On a Background of the Existence of Multi-variable Link Invariants

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Abstract. We present a quantum theoretical background of the existence of multi-variable link invariants, for example, the Kauffman polynomial, by observing the quantum \((sl(2, C), ad)-invariant\) from the Kontsevich invariant point of view. The background implies that the Kauffman polynomial can be studied by using the \(sl(N, C)\)-skein theory similar to the Jones polynomial and the HOMFLY polynomial.

1. Introduction

In 1980s–90s, many multi-variable link invariants had been successfully constructed, for example, the \(\Lambda\)-polynomial ([7]), the \(Q\)-polynomial ([1], [4]) and the Kauffman polynomial. Why was it possible? In this paper, we present a quantum theoretical background of the existence of the above multi-variable link invariants by observing the quantum \((sl(2, C), ad)-invariant\) from the Kontsevich invariant point of view.

According to [8], the quantum \((so(N), \rho_0)\)-invariant, where \(\rho_0\) is the fundamental representation of \(so(N)\), is a specialization of the Kauffman polynomial \(F(L; a, z)\) in the Laurent polynomial ring \(\mathbb{Z}[a, a^{-1}, z, z^{-1}]\), which is an invariant of unoriented unframed links defined by the following skein relations with the initial condition \(F(\bigcirc; a, z) = 1\):

\[
aF \left( \begin{array}{c} \bigcirc \\ a, z \end{array} \right) + a^{-1}F \left( \begin{array}{c} \bigcirc \\ a, z \end{array} \right) = z \left\{ F \left( \begin{array}{c} \bigcirc \\ a, z \end{array} \right) + F \left( \begin{array}{c} \bigcirc \\ a, z \end{array} \right) \right\}.
\]

In fact, we can show the equivalence of the weight systems for \((so(3), \rho_0)\) and \((sl(2, C), ad)\) by using the result in [2], and [8], where \(ad\) is the adjoint representation. Then it follows from an analytic ([8]) or a combinatorial observation ([3]) of the Kontsevich invariant that the quantum \((sl(2, C), ad)-invariant\) \(Q_{sl(2, C), ad}\) is also a specialization of the Kauffman polynomial as well as the the quantum \((so(N), \rho_0)\)-invariant:

**Theorem 1.1([3]).** The quantum \((sl(2, C), ad)-invariant\) is an unoriented framed link.

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invariant satisfying the following relations with the initial condition \( Q_{sl(2,\mathbb{C})}^{ad}(\bigcirc) = e^h + e^{-h} + 1 \):

\[
Q_{sl(2,\mathbb{C})}^{ad}(\bigotimes) - Q_{sl(2,\mathbb{C})}^{ad}(\bigotimes) = (e^h - e^{-h}) \left\{ Q_{sl(2,\mathbb{C})}^{ad}(\bigparallel) - Q_{sl(2,\mathbb{C})}^{ad}(\bigcirc) \right\} ,
\]

\[
Q_{sl(2,\mathbb{C})}^{ad}(\bigboxtimes) = e^{2h} Q_{sl(2,\mathbb{C})}^{ad}(\bigparallel) .
\]

Why can \( Q_{sl(2,\mathbb{C})}^{ad} \) be thought of as a specialization of the \( \Lambda \)-, \( Q \)- and the Kauffman polynomial? To see this, substitute \( z := e^h - e^{-h} \), \( a := e^{2h} \) to the relations in Theorem 1.1. Then we get the skein relations of the \( \Lambda \)-polynomial, except for the sign of the second terms of the both sides of the first relation. The \( \Lambda \)-polynomial induces the \( Q \)- and the Kauffman polynomial. (See [5], for example). In this sense, \( Q_{sl(2,\mathbb{C})}^{ad} \) can be regarded as a specialization of the Kauffman polynomial. Moreover, this process explains a quantum theoretical background of the existence of the above multi-variable link invariants. The process also implies a possibility that the Kauffman polynomial can be studied by using the \( sl(N,\mathbb{C}) \)-skein theory similar to the Jones polynomial and the HOMFLY polynomial. (With respect to the \( sl(N,\mathbb{C}) \)-skein theory, refer to [12]).

In this paper, we concentrate our interest on explaining what we observed in [3]. Namely, we show Theorem 1.1 in a combinatorial way using the Kontsevich invariant, which is different from the method in [8].

2. Key lemmas

To prove Theorem 1.1, we use the modified Kontsevich invariant \( \tilde{Z} \), the \((g, \rho)\)-weight system \( \tilde{W}_{g, \rho} \), its graded version \( \tilde{W}_{g, \rho}^{\text{gr}} \), quasi-tangles and Jacobi diagrams. There exists an excellent book [11] on these materials, so please refer to the book for details. The following theorem plays an important role in this paper:

**Theorem 2.1** (Kassel [6], Le and Murakami [9]). The quantum \((g, \rho)\)-invariant \( Q_{g, \rho} \) can be reconstructed by using the composition of the modified Kontsevich invariant \( \tilde{Z} \) with the \((g, \rho)\)-graded weight system \( \tilde{W}_{g, \rho}^{\text{gr}} \). Namely, \( Q_{g, \rho}(L)\big|_{q=e^h} = \tilde{W}_{g, \rho}^{\text{gr}}(\tilde{Z}(L)) \) for an arbitrary oriented framed link \( L \).

In the final section, we apply this theorem to a proof of Theorem 1.1. Before the application we first focus on the following three key lemmas to Theorem 1.1. For the sake of convenience, we often use the following notation:

\[
H := \bigotimes , \quad P := \bigotimes , \quad U := \bigboxtimes , \quad 1 := \bigparallel .
\]

where the above diagrams are Jacobi diagrams. We simply denote \( W_{sl(2,\mathbb{C})}^{ad} \) by \( W \).

**Lemma 2.1.** The \((sl(2,\mathbb{C}), \text{ad})\)-weight system \( W \) does not depend on the orientation of support (solid lines) of a Jacobi diagram and is formulated as follows:

1. \( W(\bigcirc) = 3 \),
2. \( W(D \sqcup D') = W(D) \cdot W(D') \), for any Jacobi diagrams \( D \) and \( D' \),
3. \( W(H) = 2W(P) - 2W(U) \).
Proof. The adjoint representation of $sl(2, \mathbb{C})$ is self-dual, which fact shows that $W$ does not depend on the orientation of support of a Jacobi diagram. (1) and (2) are trivial. (3) is a formula given by Chmutov and Varchenko in [2]. □

We remark that the first author generalized the formula (3) to a universal $sl(N, \mathbb{C})$-weight system via the Young symmetrizer in [10].

The formula (3) shows the equivalence of the weight system for $(so(3), \rho_0)$ and $(sl(2, \mathbb{C}), ad)$. (Refer to [8]). Although Theorem 1.1 basically follows from the fact, we explain concretely how to prove Theorem 1.1 in a combinatorial way using the Kontsevich invariant.

Lemma 2.2.

$$W((P - U)^n) = \frac{1}{2}(1 - (-1)^n)W(P) - \frac{1}{3}(1 - (-2)^n)W(U) + \frac{1}{2}(1 + (-1)^n)W(1)$$

Proof. This can be immediately shown by induction. □

Lemma 2.3. Let $\widehat{W}$ be the $(sl(2, \mathbb{C}), ad)$-graded weight system. Then there exists a non-constant element $\lambda = \lambda(h) \in \mathbb{C}[[h]]$ satisfying the following conditions:

$$\widehat{W} \circ \widehat{Z}(\bigcirc) = 3\lambda, \quad \widehat{W} \circ \widehat{Z} \left( \begin{array}{c} \bigcirc \\ \end{array} \right) = \lambda W(U),$$

where $\bigcirc$ is a quasi-tangle with an unspecified orientation and dots with its end points.

Proof. Note that the composition $\widehat{W} \circ \widehat{Z}$ does not depend on the orientation on a quasi-tangle, which property is derived from Lemma 2.1, but $\widehat{Z}$ does. Hence, at first we consider an oriented quasi-tangle for $\widehat{Z}$, then ignore the orientation later. In particular, dots with the end points of a quasi-tangle are not essential for $\widehat{W} \circ \widehat{Z}$, so we also ignore them later.

Let us first summarize the definition of the modified Kontsevich invariant $\widehat{Z}$ needed in this proof. For any monomial $w$ in the non-commutative variables $A$ and $B$, the degree of $w$ is defined by its length as a word in $A$ and $B$. Let $\varphi(A, B)$ be the formal power series in the variables $A$ and $B$ as follows:

$$\varphi(A, B) := 1 + \frac{1}{24}[A, B] - \frac{\zeta(3)}{(2\pi\sqrt{-1})^2}([A, [A, B]] + [B, [A, B]]) + (\text{terms in } A \text{ and } B \text{ with degree } \geq 4),$$

where $\zeta(z)$ is the zeta function. Let $\nu$ be the Jacobi diagram with support $\bigcirc$ as follows:

$$\nu := \left( \begin{array}{c} \varphi \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \\ \end{array} \right).$$

Then the modified Kontsevich invariant $\widehat{Z}(\bigcirc)$ is defined by

$$\widehat{Z}(\bigcirc) := \left( \begin{array}{c} \nu \bigcirc \\ \end{array} \right).$$
To get a concrete presentation of \( \nu \), let us take a closer look at \( \nu^{-1} \).

\[
\nu^{-1} = \begin{align*}
& = 1 + \frac{1}{24} \left( \begin{array}{c}
\text{a}
\text{b}
\text{c}
\text{d}
\end{array} \right) + \text{(Jacobi diagrams of degree } \geq 3) \\
& = 1 + \frac{1}{24} \left( \begin{array}{c}
\text{a}
\text{b}
\text{c}
\text{d}
\end{array} \right) + \text{(Jacobi diagrams of degree } \geq 3).
\end{align*}
\]

Let us put \( \nu := a_0 + a_1 \begin{array}{c}
\text{a}
\text{b}
\end{array} + a_2 \begin{array}{c}
\text{a}
\text{b}
\end{array} + a_3 \begin{array}{c}
\text{a}
\text{b}
\end{array} + \text{(Jacobi diagrams of degree } \geq 3) \).

For convenience, in the rest of proof, the part (Jacobi diagrams of degree \( \geq 3 \)) in the above power series is abbreviated to \( R \). Then the following equation holds:

\[
1 = \nu^{-1} \nu
\]

\[
= \left( 1 + \frac{1}{24} \left( \begin{array}{c}
\text{a}
\text{b}
\end{array} \right) + R \right) \left( a_0 + a_1 \begin{array}{c}
\text{a}
\text{b}
\end{array} + a_2 \begin{array}{c}
\text{a}
\text{b}
\end{array} + a_3 \begin{array}{c}
\text{a}
\text{b}
\end{array} + R \right)
\]

\[
= a_0 + a_1 \begin{array}{c}
\text{a}
\text{b}
\end{array} + \frac{a_0}{24} \left( \begin{array}{c}
\text{a}
\text{b}
\end{array} \right) + \text{(Jacobi diagrams of degree } \geq 3) + a_2 \begin{array}{c}
\text{a}
\text{b}
\end{array} + a_3 \begin{array}{c}
\text{a}
\text{b}
\end{array} + R
\]

So we get \( a_0 = 1, a_1 = 0, a_2 = -1/24, a_3 = 1/24 \). Then \( \nu \) has the following presentation:

\[
\nu = 1 - \frac{1}{24} \left( \begin{array}{c}
\text{a}
\text{b}
\end{array} \right) + R.
\]

We next focus on the equations below derived from Lemma 2.1,

\[
W \left( \begin{array}{c}
\text{a}
\text{b}
\end{array} \right) = 8W \left( \begin{array}{c}
\text{a}
\end{array} \right), \quad W \left( \begin{array}{c}
\text{a}
\end{array} \right) = W \left( \begin{array}{c}
\text{a}
\end{array} \right)^2 = 16W \left( \begin{array}{c}
\text{a}
\end{array} \right).
\]

Here we remark that the graded \( \mathfrak{sl}(2, \mathbb{C}) \text{-ad}-weight system } \overline{W}(D) \text{ of a Jacobi diagram } D \text{ is defined as } \hbar^{\text{deg}(D)} W(D), \text{ where } \text{deg}(D) \text{ is a half the number of uni and tri-valent vertices of the graph consisting of all the dashed edges in } D. \text{ Applying these formulas to}
the Jacobi diagram $\nu$, we get

$$
\hat{W}(\nu) = \hat{W}\left(1 - \frac{1}{24} \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} + R\right)\right)
= \left(1 + \frac{h^2}{3} + \text{terms of degree} \geq 3\right) W\left(\right).
$$

Note that the second equality of the above equations is derived from Schur's lemma. Let us put $\lambda = \lambda(h) := 1 + \frac{h^2}{3} + \text{terms of degree} \geq 3$. Then we have

$$
\hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) = \lambda^{1/2} W\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right).
$$

Moreover, we can get the same relation on $\hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right)$ as $\hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right)$, therefore we finally get the following results:

$$
\hat{W} \circ \hat{Z}(\bigcirc) = \hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) = \lambda^{1/2} W\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) = \lambda W\left(\bigcirc\right) = 3 \lambda,
$$

$$
\hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) = \hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) = \lambda^{1/2} W\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) = \lambda W\left(\bigotimes\right) = \lambda W(U).
$$

These complete the proof. □

3. Proof of Theorem

By Theorem 2.1, Lemma 2.1 and the definitions of the modified Kontsevich invariant and the weight system, we see that $Q_{sl(2,\mathbb{C}),ad}$ is an unoriented framed link invariant. Moreover, by Theorem 2.1, it suffices to show that

$$
\hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) - \hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) = (e^h - e^{-h}) \left\{\hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) - \hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) \right\},
$$

$$
\hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) = e^{2h} \hat{W} \circ \hat{Z}\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right),
$$

$$
\hat{W} \circ \hat{Z}(\bigcirc) = e^h + e^{-h} + 1,
$$

to prove Theorem 1.1. (Refer to the note at the beginning of the proof of Lemma 2.3.)

The third equation can be easily checked because

$$
\hat{W} \circ \hat{Z}(\bigcirc) = Q_{sl(2,\mathbb{C}),ad}(\bigcirc) = [3] = \frac{e^{3h} - e^{-3h}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}} = e^h + e^{-h} + 1,
$$

where $[n] = (e^{\frac{nh}{2}} - e^{-\frac{nh}{2}})/(e^{\frac{h}{2}} - e^{-\frac{h}{2}})$ is the quantum dimension. Then Lemma 2.3 shows that $\lambda = (e^h + e^{-h} + 1)/3$. 

Next, let us check the second equation. By the definition of $\tilde{Z}$,

$$\tilde{Z}\left(\begin{array}{c}
\vdots
\end{array}\right) = \exp\left(\frac{1}{2} \begin{array}{c}
\vdots
\end{array}\right).$$

Recalling the definition of $\tilde{W}$ mentioned in the proof of Lemma 2.3, we obtain the desired equation as follows:

$$\tilde{W} \circ \tilde{Z} \left(\begin{array}{c}
\vdots
\end{array}\right) = \tilde{W} \circ \tilde{Z} \left(\begin{array}{c}
\vdots
\end{array}\right) = \tilde{W} \left( \exp\left(\frac{1}{2} \begin{array}{c}
\vdots
\end{array}\right) \right) = \exp\left(\frac{1}{2} \tilde{W} \left(\begin{array}{c}
\vdots
\end{array}\right) \right)$$

$$= \exp\left(\frac{4h}{2W} \begin{array}{c}
\vdots
\end{array} \right) = e^{2h} \tilde{W} \circ \tilde{Z} \left(\begin{array}{c}
\vdots
\end{array}\right).$$

We finally focus on the first equation. By the definitions of $\tilde{W}$ and $\tilde{Z}$,

$$\tilde{W} \circ \tilde{Z} \left(\begin{array}{c}
\vdots
\end{array}\right) = \tilde{W} \circ \tilde{Z} \left(\begin{array}{c}
\vdots
\end{array}\right) = \tilde{W} \left( P \left( 1 + \frac{1}{2} H + \frac{1}{8} H^2 + \cdots + \frac{1}{n!} 2^n H^n + \cdots \right) \right)$$

$$= W \left( P \left( 1 + \frac{h}{2} H + \frac{h^2}{8} H^2 + \cdots + \frac{h^n}{n! 2^n} H^n + \cdots \right) \right)$$

$$= W \left( P e^{hH/2} \right).$$

By Lemmas 2.1 and 2.2, the following equation holds:

$$W \left( P e^{h(P-U)} \right) = W \left( P e^{h(P-U)} \right)$$

$$= W \left( P \sum \frac{h^n}{n!} (P-U)^n \right)$$

$$= W \left( P \sum \frac{h^n}{n!} \left\{ \frac{1}{2} (1-(1)^n)P - \frac{1}{3} (1-(2)^n)U + \frac{1}{2} (e^h + e^{-h}) \cdot 1 \right\} \right)$$

$$= W \left( \sum \frac{h^n}{n!} \left\{ \frac{1}{2} (1-(1)^n) \cdot 1 - \frac{1}{3} (1-(2)^n)U + \frac{1}{2} (1 + (1)^n)P \right\} \right)$$

$$= \frac{1}{2} (e^h - e^{-h}) W(1) - \frac{1}{3} (e^h - e^{-2h}) W(U) + \frac{1}{2} (e^h + e^{-h}) W(P).$$

Similarly, we can get the following relation:

$$\tilde{W} \circ \tilde{Z} \left(\begin{array}{c}
\vdots
\end{array}\right) = \tilde{W} \circ \tilde{Z} \left(\begin{array}{c}
\vdots
\end{array}\right)$$

$$= \frac{1}{2} (e^{-h} - e^h) W(1) - \frac{1}{3} (e^{-h} - e^{2h}) W(U) + \frac{1}{2} (e^{-h} + e^h) W(P).$$
Hence, by Lemma 2.3, we obtain the following equation:

\[
\tilde{W} \circ \tilde{Z} \left( \begin{array}{c} \times \\ \times \end{array} \right) - \tilde{W} \circ \tilde{Z} \left( \begin{array}{c} \times \\ \times \end{array} \right) = (e^h - e^{-h})W(1) - \frac{1}{3}(e^{2h} - e^{-2h} + e^h - e^{-h})W(U)
\]

\[
= (e^h - e^{-h}) \left\{ W(1) - \frac{1}{3}(e^h + e^{-h} + 1)W(U) \right\}
\]

\[
= (e^h - e^{-h}) \left\{ \tilde{W} \circ \tilde{Z!} \left( \begin{array}{c} | \\ \vert \end{array} \right) - \frac{1}{3\lambda}(e^h + e^{-h} + 1)\tilde{W} \circ \tilde{Z!} \left( \begin{array}{c} \bigcap \\ \bigcap \end{array} \right) \right\}.
\]

Recall that \( \lambda = (e^h + e^{-h} + 1)/3 \). Therefore this equation completes the proof of the first equation and Theorem 1.1.

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**References**


