Normal Families and Shared Values of Meromorphic Functions

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Abstract. Some criteria for determining the normality of the family $F$ of meromorphic functions in the unit disc, which share values depending on $f \in F$ with their derivatives is obtained. The new results in this paper improve some earlier related results given by Pang and Zalcman [3], Fang and Zalcman [2], A. P. Singh and A. Singh [5].

1. Introduction, definitions and main results

Let $f$ and $g$ be meromorphic functions on a domain $D$ in $C$, and let $a$ and $b$ be complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write

$$f(z) = a \Rightarrow g(z) = b.$$  

In a different notation, we have $E_f(a) \subset E_g(b)$, where $E_h(c) = h^{-1}(c) \cap D = \{ z \in D : h(z) = c \}$. If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write

$$f(z) = a \Leftrightarrow g(z) = b.$$  

If $f(z) = a \Leftrightarrow g(z) = a$, we say that $f$ and $g$ share $a$ on $D$.

Schwick is probably the first to find a connection between the normality criterion and shared values of meromorphic functions. He proved the following theorem.

**Theorem A([4]).** Let $F$ be a family of meromorphic functions in the unit disc $\Delta$, and let $a_1$, $a_2$, $a_3$ be distinct complex numbers. If $f$ and $f'$ share $a_1$, $a_2$ and $a_3$ for every $f \in F$, then $F$ is normal in $\Delta$.

Pang and Zalcman extended the above result as follows.

**Theorem B([3]).** Let $F$ be a family of meromorphic functions in the unit disc $\Delta$ and let $a$ and $b$ be distinct complex numbers and $c$ be a nonzero complex number. If for every $f \in F$, $f(z) = 0 \Leftrightarrow f'(z) = a$ and $f(z) = c \Leftrightarrow f'(z) = b$, then $F$ is normal in $\Delta$.

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In 2001, Fang and Zalcman proved the following result.

**Theorem C([2]).** Let $F$ be a family of meromorphic functions in the unit disc $\Delta$ and let $b$, $c$ and $d$ be nonzero complex numbers such that $d \neq b$. Suppose that for each $f \in F$, $f(z) = 0 \Rightarrow f'(z) = b$ and $f'(z) = d \Rightarrow f(z) = c$. Then $F$ is normal in $\Delta$ so long as $b \neq (m + 1)d$, $m = 1, 2, 3, \cdots$.

In Theorem B and Theorem C the constants are the same for each $f \in F$. In 2004, A. P. Singh and A. Singh proved that the condition for the constants to be the same can be relaxed to some extent. More precisely, they proved the following theorem.

**Theorem D([5]).** Let $F$ be a family of meromorphic functions in the unit disc $\Delta$. For each $f \in F$ let $a_f$, $b_f$, $c_f$ be distinct nonzero complex numbers such that $\left(\frac{a_f b_f}{c_f^2}\right) = M$ for some constant $M$. Let the spherical distance $\sigma$ between the points $a_f$, $b_f$, $c_f$ satisfy

$$\min\{\sigma(a_f, b_f), \sigma(b_f, c_f), \sigma(c_f, a_f)\} \geq m$$

for some $m > 0$. Let $f(z) = 0 \Leftrightarrow f'(z) = a_f$ and $f'(z) = c_f \Leftrightarrow f'(z) = b_f$. Let $M = (ab/c^2)$, where $a$, $b$, $c$ are distinct. If the elements of $E_f(c_f)$ and $E_f(0)$ are the only solutions of

$$f'(z) = \frac{a_f b_f}{a}(1 - \frac{1}{c_f} - \frac{a}{ca_f})f(z))^2$$

and

$$f'(z) = a_f(1 - \frac{1}{c_f} - \frac{a}{ca_f})f(z))^2$$

respectively, then $F$ is normal in $\Delta$.

Now the following problem is considered: Is it possible to relax the nature of sharing values in Theorem D? In this paper, we prove the following theorem which answers the above question.

**Theorem 1.** Let $F$ be a family of meromorphic functions in the unit disc $\Delta$. For each $f \in F$ let $a_f$, $b_f$, $c_f$ be distinct nonzero complex numbers such that $\left(\frac{a_f b_f}{c_f^2}\right) = M$ for some constant $M$. Let the spherical distance $\sigma$ between the points $a_f$, $b_f$, $c_f$ satisfy

$$\min\{\sigma(a_f, b_f), \sigma(b_f, c_f), \sigma(c_f, a_f)\} \geq m$$

for some $m > 0$. Let $f(z) = 0 \Rightarrow f'(z) = a_f$ and $f'(z) = b_f \Rightarrow f(z) = c_f$. Let $M = (ab/c^2)$, where $a$, $b$, $c$ are distinct and $a \neq (m + 1)b$, $m = 1, 2, 3, \cdots$. If the elements of $E_f(c_f)$ are the only solutions of

$$f'(z) = \frac{a_f b_f}{a}(1 - \frac{1}{c_f} - \frac{a}{ca_f})f(z))^2,$$
then \( F \) is normal in \( \Delta \).

**Remark.** Theorem 1 removes the restriction on \( E_f(0) \) of Theorem D and the nature of sharing values is relaxed.

If the meromorphic functions in \( F \) and their derivatives share \( a_f \) bounded by some constant \( M \), and \( \alpha a_f \) respectively, then we shall prove the following theorem.

**Theorem 2.** Let \( F \) be a family of meromorphic functions on the unit disc \( \Delta \). For each \( f \in F \) let there exist \( a_f (0 < |a_f| \leq M \) for some constant \( M \)) and \( \alpha a_f (\alpha \neq 1, 2, 3, \ldots \) is a constant) and \( \sigma_f(z) = a_f \Rightarrow f'(z) = a_f, f'(z) = \alpha a_f \Rightarrow f(z) = \alpha a_f \). Further let the elements of \( E_f(\alpha a_f) \) be the only solutions of

\[
\sigma_f'(z) = \frac{a_f b}{a^2}(1 - A_f(f(z) - a_f))^2,
\]

where \( A_f = (1/(\alpha - 1)a_f) - (a/\alpha a_f) \), \( a \) is any nonzero constant, \( b = \alpha a \) and \( c = (\alpha - 1)a \). Then \( F \) is normal in \( \Delta \).

**2. Some lemmas**

We need the following lemmas in the proof of Theorem 1 and Theorem 2.

**Lemma 1([1]).** The Mobius map \( g(z) = \frac{az + b}{cz + d}, ad - bc = 1 \) satisfies the Lipschitz condition

\[
\sigma(g(z), g(w)) \leq \frac{\pi}{2} ||g||^2 \sigma(z, w),
\]

where \( ||g||^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 \).

The proof is immediate since from [5]

\[
\sigma_0(g(z), g(w)) \leq ||g||^2 \sigma_0(z, w),
\]

where \( \sigma_0 \) is the spherical metric, and the spherical metric and chordal metric \( \sigma \) are related by

\[
\frac{2}{\pi} \sigma_0(z, w) \leq \sigma(z, w) \leq \sigma_0(z, w).
\]

**Lemma 2([1]).** Let \( m \) be any positive number. Then the Mobius transformation \( g \) which satisfies \( \sigma(g(a), g(b)) \geq m \), \( \sigma(g(b), g(c)) \geq m \), \( \sigma(g(c), g(a)) \geq m \) for some constant \( a, b \) and \( c \), also satisfies the uniform Lipschitz condition

\[
\sigma(g(z), g(w)) \leq k_m \sigma(z, w),
\]

where \( k_m \) is a constant depending on \( m \).
3. Proof of Theorem 1

For each $f \in F$, define a Mobius map $g_f$ by

$$g_f(z) = \frac{z}{Az + B},$$

where $A = \frac{1}{c_f} - \frac{a}{ca_f}$ and $B = \frac{a}{a_f}$. Then clearly we have

$$g_f^{-1}(z) = \frac{Bz}{1 - Az}$$

and

$$(g_f^{-1})'(z) = \frac{B}{(1 - Az)^2}$$

so that $g_f^{-1}(0) = 0$, $g_f^{-1}(c_f) = c$, $(g_f^{-1})'(0) = \frac{a}{a_f}$, $(g_f^{-1})'(c_f) = \frac{b}{b_f}$.

Now if $z_0$ is such that $f(z_0) = 0$ then since $f(z) = 0 \Rightarrow f'(z) = a_f$ we have $f'(z_0) = a_f$ and so $(g_f^{-1} \circ f)(z_0) = g_f^{-1}(0) = 0$ and $(g_f^{-1} \circ f)'(z_0) = (g_f^{-1})'(f(z_0))f'(z_0) = a$.

Now we show that $g_f^{-1} \circ f(z_0) = 0 \Rightarrow (g_f^{-1} \circ f(z))' = a$. Let $z_1$ is such that $(g_f^{-1} \circ f)(z_1) = 0$, then $(g_f^{-1} \circ f)(z_1) = g_f^{-1}(0)$. Also $g_f^{-1}$ being a Mobius map, is one-to-one so that $f(z_1) = 0$ and so $f'(z_1) = a_f$. Thus $(g_f^{-1} \circ f(z_1))' = a$.

Next we show that $(g_f^{-1} \circ f(z_2))' = b \Rightarrow g_f^{-1} \circ f(z) = c$. Let $z_2$ is such that $(g_f^{-1} \circ f(z_2))' = b$. Then $(g_f^{-1})'(f(z_2))f'(z_2) = b$ and so

$$f'(z_2) = \frac{a_f b}{a} \left(1 - \left(\frac{1}{c_f} - \frac{a}{ca_f}\right)f(z_2)\right)^2.$$ 

Since only the elements of $E_f(c_f)$ satisfy (1), it follows that $f(z_2) = c_f$ and so $(g_f^{-1} \circ f)(z_2) = c$. Thus $(g_f^{-1} \circ f(z))' = b \Rightarrow g_f^{-1} \circ f(z) = c$.

Thus by Theorem C, the family $G = \{(g_f^{-1} \circ f) : f \in F\}$ is normal and hence equicontinuous in $\Delta$. Therefore given $(\epsilon/k_m) > 0$, where $k_m$ is the constant of Lemma 2, there exist $\delta > 0$ such that for the spherical distance $\sigma(x, y) < \delta$,

$$\sigma((g_f^{-1} \circ f)(x), (g_f^{-1} \circ f)(y)) < \frac{\epsilon}{k_m}$$

for each $f \in F$. Hence by Lemma 2

$$\sigma(f(x), f(y)) = \sigma((g_f \circ g_f^{-1} \circ f)(x), (g_f \circ g_f^{-1} \circ f)(y))$$

$$\leq k_m \sigma((g_f^{-1} \circ f)(x), (g_f^{-1} \circ f)(y)) < \epsilon.$$
Thus the family $F$ is equicontinuous in $\Delta$. This completes the proof of Theorem 1.

4. Proof of Theorem 2

Let $b_f = \alpha a_f$ and $c_f = (\alpha - 1)a_f$ so that $a_f b_f / c_f^2$ is the constant $\alpha/(\alpha - 1)^2$. For each $f \in F$, define $g_f(z) = f(z) - a_f$. Let $G = \{g_f : f \in F\}$. Then clearly

$$g_f(z) = 0 \Rightarrow f(z) = a_f \Rightarrow f'(z) = a_f \Rightarrow g'_f(z) = a_f$$

and

$$g'_f(z) = b_f \Rightarrow f'(z) = b_f \Rightarrow f(z) = b_f \Rightarrow g_f(z) = c_f.$$  

From the assumption of Theorem 2, we get that the elements of $E_{g_f}(c_f)$ are the only solutions of

$$g'_f(z) = \frac{a_f b_f}{a} (1 - A_f g_f(z))^2.$$  

Hence by Theorem 1, $G$ is normal in $\Delta$, and hence given $\epsilon > 0$, there exists $\delta > 0$ for all $x, y$ such that $\sigma(x, y) < \delta$, then we have $\sigma(g_f(x), g_f(y)) < \epsilon$ for every $f \in F$. Now define $R_f(z) = z + a_f$, then each $R_f(z)$ is a Mobius map and $R_f(g_f(z)) = f(z)$. Hence by Lemma 1

$$\sigma(f(x), f(y)) = \sigma(R_f(g_f(x)), R_f(g_f(y))) \leq \|R_f\| \frac{\pi}{2} \sigma(g_f(x), g_f(y)) \leq (2 + M^2) \frac{\pi}{2} \epsilon.$$  

Hence $F$ is normal in $\Delta$. This completes the proof of Theorem 2.

References