On the Ideal Extensions in $\Gamma$-Semigroups

Manoj Siripitukdet and Aiyared Iampan
Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Abstract. In 1981, Sen [4] have introduced the concept of $\Gamma$-semigroups. We have known that $\Gamma$-semigroups are a generalization of semigroups. In this paper, we introduce the concepts of the extensions of $s$-prime ideals, prime ideals, $s$-semiprime ideals and semiprime ideals in $\Gamma$-semigroups and characterize the relationship between the extensions of ideals and some congruences in $\Gamma$-semigroups.

1. Preliminaries

Let $M$ and $\Gamma$ be any two nonempty sets. $M$ is called a $\Gamma$-semigroup [5], [7] if for all $a, b, c \in M$ and $\gamma, \mu \in \Gamma$, we have (i) $a\gamma b \in M$ and (ii) $(a\gamma b)\mu c = a\gamma (b\mu c)$. A $\Gamma$-semigroup $M$ is called a commutative $\Gamma$-semigroup if $a\gamma b = b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$. A nonempty subset $K$ of a $\Gamma$-semigroup $M$ is called a sub-$\Gamma$-semigroup of $M$ if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

For examples of $\Gamma$-semigroups, see [1], [3], [5], [6], [7].

Let $S$ be a semigroup and $\Gamma = \{1\}$. We define a mapping $S \times \Gamma \times S \rightarrow S$ by $a_1 b = ab$ for all $a, b \in S$. Then $S$ is a $\Gamma$-semigroup. Hence we have known that $\Gamma$-semigroups are a generalization of semigroups.

For nonempty subsets $A$ and $B$ of a $\Gamma$-semigroup $M$ and a nonempty subset $\Gamma'$ of $\Gamma$, let $A\Gamma' B := \{a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $\{a\}\Gamma' B$ as $a\Gamma' B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$. A nonempty subset $I$ of a $\Gamma$-semigroup $M$ is called an ideal of $M$ if $MTI \subseteq I$ and $I\Gamma M \subseteq I$. The intersection of all ideals of a $\Gamma$-semigroup $M$ containing a nonempty subset $A$ of $M$ is the ideal of $M$ generated by $A$, and will be denoted by $I(A)$. If $A = \{x\}$, then we also write $I(\{x\})$ as $I(x)$. An ideal $I$ of a $\Gamma$-semigroup $M$ is called an $s$-prime ideal [3] of $M$ if for any $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b \in I$ implies $a \in I$ or $b \in I$. Equivalently, for any $A, B \subseteq M$ and $\gamma \in \Gamma$, $A\gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. An ideal $I$ of a $\Gamma$-semigroup $M$ is called a prime ideal of $M$ if for any $a, b \in M$, $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in I$. Equivalently, for any $A, B \subseteq M$, $A\Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. 

Received June 30, 2006, and, in revised form, January 15, 2008.

2000 Mathematics Subject Classification: 20M99, 06B10.

Key words and phrases: $\Gamma$-semigroup, extension of $s$-prime ideal, prime ideal, $s$-semiprime ideal and semiprime ideal.

585
An ideal $I$ of a $\Gamma$-semigroup $M$ is called a $s$-semiprime ideal of $M$ if for any $a \in M$ and $\gamma \in \Gamma$, $a \gamma a \in I$ implies $a \in I$. Equivalently, for any $A \subseteq M$ and $\gamma \in \Gamma$, $A \gamma A \subseteq I$ implies $A \subseteq I$. An ideal $I$ of a $\Gamma$-semigroup $M$ is called a semiprime ideal of $M$ if for any $a \in M$, $a \Gamma a \subseteq I$ implies $a \in I$. Equivalently, for any $A \subseteq M$, $A \Gamma A \subseteq I$ implies $A \subseteq I$. Hence we have the following statements for $\Gamma$-semigroups.

1. Every $s$-prime ideal is a prime ideal.
2. Every prime ideal is a semiprime ideal.
3. Every $s$-prime ideal is an $s$-semiprime ideal.
4. Every $s$-semiprime ideal is a semiprime ideal.

For a $\Gamma$-semigroup $M$, let

$$
P(M) := \{A: A \text{ is a prime ideal of } M\},
$$

$$
SP(M) := \{A: A \text{ is an } s\text{-prime ideal of } M\}.
$$

Then $\emptyset \neq SP(M) \subseteq P(M)$. A sub-$\Gamma$-semigroup $F$ of a $\Gamma$-semigroup $M$ is called a filter [3] of $M$ if for any $a, b \in M$ and $\gamma \in \Gamma, a \gamma b \in F$ implies $a, b \in F$. The intersection of all filters of a $\Gamma$-semigroup $M$ containing a nonempty subset $A$ of $M$ is the filter of $M$ generated by $A$. For $A = \{x\}$, let $\mathcal{I}(x)$ denote the filter of $M$ generated by $\{x\}$. An equivalence relation $\sigma$ on a $\Gamma$-semigroup $M$ is called a congruence [2], [6] if for any $a, b, c \in M$ and $\gamma \in \Gamma, (a, b) \in \sigma$ implies $(a \gamma c, b \gamma c) \in \sigma$ and $(c \gamma a, c \gamma b) \in \sigma$. Let $\sigma$ be a congruence on a $\Gamma$-semigroup $M$ and $M/\sigma := \{(x)_\sigma : x \in M\}$. We define $(x)_\sigma (y)_\sigma = (x \gamma y)_\sigma$ for all $(x)_\sigma, (y)_\sigma \in M/\sigma$ and $\gamma \in \Gamma$. It is easy to verify that the definition is well-defined and $M/\sigma$ is a $\Gamma$-semigroup. A congruence $\sigma$ on a $\Gamma$-semigroup $M$ is called a semilattice congruence [8] if for all $a, b, c \in M$ and $\gamma \in \Gamma, (a \gamma b, b \gamma a) \in \sigma$ and $(a \gamma a, a) \in \sigma$. For an ideal $I$ of a $\Gamma$-semigroup $M$ and $A \subseteq M$, the set $< A, I > := \{x \in M : A \Gamma x \subseteq I\}$ is called the extension of $I$ by $A$. If $A = \{a\}$, then we also write $< \{a\}, I >$ as $< a, I >$. For an ideal $I$ of a $\Gamma$-semigroup $M$, we define equivalence relations on $M$ as follows:

$$
\sigma_I := \{(x, y) \in M \times M : x, y \in I \text{ or } x, y \notin I\},
$$

$$
\phi_I := \{(x, y) \in M \times M : < x, I > = < y, I >\},
$$

$$
n := \{(x, y) \in M \times M : n(x) = n(y)\}.
$$

Example 1.[[3]] Let $M = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ with the multiplication defined by

$$
x \gamma y = \begin{cases} b & \text{if } x, y \in \{a, b\}, \\ c & \text{otherwise}. \end{cases}
$$

Then $M$ is a $\Gamma$-semigroup. We can easily get all ideals of $M$ as follows:

$$
P_1 = M, P_2 = \{c, d\}, P_3 = \{b, c\}, P_4 = \{c\}, P_5 = \{a, b, c\}, P_6 = \{b, c, d\}.
$$

It is easy to see that $P_1$ and $P_2$ are $s$-prime ideals of $M$, so $P_1$ and $P_2$ are semiprime ideals of $M$. Let

$$
\sigma_1 = M \times M,
$$

$$
\sigma_2 = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\}.
$$
It is easy to see that $\sigma_1$ and $\sigma_2$ are semilattice congruences on $M$.

**Example 2.** For $n \in \{1, 2\}$, let $M = \{n, n + 1, n + 2, \cdots\}$ and $\Gamma = \{-n\}$. Then $M$ is a $\Gamma$-semigroup under usual addition. Let $I = \{2n, 2n + 1, 2n + 2, \cdots\}$. It is easy to verify that $I$ is a semiprime ideal of $M$ and $\sigma = \{(n, n)\}$ is a semilattice congruence on $M$.

The following theorem is obtained similarly in [3] and the following lemmas will be used frequently in this paper.

**Theorem 1.1.** If $M$ is a $\Gamma$-semigroup, then

$$n = \bigcap_{I \in SP(M)} \sigma_I.$$

In this paper, we consider the ideal extensions in a commutative $\Gamma$-semigroup. From now on, $M$ stands for a commutative $\Gamma$-semigroup. The next two lemmas are easy to verify.

**Lemma 1.2.** If $A$ is a subset of $M$, then $I(A) = A \cup MA$.

**Lemma 1.3.** Let $I$ be an ideal of $M$ and $A \subseteq B \subseteq M$. Then $< B, I > \subseteq < A, I >$.

**Lemma 1.4.** Let $I$ be an ideal of $M$, $A \subseteq M$ and $\gamma \in \Gamma$. Then we have the following statements:

(a) $< A, I >$ is an ideal of $M$.

(b) $I \subseteq< A, I > \subseteq< A\Gamma A, I > \subseteq< A\gamma A, I >$.

(c) If $A \subseteq I$, then $< A, I > = M$.

**Proof.**

(a) Let $x \in< A, I >, y \in M$ and $\gamma \in \Gamma$. Then $A\gamma y = (A\Gamma y) \subseteq I \Gamma M \subseteq I$, so $x\gamma y \in< A, I >$. Hence $< A, I >$ is an ideal of $M$.

(b) If $x \in I$, then $A\Gamma x \subseteq M\Gamma I \subseteq I$. Thus $x \in< A, I >$. If $x \in< A, I >$, then $(A\Gamma A)\Gamma x = A\Gamma (A\Gamma x) \subseteq M\Gamma I \subseteq I$. Thus $x \in< A\Gamma A, I >$. If $x \in< A\Gamma A, I >$, then $(A\gamma A)\Gamma x \subseteq (A\Gamma A)\Gamma x \subseteq I$. Thus $x \in< A\Gamma A, I >$. Hence $I \subseteq< A, I > \subseteq< A\Gamma A, I > \subseteq< A\gamma A, I >$.

(c) Let $A \subseteq I$ and $x \in M$. Then $A\Gamma x \subseteq I\Gamma M \subseteq I$, so $x \in< A, I >$. Hence $< A, I > = M$. \qed

**Lemma 1.5.** Let $I$ be an ideal of $M$ and $A \subseteq M$. Then

$$< A, I > = \bigcap_{a \in A} < a, I > =< A \setminus I, I >.$$

**Proof.** By Lemma 1.3, we have $< A, I > \subseteq \bigcap_{a \in A} < a, I >$. Let $x \in \bigcap_{a \in A} < a, I >$. Then $a\Gamma x \subseteq I$ for all $a \in A$, so $A\Gamma x \subseteq I$. Thus $x \in< A, I >$, so $\bigcap_{a \in A} < a, I > \subseteq< A\setminus I, I >$. \qed
Manoj Siripitukdet and Aiyared Iampan

\[ A, I >. \] Hence \(< A, I > = \bigcap_{a \in A} < a, I >. \] By Lemma 1.4 (c), we have \(< A, I > = \bigcap_{a \in A} < a, I > = < A \setminus I, I >. \]

\[ A, I >. \] By Lemma 1.4 (c), we have \(< A, I > = \bigcap_{a \in A} < a, I > = < A \setminus I, I >. \]

\[ < A, I > = < A \setminus I, I >. \]

\[ \square \]

**Lemma 1.6.** Let \( I \) be an ideal of \( M \). Then \( I \) is a prime ideal of \( M \) if and only if \(< A, I > = I \) for all \( A \not\subseteq I \).

**Proof.** Assume that \( I \) is a prime ideal of \( M \) and \( A \not\subseteq I \). Let \( x \in < A, I >. \) Then \( A^\Gamma x \subseteq I. \) By hypothesis and \( A \not\subseteq I \), \( x \in I \). Thus \(< A, I > \subseteq I. \) By Lemma 1.4 (b), \(< A, I > = I \).

Conversely, assume that \(< A, I > = I \) for all \( A \not\subseteq I \). Let \( A, B \subseteq M \) be such that \( A^\Gamma B \subseteq I \) and \( A \not\subseteq I \). Then \( B \subseteq < A, I > = I. \) Hence \( I \) is a prime ideal of \( M \).

\[ \square \]

We can easily prove the last lemma.

**Lemma 1.7.** Let \( A \) and \( B \) be two nonempty subfamilies of \( P(M) \) and \( SP(M) \), respectively. Then we have the following statements:

(a) \( \bigcap_{P \in A} P \) is a semiprime ideal of \( M \) if \( \bigcap_{P \in A} P \not= \emptyset \).

(b) \( \bigcup_{P \in B} P \) is a prime ideal of \( M \).

(c) \( \bigcap_{P \in B} P \) is an \( s \)-semiprime ideal of \( M \) if \( \bigcap_{P \in B} P \not= \emptyset \).

(d) \( \bigcup_{P \in B} P \) is an \( s \)-prime ideal of \( M \).

2. **Main theorems**

In this section, we give some characterizations of the relationship between the extensions of ideals and some congruences in \( \Gamma \)-semigroups.

**Theorem 2.1.** Let \( P \) be a prime ideal of \( M \) and \( A \subseteq M \). Then \(< A, P > \) is a prime ideal of \( M \). Furthermore, \(< A, \bigcap_{P \in P(M)} P > \) is a semiprime ideal of \( M \) if \( \bigcap_{P \in P(M)} P \not= \emptyset \).

**Proof.** If \( A \subseteq P \), then it follows from Lemma 1.4 (c) that \(< A, P > = M \). If \( A \not\subseteq P \), then it follows from Lemma 1.6 that \(< A, P > = P \). Hence \(< A, P > \) is a prime
ideal of $M$. Now,

\[ x \in< A, \bigcap_{P \in P(M)} P > \quad \Leftrightarrow \quad A \Gamma x \subseteq \bigcap_{P \in P(M)} P \]
\[ \quad \Leftrightarrow \quad A \Gamma x \subseteq P \text{ for all } P \in P(M) \]
\[ \quad \Leftrightarrow \quad x \in< A, P > \text{ for all } P \in P(M) \]
\[ \quad \Leftrightarrow \quad x \in \bigcap_{P \in P(M)} < A, P > . \]

Hence

\[ < A, \bigcap_{P \in P(M)} P > = \bigcap_{P \in P(M)} < A, P > . \]

It follows from Lemma 1.7 (a) that $< A, \bigcap_{P \in P(M)} P >$ is a semiprime ideal of $M$. □

Theorem 2.2. Let $A, B \subseteq M$ and $A \subseteq M \Gamma A$. Then $I(A) \subseteq I(B)$ if and only if for every ideal $J$ of $M$, $< B, J > \subseteq< A, J >$.

Proof. Assume that $I(A) \subseteq I(B)$. Let $J$ be an ideal of $M$ and $x \in< B, J >$. By Lemma 1.2, we have $A \subseteq I(B) = B \cup M \Gamma B$. For any $a \in A$, if $a = yab$ for some $y \in M, b \in B$ and $\alpha \in \Gamma$, then $a \gamma x = (yab) \gamma x = y\alpha(b \gamma x) \in M \Gamma J \subseteq J$ for all $\gamma \in \Gamma$. Hence $a \gamma x \in J$ for all $\gamma \in \Gamma$, so $x \in< a, J >$. If $a = b$ for some $b \in B$, then $a \gamma x = b \gamma x \in J$ for all $\gamma \in \Gamma$. Hence $a \gamma x \in J$ for all $\gamma \in \Gamma$, so $x \in< a, J >$. Therefore

\[ < B, J > \subseteq \bigcap_{a \in A} < a, J > . \]

It follows from Lemma 1.5 that $< B, J > \subseteq< A, J >$.

Conversely, assume that $< B, J > \subseteq< A, J >$ for all ideal $J$ of $M$. Then $< B, I(B) > \subseteq< A, I(B) >$. Since $B \subseteq I(B)$, it follows from Lemma 1.4 (c) that $< B, I(B) > = M$. Thus $< A, I(B) > = M$, so $M \Gamma A \subseteq I(B)$. Hence $A \subseteq M \Gamma A \subseteq I(B)$. This implies that $I(A) \subseteq I(B)$. □

Theorem 2.3. If $I$ is an $s$-semiprime ideal of $M$, then $\phi_I$ is a semilattice congruence on $M$.

Proof. Let $(x, y) \in \phi_I, c \in M$ and $\gamma \in \Gamma$. Then $< x, I > =< y, I >$. Thus

\[ a \in< x \gamma c, I > \quad \Leftrightarrow \quad (x \gamma c) \Gamma a \subseteq I \]
\[ \quad \Leftrightarrow \quad x \Gamma (c \gamma a) \subseteq I \]
\[ \quad \Leftrightarrow \quad c \gamma a \in< x, I > \]
\[ \quad \Leftrightarrow \quad c \gamma a \in< y, I > \]
\[ \quad \Leftrightarrow \quad y \Gamma (c \gamma a) \subseteq I \]
\[ \quad \Leftrightarrow \quad (y \gamma c) \Gamma a \subseteq I \]
\[ \quad \Leftrightarrow \quad a \in< y \gamma c, I > . \]
Hence \((x\gamma c, y\gamma c) \in \phi_I\). Similarly, we can show that \((c\gamma x, c\gamma y) \in \phi_I\). Hence \(\phi_I\) is a congruence on \(M\). Let \(x \in M\) and \(\gamma \in \Gamma\). Then
\[
\begin{align*}
 a \in < x\gamma x, I > & \Rightarrow (x\gamma x)\Gamma a \subseteq I \\
 & \Rightarrow (x\gamma x)\Gamma a \subseteq I \Gamma M \subseteq I \\
 & \Rightarrow (x\Gamma a)\gamma (x\Gamma a) \subseteq I \\
 & \Rightarrow x\Gamma a \subseteq I \\
 & \Rightarrow a \in < x, I > .
\end{align*}
\]
Thus \(< x\gamma x, I > \subseteq < x, I >\). By Lemma 1.4 \((b)\), \(< x, I >\subseteq < x\gamma x, I >\). Hence \(< x\gamma x, I > = < x, I >\), so \((x\gamma x, x) \in \phi_I\). Therefore \(\phi_I\) is a semilattice congruence on \(M\). \(\square\)

**Theorem 2.4.** If \(I\) is an \(s\)-prime ideal of \(M\), then \(\phi_I = \sigma_I\) and \(n \subseteq \phi_I\).

**Proof.** Let \((x, y) \in \phi_I\). Then \(< x, I > = < y, I >\). Suppose that \((x, y) \notin \sigma_I\). Without loss of generality, we may assume that \(x \in I\) but \(y \notin I\). By Lemma 1.4 \((c)\) and Lemma 1.6, we have \(< x, I > = M\) and \(< y, I > = I\). Thus \(I = M\), so \(y \notin M\). This is a contradiction. Hence \((x, y) \in \sigma_I\), so \(\phi_I \subseteq \sigma_I\). Let \((x, y) \in \sigma_I\). If \(x \in I\), then \(y \in I\). By Lemma 1.4 \((c)\), \(< x, I > = M = < y, I >\). If \(x \notin I\), then \(y \notin I\). By Lemma 1.6, \(< x, I > = \phi_I = < y, I >\). Hence \((x, y) \in \phi_I\), so \(\sigma_I \subseteq \phi_I\). Therefore \(\phi_I = \sigma_I\). It follows from Theorem 1.1 that
\[
n = \bigcap_{J \in SP(M)} \sigma_J = \bigcap_{J \in SP(M)} \phi_J \subseteq \phi_I .
\]
Hence the proof is completed. \(\square\)

**Acknowledgment.** The authors would like to thank the referees for the useful and helpful suggestions.

**References**


