Integrability and $L^1$-convergence of Certain Cosine Sums

JATINDERDEEP KAUR
School of Mathematics and Computer Applications, Thapar Institute of Engg. and Tech., (Deemed University), Patiala(Pb.)-147004, India
e-mail: jatinkaur4u@yahoo.co.in

SATVINDER SINGH BHATIA
School of Mathematics and Computer Applications, Thapar Institute of Engg. and Tech., (Deemed University), Patiala(Pb.)-147004, India
e-mail: ssbhatia63@yahoo.com

Abstract. In this paper, we extend the result of Ram [3] and also study the $L^1$-convergence of the $r^{th}$ derivative of cosine series.

1. Introduction

Consider cosine series

\[ \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx. \]

Let the partial sum of (1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n \to \infty} S_n(x)$. Further, let $f'(x) = \lim_{n \to \infty} S'_n(x)$ where $S'_n(x)$ represents $r^{th}$ derivative of $S_n(x)$.

Definition ([6]). A null sequence $\{a_k\}$ is said to belong to class $S$ if there exists a sequence $\{A_k\}$ such that

(1.2) \[ A_k \downarrow 0, \; k \to \infty, \]

(1.3) \[ \sum_{k=0}^{\infty} A_k < \infty, \]

and

(1.4) \[ |\Delta a_k| \leq A_k, \; \forall \; k. \]

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\[
g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx.
\]

Ram [3] proved the following result:

**Theorem A.** If (1.1) belongs to class $S$. Then $\|f - g_n\|_{L^1} = o(1), \ n \to \infty$.

Recently, Tomovski [7] extended the Sidon class to a new class $S_r$, $r = 1, 2, 3, \ldots$ as follows:

**Definition.** A null sequence $\{a_k\}$ is said to belong to class $S_r$ if there exists a sequence $\{A_k\}$ such that

\[
A_k \downarrow 0, \ k \to \infty,
\]

\[
\sum_{k=0}^{\infty} k^r A_k < \infty,
\]

and

\[
|\Delta a_k| \leq A_k, \ \forall \ k.
\]

Clearly $S_{r+1} \subset S_r, \ \forall \ r = 1, 2, 3, \ldots$.

Note that by $A_k \downarrow 0, \ k \to \infty$ and $\sum_{k=0}^{\infty} k^r A_k < \infty$, we have

\[
k^{r+1} A_k = o(1), \ k \to \infty.
\]

For $r = 0$, this class reduces to class $S$.

The aim of this paper is to generalize Theorem A for the cosine series with extended class $S_r$, $r = 1, 2, 3, \ldots$ of coefficient sequences and also to study the $L^1$-convergence of the $r$th derivative of cosine series.

2. **Lemma**

The proofs of our results are based on the following lemmas:

**Lemma 2.1 ([2]).** If $|a_k| \leq 1$, then

\[
\int_0^{\pi} \left| \sum_{k=0}^{n} a_k D_k(x) \right| \ dx \leq C(n + 1),
\]

where $C$ is positive absolute constant.
Lemma 2.2. Let \(\{a_k\}\) be a sequence of real numbers such that \(|a_k| \leq 1, \forall k\). Then there exists a constant \(C > 0\) such that for any \(n \geq 0\) and \(r = 0, 1, 2, 3, \cdots\)

\[
\int_0^{\pi} \left| \sum_{k=0}^{n} a_k D^r_k(x) \right| \, dx \leq C(n + 1)^{r+1}.
\]

Proof. We note that \(\sum_{k=0}^{n} a_k D_k(x)\) is a cosine trigonometric polynomial of order \(n\).

Applying first Bernstein’s inequality ([8], vol. II, p. 11) and then using lemma 2.1, we have

\[
\int_0^{\pi} \left| \sum_{k=0}^{n} a_k D^r_k(x) \right| \, dx \leq (n + 1)^r \int_0^{\pi} \left| \sum_{k=0}^{n} a_k D_k(x) \right| \, dx \leq C(n + 1)^{r+1},
\]

where \(C > 0\).

Lemma 2.3 ([5]). \(|D^r_n(x)||_{L^1} = O(n^r \log n), r = 0, 1, 2, 3, \cdots, \) where \(D^r_n(x)\) represents the \(r^{th}\) derivative of Dirichlet-Kernel.

3. Results

Theorem 3.1. If (1.1) belongs to class \(S_r\), then \(||f - g_n||_{L^1} = o(1), n \to \infty||\).

Proof. Consider,

\[
g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx
\]

\[
= \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x)
\]

Thus, \(\lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} S_n(x) = f(x)\) (since, \(D_n(x)\) is bounded in \((0, \pi)\) and \(\{a_k\} \in S_r\)).

Now, we consider

\[
f(x) - g_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + a_{n+1} D_n(x)
\]
Making use of Abel’s transformation and lemma 2.1, we have
\[
\int_0^\pi |f(x) - g_n(x)| \, dx = \int_0^\pi \left| \sum_{k=n+1}^{\infty} \triangle a_k D_k(x) \right| \, dx
\]
\[
= \int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\triangle a_k}{A_k} D_k(x) \right| \, dx
\]
\[
\leq \int_0^\pi \left| \sum_{k=n+1}^{\infty} \triangle A_k \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} D_i(x) \right| \, dx
\]
\[
\leq \int_0^\pi \left| \sum_{k=n+1}^{\infty} k^r \triangle A_k \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} D_i(x) \right| \, dx
\]
\[
\leq \frac{1}{(n+1)^r} \int_0^\pi \left| \sum_{k=n+1}^{\infty} k^r \triangle A_k \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} D_i(x) \right| \, dx
\]
\[
\leq C \sum_{k=n+1}^{\infty} (k+1)^{r+1} \triangle A_k
\]
(1.6) and (1.7) now imply the conclusion of the Theorem 3.1.

**Corollary.** If (1.1) belongs to class $S_r$, $r = 1, 2, 3, \ldots$ then $||f - S_n||_{L^1} = o(1)$, $n \to \infty$ if and only if $a_n \log n = o(1)$, $n \to \infty$.

**Proof.** Consider,
\[
||f - S_n|| = ||f - g_n + g_n - S_n||
\]
\[
\leq ||f - g_n|| + ||g_n - S_n||
\]
\[
\leq ||f - g_n|| + ||a_{n+1}D_n(x)||
\]
\[
\leq ||f - g_n|| + ||a_{n+1}|| \int_0^\pi |D_n(x)| \, dx
\]
Further, $||f - g_n||_{L^1} = o(1)$, $n \to \infty$ (by Theorem 3.1) and $||D_n(x)|| = O(\log n)$ (by Zygmund’s Theorem ([1], p. 458)).

The conclusion of corollary follows.

**Remark.** Case $r = 0$ yields the result of B. Ram [3].

**Theorem 3.2.** If (1.1) belongs to class $S_r$, then $||f^r - g_n^r||_{L^1} = o(1)$, $n \to \infty$.

**Proof.** Consider,
\[
g_n(x) = S_n(x) - a_{n+1}D_n(x)
\]
We have then
\[
g_n^r(x) = S_n^r(x) - a_{n+1}D_n^r(x)
\]
Where $g_n^r(x)$ represents the $r$th derivative of $g_n(x)$ and $D_n^r(x)$ represents the $r$th derivative of Dirichlet kernel. Since $\{a_k\}$ is a null sequence and $D_n^r(x)$ is bounded in $(0, \pi)$.

Therefore,

$$\lim_{n \to \infty} g_n^r(x) = \lim_{n \to \infty} S_n^r(x) = f^r(x)$$

For $x \neq 0$, it follows from (3.1) that

$$f^r(x) - g_n^r(x) = \sum_{k=n+1}^{\infty} a_k k^r \cos \left( kx + \frac{r\pi}{2} \right) + a_{n+1} D_n^r(x)$$

Making use of Abel’s transformation, we get

$$f^r(x) - g_n^r(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x)$$

Now consider,

$$\int_0^\pi |f^r(x) - g_n^r(x)| \, dx = \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^r(x) \right| \, dx$$

$$= \int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k^r(x) \right| \, dx$$

$$\leq \int_0^\pi \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i^r(x) \, dx$$

$$\leq \int_0^\pi \sum_{k=n+1}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i^r(x) \, dx$$

$$\leq C \sum_{k=n+1}^{\infty} (k+1)^{r+1} \Delta A_k$$

(1.6), (1.7) and lemma 2.2 imply the conclusion of the Theorem (3.2). □

**Corollary.** If (1.1) belongs to class $S_r$, $r = 1, 2, 3, \cdots$ then $||f^r - S_n^r||_{L^1} = o(1)$, $n \to \infty$ if and only if $a_n n^r \log n = o(1)$, $n \to \infty$.

**Proof.** Consider,

$$||f^r - S_n^r|| = ||f^r - g_n^r + g_n^r - S_n^r||$$

$$\leq ||f^r - g_n^r|| + ||g_n^r - S_n^r||$$

$$= ||f^r - g_n^r|| + ||a_{n+1} D_n^r(x)||$$

$$\leq ||f^r - g_n^r|| + |a_{n+1}| \int_0^\pi |D_n^r(x)|$$
Since, \( \|f^r - g_n^r\|_{L^1} = o(1) \), \( n \to \infty \) (by Theorem 3.2) and using lemma 2.3, we get the conclusion of corollary. \( \square \)

References