2
\]
be an oriented 2-bridge link as shown Figure 4. Then the diagram, \(D^{(p)}\), shown in Figure 5 is a canonical oriented \(p\)-periodic diagram of the oriented \(p\)-periodic link \((\phi_{U_2}^p)^{-1}(U_1)\) with rational quotient
\[
L = \overrightarrow{C}[n_1, n_2, \ldots, n_r].
\]
In what follows, we shall denote the oriented \(p\)-periodic link \((\phi_{U_2}^p)^{-1}(U_1)\) by \(L^{(p)}\) or \(\overrightarrow{C}[n_1, n_2, \ldots, n_r]^{(p)}\) for our convenience. Then any \(p\)-periodic link with rational quotient can be represented by \(\overrightarrow{C}[n_1, n_2, \ldots, n_r]^{(p)}\) for some nonzero integers \(n_1, n_2, \ldots, n_r\) [7], [10].

3. Seifert matrices

We begin with a brief review of Seifert matrix of a link in \(S^3\) from Chapter 5 in [13].

A Seifert surface for a link in \(S^3\) is a connected compact orientable surface embedded in \(S^3\) with its boundary \(L\). In [15], Seifert proved the existence of Seifert surface for a link \(L\) applying \(L\) an algorithm, called Seifert’s algorithm, on a diagram of \(L\). Let \(L\) be a link and \(F\) its Seifert surface. There is an embedding \(F \times [-1, 1] \rightarrow S^3\) such that \(b(F \times \{0\}) = F\) and \(b(F \times \{1\})\) lies on the positive side of \(F\). For any simple closed curve \(x \in F\), let \(x^+ = b(x \times \{1\})\) and \(x^- = b(x \times \{-1\})\). Since \(H_1(F)\) is a free abelian group of finite rank \(n\) and is generated by simple closed oriented curves \(x_1, \ldots, x_n\), we can define a bilinear form \(\phi : H_1(F) \times H_1(F) \rightarrow \mathbb{Z}\)
This is called the Seifert pairing or linking form of \( F \). The \( n \times n \) matrix \( M = (m_{i,j}) \) defined by

\[
m_{i,j} = \phi(x_i, x_j)
\]

is called a Seifert matrix of \( L \) associated to \( F \). The Seifert matrix of \( L \) depends on the Seifert surface \( F \) and the choice of generators of \( H_1(F) \).

**Theorem 3.1** ([13]). Two Seifert matrices obtained from two equivalent links can be changed from one to the other by applying, a finite number of times, the following two operations \( \Lambda_1 \) and \( \Lambda_2 \), and their inverses:

\( \Lambda_1 : M_1 \longrightarrow PM_1P^T \), where \( P \) is an invertible matrix with \( \det P = \pm 1 \) and \( P^T \) denotes the transpose matrix of \( P \).

\( \Lambda_2 : M_1 \longrightarrow M_2 = \begin{pmatrix} M_1 & \mathbf{v} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \) or \( \begin{pmatrix} M_1 & 0 & 0 \\ \mathbf{v} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \),

where \( \mathbf{v} \) denotes an arbitrary integral row or column vector, and \( \mathbf{0} \) the row or column zero vector.

Two square matrices \( M \) and \( M' \) are said to be \( S \)-equivalent if one is obtained from the other by applying the operations \( \Lambda_1, \Lambda_2 \) and the inverse \( \Lambda_2^{-1} \), a finite number of times.

For any real number \( y \), let \( \lfloor y \rfloor \) denote the largest integer less than or equal to \( y \).

**Theorem 3.2.** For given nonzero integers \( n_1, n_2, \ldots, n_r \) (\( r \geq 1 \)) and a positive
integer $p \geq 1$, let $A$, $B$ and $C$ be $r \times r$ tridiagonal matrices with integral entries given by

\[
A = \begin{bmatrix}
\alpha_1 & \epsilon_1 - 1 & & & \\
& \alpha_2 & \epsilon_2 - 1 & & \\
& & \ddots & \ddots & \\
& & & \alpha_{r-1} & \epsilon_{r-1} - 1 \\
& & & & \alpha_r
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
\beta_1 & & & & \\
\epsilon_1 & \beta_2 & & & \\
& \epsilon_2 & \ddots & & \\
& & \ddots & \beta_{r-1} & \\
& & & \epsilon_{r-1} & \beta_r
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
-n_1 & 1 - \epsilon_1 & & & \\
\epsilon_1 & -n_2 & 1 - \epsilon_2 & & \\
& \ddots & \ddots & \ddots & \\
& & \epsilon_{r-2} & -n_{r-1} & 1 - \epsilon_{r-1} \\
& & & \epsilon_{r-1} & -n_r
\end{bmatrix},
\]

where $\alpha_i = \left\lfloor \frac{n_i + 1 - \epsilon_i - 1}{2} \right\rfloor$, $\beta_i = \left\lfloor \frac{n_i + \epsilon_i - 1}{2} \right\rfloor$ and $\epsilon_i = 1$ if $n_1 + n_2 + \cdots + n_i + i$ is even and $\epsilon_i = 0$ otherwise. Suppose that $L^{(p+1)}$ is the $(p+1)$-periodic link in $S^3$ with rational quotient $L = C[[n_1, n_2, \ldots, n_r]]$. Then a Seifert matrix of $L^{(p+1)}$ is
S-equivalent to the $p \times p$ block tridiagonal matrix

$$M = \begin{bmatrix}
C & B & & & \\
A & C & B & & \\
& A & C & B & \\
& & \ddots & \ddots & \ddots \\
& & & A & C & B \\
& & & & A & C
\end{bmatrix}.$$ 

**Proof.** Let $D^{(p+1)}$ be the diagram of $L^{(p+1)}$ as shown in Figure 5 and let $F$ be the Seifert surface of $L^{(p+1)}$ obtained by applying Seifert algorithm to $D^{(p+1)}$. Let $\{x_{i,j} | 1 \leq i \leq p, 1 \leq j \leq r\}$ be the set of simple closed curves which represent the generators of $H_1(F)$. We assign the clockwise orientation to each curves $x_{i,j}$. For example, see Figure 6. In Figure 6, there are the Seifert surface $F$ of $C[[2,1, -2]]^{(3)}$ obtained by applying Seifert algorithm and simple closed curves representing the generators of $F$. For each 4-tuple $(i,j,k,l)$ with $1 \leq i,k \leq p$ and $1 \leq j,l \leq r$, we can calculate that

$$lk(x_{i,j}, x_{k,l}^+) = \begin{cases} 
-n_j & \text{if } k = i, l = j, \\
\epsilon_{j-1} & \text{if } k = i, l = j - 1, \\
-\epsilon_{j-1} & \text{if } k = i+1, l = j - 1, \\
1 - \epsilon_j & \text{if } k = i, l = j + 1, \\
-1 + \epsilon_j & \text{if } k = i - 1, l = j + 1, \\
\alpha_j & \text{if } k = i - 1, l = j, \\
\beta_j & \text{if } k = i + 1, l = j, \\
0 & \text{otherwise.}
\end{cases}$$

(3.1)

Consider the simple closed curves $x_{1,1}, x_{1,2}, \cdots, x_{1,r}, \ x_{2,1}, x_{2,2}, \cdots, x_{2, r}, \cdots, x_{p,1}, x_{p,2}, \cdots, x_{p, r}$ and let $M = (m'_{a,b})$ be the $rp \times rp$ Seifert matrix defined by

$$m'_{a,b} = lk(x_{i,j}, x_{k,l}^+)$$
where \( a = r(i - 1) + j \) and \( b = r(k - 1) + l \). We can partition the matrix \( M' \) into \( r \times r \) submatrices of \( M' \) as follows:

\[
M' = (M'_{i,j}), \quad M'_{i,j} = (m''_{k,l}) \]

where \( m''_{k,l} = m'_{r(i-1)+k,r(j-1)+l} \). From (3.1), we can see that \( M'_{i,j} = \)

\[
\begin{cases}
  C & \text{if } j = i, \\
  A & \text{if } j = i - 1, \\
  B & \text{if } j = i + 1, \\
  O & \text{otherwise},
\end{cases}
\]

where \( O \) is the \( r \times r \) zero matrix. Hence \( M' = M \). This completes the proof. □

**Example 3.3.** Let \( L^{(3)} \) be the 3-periodic link with rational quotient \( L = \mathbb{C}[[2,1,-2]] \). Let \( F \) be a Seifert surface obtained by applying Seifert algorithm to a diagram as described in Figure 6. Consider the simple closed curves representing the generator of \( H_1(F) \) as depicted in Figure 6. Then any Seifert matrix \( M \) of \( L^{(3)} \) is \( S \)-equivalent to the matrix of the form:

\[
M = \begin{bmatrix}
-2 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & -1 \\
1 & -1 & 0 & -2 & 1 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 \\
0 & 0 & -1 & 0 & 0 & 2
\end{bmatrix}
\]

4. Invariants of Seifert matrices

For a real symmetric matrix \( A \), there exists an invertible matrix \( P \) such that \( PAP^T = B \) is a diagonal matrix. By Sylvester’s Theorem in Linear Algebra, the sum of signs of the entries in the diagonal of \( B \), called the signature of \( A \) and is denoted by \( \sigma(A) \), is independent on the diagonalization. It is well known that two \( S \)-equivalent symmetric matrices have the same signature. Now let \( M \) be a Seifert matrix of a link \( L \). Then the signature \( \sigma(L) \) of \( L \) is defined by

\[
\sigma(L) = \sigma(M + M^T).
\]

Note that \( \sigma(L) \) is a link invariant [13].

For given nonzero integers \( n_1, n_2, \ldots, n_r \) \( (r \geq 1) \), let \( L^{(p+1)} \) be the \( (p+1) \)-periodic link in \( S^3(p \geq 1) \) with rational quotient \( L = \mathbb{C}[[n_1,n_2,\ldots,n_r]] \). Let \( M \) be the Seifert matrix of \( L^{(p+1)} \) given by Theorem 3.2 above and \( S = M + M^T \). Then
$S$ is the $p \times p$ symmetric block tridiagonal matrix given by

$$S = \begin{bmatrix} E & F^T & & & & \\ F & E & F^T & & & \\ & F & E & F^T & & \\ & & \ddots & \ddots & \ddots & \\ & & & F & E & F^T \\ & & & & F & E \end{bmatrix},$$

(4.2)

where $E$ and $F$ are $r \times r$ tridiagonal matrices given by

$$E = \begin{bmatrix} -2n_1 & 1 & & & & \\ 1 & -2n_2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & -2n_{r-1} & 1 & \\ & & & 1 & -2n_r \end{bmatrix},$$

$$F = \begin{bmatrix} n_1 & -1 & & & & \\ n_2 & -1 & & & & \\ & \ddots & \ddots & \ddots & & \\ & & n_{r-1} & -1 & & \\ & & & n_r & \end{bmatrix}.$$

4.1. Signatures of 2-periodic links with rational quotients. For given nonzero integers $n_1, n_2, \ldots, n_r$ ($r \geq 1$), let $L^{(2)}$ be the 2-periodic link in $S^3$ with rational quotient $L = \overline{C}([n_1, n_2, \ldots, n_r])$. Let $M$ denote the Seifert matrix of $L^{(2)}$ given by Theorem 3.2 above and set $S = M + M^T$. From (4.2), we know that $S = E$.

In [7], the authors show that $L^{(2)}$ is the 2-bridge knot with Conway normal form $C(-2n_1, -2n_2, \ldots, -2n_r)$. For each $k = 1, 2, \ldots, r$, we define a rational number $<k>$ by

$$<k> = \begin{cases} -2n_1 & \text{if } k = 1, \\ -2n_k - \frac{1}{k-1} & \text{if } k = 2, 3, \ldots, r. \end{cases}$$

We know that all $<k>$ is not equal to zero. We can calculate that

$$S_2 = VDV^T,$$

where $D$ is the diagonal matrix with diagonal entries $<1>, <2>, \ldots, <r>$ and $V$ is the
$r \times r$ tridiagonal matrices given by

$$V = \begin{bmatrix}
\frac{1}{n_1} & 1 & \frac{1}{2} & 1 \\
& \ddots & \ddots & \ddots \\
& & \frac{1}{r-2} & 1 \\
& & & \frac{1}{r-1} & 1
\end{bmatrix}.$$ 

Since all $n_k$ are nonzero, it follows that $0 < \frac{1}{|n_k|} < 1$ and hence the sign of $\langle k \rangle$ is opposite to the sign of $n_k$. Therefore the signature of 2-periodic link $L^{(2)}$ with rational quotient $L = \mathbb{C}[[n_1, n_2, \cdots, n_r]]$ is given by

$$\sigma(L^{(2)}) = -\sum_{i=1}^{r} \frac{n_i}{|n_i|}.$$ 

### 4.2. Signatures of 3-periodic links with rational quotients.

For given nonzero integers $n_1, n_2, \cdots, n_r$ ($r \geq 1$), let $L^{(3)}$ be the 3-periodic link in $S^3$ with rational quotient $L = \mathbb{C}[[n_1, n_2, \cdots, n_r]]$. Let $M$ denote the Seifert matrix of $L^{(3)}$ given by Theorem 3.2 above and set $S = M + M^T$. From (4.2), we know that $S$ is given by

$$S = \begin{bmatrix}
E & F^T \\
F & E
\end{bmatrix}.$$ 

For given nonzero integers $n_1, n_2, \cdots, n_r$, we define the rational numbers $d_1, d_2, \cdots, d_r, w_1, w_2, \cdots, w_{r-2}$ by

$$d_1 = -\frac{3n_1}{2} + \frac{1}{2n_2},$$
$$d_2 = \frac{1}{2n_1} - \frac{3n_2}{2} + \frac{1}{2n_3},$$
$$d_i = \frac{1}{2n_{i-1}} - \frac{3n_i}{2} + \frac{1}{2n_{i+1}} - \frac{1}{4n_{i-1}^2 d_{i-2}}, \quad i = 3, 4, \cdots, r-1,$$
$$d_r = \frac{1}{2n_{r-1}} - \frac{3n_r}{2} - \frac{1}{4n_{r-1}^2 d_{r-2}},$$
$$w_j = \frac{\tau_j}{2n_{j+1} d_j}, \quad j = 1, 2, \cdots, r-2,$$

where $\tau_j = -1$ if $j - 1 \equiv 0 \pmod{3}$ and $\tau_j = 1$ otherwise. Note that if $d_i = 0$, then $w_j$ and $d_{j+2}$ are not defined for all $j = i, i + 1, \cdots, r - 2$.

**Theorem 4.1.** Let $n_1, n_2, \cdots, n_r$ be given nonzero integers ($r \geq 1$) and let $L^{(3)}$
be the 3-periodic link in $S^3$ with rational quotient $L = \overline{C}[n_1, n_2, \ldots, n_r]$. Let $M$ be the Seifert matrix of $L^{(3)}$ given by Theorem 3.2 above and set $S = M + M^T$. Suppose that $d_i \neq 0$ for all $i = 1, 2, \ldots, r$. Then there exists an invertible matrix $P$ such that $det P = \pm 1$ and

$$S = PDP^T,$$

where $D$ is the $2r \times 2r$ diagonal matrix with diagonal entries $-2n_1, -2n_2, \ldots, -2n_r, d_1, d_2, \ldots, d_r$.

Proof. Let $D_1$ and $D_2$ be the $r \times r$ diagonal matrices with diagonal entries $-2n_1, -2n_2, \ldots, -2n_r$ and $d_1, d_2, \ldots, d_r$, respectively, and let $G = (g_{ij})$ be the $r \times r$ tridiagonal matrix given by

$$g_{ij} = \begin{cases} -n_i & \text{if } j = i, \\ -1 & \text{if } j = i + 1 \text{ and } i \not\equiv 0 \pmod{3}, \\ 1 & \text{if } j = i + 1 \text{ and } i \equiv 0 \pmod{3}, \\ 1 & \text{if } j = i - 1 \text{ and } j \not\equiv 0 \pmod{3}, \\ -1 & \text{if } j = i - 1 \text{ and } j \equiv 0 \pmod{3}, \\ 0 & \text{otherwise}. \end{cases}$$

Then $D = D_1 \oplus D_2$ and we have that

$$(4.4) \quad \begin{bmatrix} U_1 & U_3 \\ U_2 & U_1 \end{bmatrix} S \begin{bmatrix} U_1 & U_3 \\ U_2 & U_1 \end{bmatrix}^T = \begin{bmatrix} D_1 & G \\ G & D_1 \end{bmatrix},$$

where $U_3 = U_1 - U_2$, $U_1 = (u_{ij})$ and $U_2 = (v_{ij})$ are $r \times r$ diagonal matrices with entries

$$u_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i \not\equiv 0 \pmod{3}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$v_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i \not\equiv 1 \pmod{3}, \\ 0 & \text{otherwise}. \end{cases}$$

Now let $W = (w_{ij})$ be the $r \times r$ matrix given by

$$w_{ij} = \begin{cases} 1 & \text{if } j = i, \\ w_j & \text{if } j = i - 2, \\ 0 & \text{otherwise}. \end{cases}$$

By elementary calculations, we obtain that

$$D_1 = GD_1^{-1}G^T + WD_2W^T.$$ 

Hence it follows that

$$(4.5) \quad \begin{bmatrix} D_1 & G^T \\ G & D_1 \end{bmatrix} = \begin{bmatrix} I & O \\ GD_1^{-1}W & I \end{bmatrix} \begin{bmatrix} D_1 & O \\ O & D_2 \end{bmatrix} \begin{bmatrix} I & O \\ GD_1^{-1}W & I \end{bmatrix}^T.$$
From (4.4) and (4.5), we have \( S = PDPT \), where
\[
P = \begin{bmatrix}
U_1 & U_3 \\
U_2 & U_1
\end{bmatrix}^{-1} \begin{bmatrix}
I & 0 \\
GD_1 & W
\end{bmatrix}.
\]

This completes the proof. \( \square \)

**Corollary 4.2.** Let \( n_1, n_2, \ldots, n_r \) be given nonzero integers \((r \geq 1)\) and let \( L^{(3)} \) be the 3-periodic link in \( S^3 \) with rational quotient \( L = \overline{C}[n_1, n_2, \ldots, n_r] \). Suppose that \( d_i \neq 0 \) for all \( i = 1, 2, \ldots, r \). Then
\[
\sigma(L^{(3)}) = \sum_{i=1}^{r} \left( \frac{d_i}{|d_i|} - \frac{n_i}{|n_i|} \right).
\]

**Proof.** The result follows from Theorem 4.1 at once. \( \square \)

**Corollary 4.3.** Let \( n_1, n_2, \ldots, n_r \) be given nonzero integers \((r \geq 1)\) and let \( L^{(3)} \) be the 3-periodic link in \( S^3 \) with rational quotient \( L = \overline{C}[n_1, n_2, \ldots, n_r] \). Suppose that \(|n_1n_2n_3| \geq 2\) for each \( i = 1, 2, \ldots, r-3 \). Then the signature \( \sigma(L^{(3)}) \) of \( L^{(3)} \) is given by
\[
\sigma(L^{(3)}) = -2 \sum_{i=1}^{r} \frac{n_i}{|n_i|} = 2\sigma(L^{(2)}).
\]

**Proof.** We will claim that the sign of \( d_i \) is opposite of the sign of \( n_i \) and the absolute value of \( d_i \) is greater than or equal to \( \frac{1}{4} \) for all \( i = 1, 2, \ldots, r \). Since \( n_1 \) and \( n_2 \) are nonzero integers and \( d_1 = -\frac{3n_1}{2} + \frac{1}{2n_2} \), the sign of \( d_1 \) is opposite of the sign of \( n_1 \) and
\[
|d_1| \geq \frac{3}{2} - \frac{1}{2} = 1 \geq \frac{1}{4}.
\]

Since \( n_1, n_2 \) and \( n_3 \) are nonzero integers and \( d_2 = \frac{1}{2n_1} - \frac{3n_2}{2} + \frac{1}{2n_3} \), the sign of \( d_2 \) is opposite of the sign of \( n_2 \) and
\[
|d_2| \geq \frac{3}{2} - \frac{1}{2} - \frac{1}{2} = \frac{1}{2} \geq \frac{1}{4}.
\]

Suppose that the sign of \( d_i \) is opposite of the sign of \( n_i \) and \(|d_i| \geq \frac{1}{4}\) for all \( i = 1, 2, \ldots, k \). Now we claim that the sign of \( d_{k+1} \) is opposite of the sign of \( n_{k+1} \) and \(|d_{k+1}| \geq \frac{1}{4}\). We recall that, for \( 2 \leq k \leq r-2 \),
\[
d_{k+1} = \frac{1}{2n_k} - \frac{3n_{k+1}}{2} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2d_{k-1}}.
\]

If \(|n_{k+1}| \geq 2\), then \(|\frac{3n_{k+1}}{2}| \geq 3\). Since \(|d_{k-1}| \geq \frac{1}{4}, \ |\frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2d_{k-1}}| \leq \frac{1}{2} + \frac{1}{2} + 1 = 2\). Hence the sign of \( d_{k+1} \) is opposite of the sign of \( n_{k+1} \) and
\[
|d_{k+1}| \geq 3 - 2 = 1 \geq \frac{1}{4}.
\]
If \(|n_{k+1}| = 1\) and \(|n_k| \geq 2\), then \(|\frac{3n_{k+1}}{2} - \frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2d_{k-1}}| \leq \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1\). Hence the sign of \(d_{k+1}\) is opposite of the sign of \(n_{k+1}\) and
\[|d_{k+1}| \geq \frac{3}{2} - 1 = \frac{1}{2} \geq \frac{1}{4}.\]

If \(|n_{k+1}| = 1\), \(|n_k| = 1\) and \(|n_{k-1}| \geq 2\), then \(|d_{k-1}| \geq 1\). Since \(|\frac{3n_{k+1}}{2} - \frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2d_{k-1}}| \leq \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3}{4}\), the sign of \(d_{k+1}\) is opposite of the sign of \(n_{k+1}\) and
\[|d_{k+1}| \geq \frac{3}{2} - \frac{5}{4} = \frac{1}{4}.\]

If \(|n_{k+1}| = 1\), \(|n_k| = 1\) and \(|n_{k-1}| = 1\), then \(|n_{k-2}| \geq 2\) and \(|n_{k+2}| \geq 2\). If \(4 \leq k \leq r-2\), then \(|d_{k-1}| \geq \frac{3n_{k+1}}{2} - \frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2d_{k-1}}| \geq \frac{1}{2} - (\frac{1}{4} + \frac{1}{2} + \frac{1}{4}) = \frac{1}{2}\) and hence \(\frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2d_{k-1}}| \leq \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{4}\). If \(k = 3\), then \(|d_{k-1}| = |d_2| \geq \frac{1}{2}\) and hence \(\frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2d_{k-1}}| \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{5}{4}\). If \(k = 2\), then \(|d_{k-1}| = |d_1| \geq 1\) and hence \(\frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2d_{k-1}}| \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1\). Since \(|\frac{3n_{k+1}}{2} - \frac{1}{2n_k} + \frac{1}{2n_{k+2}} - \frac{1}{4n_k^2d_{k-1}}| = \frac{1}{2}\), the sign of \(d_{k+1}\) is opposite of the sign of \(n_{k+1}\) and
\[|d_{k+1}| \geq \frac{3}{2} - \frac{5}{4} = \frac{1}{4}.\]

If \(k+1 = r\), then we can also see that the sign of \(d_r\) is opposite of the sign of \(n_r\) and \(|d_r| \geq \frac{1}{4}\) by the similar argument.

Therefore \(d_r\) is nonzero and the sign of \(d_i\) is opposite of the sign of \(n_i\) for all \(i = 1, 2, \ldots, r\). From Corollary 4.2, the signature \(\sigma(L^{(3)})\) of \(L^{(3)}\) is given by
\[\sigma(L^{(3)}) = -2 \sum_{i=1}^{r} \frac{n_i}{|n_i|}.\]

From (4.3), \(\sigma(L^{(3)}) = 2(-\sum_{i=1}^{r} \frac{n_i}{|n_i|}) = 2\sigma(L^{(2)})\). This completes the proof. \(\square\)

**Corollary 4.4.** Let \(n_1, n_2, \ldots, n_r\) be given nonzero integers \((r \geq 1)\) and let \(L^{(3)}\) be the 3-periodic link in \(S^3\) with rational quotient \(L = \overrightarrow{C}[n_1, n_2, \ldots, n_r]\). Suppose that \(d_i \neq 0\) for all \(i = 1, 2, \ldots, r\). Then
\[\det(L^{(3)}) = |\Delta_{L^{(3)}}(-1)| = 2^r |n_1n_2 \cdots n_r| d_1d_2 \cdots d_r|.\]

**Proof.** The result follows from Theorem 4.1 at once. \(\square\)

**Example 4.5.** The symmetric matrix \(S\) of the 3-periodic link \(L^{(3)}\) with rational quotient \(L = \overrightarrow{C}[2, 1, -2]\) is given by
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\[
S = \begin{bmatrix}
-4 & 1 & 0 & 2 & 0 & 0 \\
1 & -2 & 1 & -1 & 1 & 0 \\
0 & 1 & 4 & 0 & -1 & -2 \\
2 & -1 & 0 & -4 & 1 & 0 \\
0 & 1 & -1 & 1 & -2 & 1 \\
0 & 0 & -2 & 0 & 1 & 4
\end{bmatrix}.
\]

We have that \( S = PDP^T \), where

\[
D = \text{diag}(-4, -2, 4, -\frac{3}{2}, -\frac{5}{2}, -\frac{3}{2}) \quad \text{and}
\]

\[
P = \begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}.
\]

This implies that \( \sigma(L(3)) = -2 = -2\sum_{i=1}^{3} \frac{n_i}{|n_i|} = 2\sigma(L(2)) \) (cf. Corollary 4.3) and \( \det(L(3)) = 432 \).

**Remarks 4.6.**

(1) In order to generalize Theorem 4.1 for the case \( p \geq 4 \), we need to diagonalize the symmetric matrix \( S = M + M^T \) in (4.2) of a \( p \)-periodic link with rational quotient \( \mathbb{C}[[n_1, n_2, \ldots, n_r]] \) so that the diagonal entries are completely expressed as the integer \( n_1, n_2, \ldots, n_r \). The authors have no such a diagonalization of \( S \) and so we leave this an open question.

It should be noted that if \( |n_in_{i+1}n_{i+2}n_{i+3}| = 1 \) for some \( i = 1, 2, \ldots, r-3 \), Corollary 4.3 may not hold. For example, if \( n_1 = n_2 = n_3 = n_4 = n_6 = n_8 = n_9 = 1 \) and \( n_5 = 2 \), then \( d_9 = 0 \).

(2) The signatures of more general periodic knots and links in \( S^3 \) have been studied by several authors, for example, see [2], [3], [4], [8], [9], [12].

(3) It is well known that the Alexander polynomial is given by the formula \( \Delta_K(t) = \det(M - tM^T) \). In [10], with Fox’s free differential calculus, Lee and Seo gave a recurrence formula for calculating the Alexander polynomials of 2-bridge links by using a special type of Conway diagram as shown in Figure 3 and the reduced Alexander polynomials of \( p \)-periodic links with rational quotient \( \mathbb{C}[[n_1, n_2, \ldots, n_r]] \) in terms of \( n_1, n_2, \ldots, n_r \) and \( p \).
References


