On Sufficient Conditions for Certain Subclass of Analytic Functions Defined by Convolution

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Abstract. In the present investigation sufficient conditions are found for certain subclass of normalized analytic functions defined by Hadamard product. Differential sandwich theorems are also obtained. As a special case of this we obtain results involving Ruscheweyh derivative, Sălăgean derivative, Carlson-Shaffer operator, Dziok-Srivatsava linear operator, Multiplier transformation.

1. Introduction

Let \( A \) denote the class of analytic functions of the form

\[
(1.1) \quad f(z) := z + \sum_{n=2}^{\infty} a_n z^n.
\]

For two functions \( f(z) \) defined as in (1.1) and \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) the Hadamard product or convolution of \( f(z) \) and \( g(z) \), denoted by \((f * g)(z)\), is defined by

\[
(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.
\]

For \( \alpha_j \in \mathbb{C}, (j = 1, 2, \cdots, l) \) and \( \beta_j \in \mathbb{C} \setminus \{0, -1, -2, -3, \cdots\}, (j = 1, 2, \cdots, m) \), the Dziok-Srivatsava linear operator [7] for functions in \( A \) is defined as follows:

\[
H_{l,m}^{\alpha}(\alpha_1) f(z) := z + \sum_{n=2}^{\infty} \Gamma_n a_n z^n,
\]

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where

\[(1.2)\quad \Gamma_n := \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}(1)_{n-1}},\]

where \((\lambda)_n\) is the Pochhammer symbol defined by

\[(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1) & (n = 1, 2, 3, \cdots). \end{cases} \]

On defining \(g(z) = z + \sum_{n=2}^{\infty} \Gamma_n z^n\), we see that \((f * g)(z) = H_{10}^m(a_1)f(z)\).

By taking \(l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1\) and \(\beta_1 = c\) we see that

\[(f * g)(z) = H_{21}^1(a)f(z) = L(a, c)f(z),\]

where \(L(a, c)f(z)\) denotes the Carlson-Shaffer linear operator [5].

On choosing \(\frac{z}{1 - z} (1 - z)^{m+1} (\lambda > -1), z + \sum_{n=2}^{\infty} n^m a_n z^n + z + \sum_{n=2}^{\infty} \left(\frac{n - \lambda}{1 + \lambda}\right)^m z^n\)
as \(g(z)\), we find \((f * g)(z) = D^\lambda f(z), D^m f(z)\) and \(I(r, \lambda)f(z)\) respectively, where \(D^\lambda, D^m,\) and \(I(m, \lambda)\) denotes Ruscheweyh derivative of order \(\lambda\), Sălăgean derivative of order \(m\) and Multiplier transformation.

Let \(H\) denotes the class of all analytic functions defined on the open unit disk \(\Delta := \{z \in \mathbb{C} : |z| < 1\}\) and \(H[a, n]\) be the subclass of \(H\) consisting of functions of the form \(f(z) = a + a_1 z + a_{n+1} z^{n+1} + \cdots\). For two analytic functions \(f\) and \(F\), we say \(F\) is superordinate to \(f\), if \(f\) is subordinate to \(F\). Let \(p, h \in H\) and let \(\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}\). If \(p\) and \(\phi(p(z), z^p(z), z^2 p''(z) ; z)\) are univalent and if \(p\) satisfies the second order superordination

\[(1.3)\quad h(z) \prec \phi(p(z), z^p(z), z^2 p''(z); z),\]

then \(p\) is the solution of the differential superordination (1.3). An analytic function \(g(z)\) is called subordinant, if \(g(z) \prec p(z)\) for all \(p(z)\) satisfying (1.3). A univalent subordinant \(g(z)\) that satisfies \(g(z) \prec q(z)\) for all subordinants \(q(z)\) of (1.3), is said to be best subordinant. Recently Miller and Mocanu [3] considered certain first and second order differential superordinations. Using the results of Miller and Mocanu [3], Bulboacă have considered certain classes of first order differential superordinations [2] as well as superordination preserving integral operators [1].

In the present investigation we obtain the sufficient conditions for normalized analytic functions \(f(z)\) to satisfy

\[q_1(z) \prec \frac{z^2(f * g)'(z)}{[(f * g)(z)]^2} \prec q_2(z),\]

where \(g(z)\) is the fixed analytic function in \(A\).
2. Preliminaries

For the present study we may need the following definitions and results.

**Definition 2.1** ([3, Definition 2, p.817]). Denote by $Q$, the set of all functions $f(z)$ that are analytic and univalent in $\Delta \setminus E(f)$, where

$$E(f) := \{ \zeta \in \partial \Delta : \lim_{z \to \zeta} f(z) = \infty \}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \Delta \setminus E(f)$.

**Theorem 2.1** (cf. Miller and Mocanu [4, Theorem 3.4h, p.132]). Let $q(z)$ be univalent in $\Delta$ and $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(\Delta)$ with $\phi(w) \neq 0$, when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(i) $Q(z)$ is starlike univalent in $\Delta$ and

(ii) $\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0$ for $z \in \Delta$.

If $p$ is analytic in $\Delta$ with $p(\Delta) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z))$$

then

$$p(z) \prec q(z)$$

and $q(z)$ is the best dominant.

**Theorem 2.2** ([2]). Let $q(z)$ be univalent in $\Delta$ and $\theta$ and $\phi$ be analytic in domain $D$ containing $q(\Delta)$. Suppose that

(i) $\Re \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} \geq 0$ for $z \in \Delta$ and

(ii) $Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in $\Delta$.

If $p \in H[q(0), 1] \cap Q$ with $p(\Delta) \subseteq D$ and $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\Delta$, and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)),$$

then

$$q(z) \prec p(z)$$

and $q(z)$ is the best subordinant.

3. Main results

Throughout this paper we assume that $\alpha, \beta, \gamma$ and $\delta$ are complex numbers and $\delta \neq 0$.

**Theorem 3.1.** Let $q(z)$ be a convex univalent in $\Delta$ with $q(0) = 1$. Assume that

$$\Re \left\{ \frac{\beta q(z) + 2\gamma q^2(z)}{\delta} - \frac{zq'(z)}{q(z)} \right\} > 0.$$
Let

\[
\psi(z) := \alpha + \beta z^2 \frac{(f \ast g)'(z)}{([f \ast g](z))^2} + \gamma \left( \frac{z^2(f \ast g)'(z)}{([f \ast g](z))^2} \right)^2 + \delta \left[ \frac{(z(f \ast g)(z))^\prime}{(f \ast g)(z)} - \frac{2z(f \ast g)'(z)}{(f \ast g)(z)} \right].
\]

If \( f \in A \) and

\[
\psi(z) \prec \alpha + \beta q(z) + \gamma q^2(z) + \delta \frac{zq'(z)}{q(z)},
\]

then

\[
\frac{z^2(f \ast g)'(z)}{([f \ast g](z))^2} \prec q(z)
\]

and \( q(z) \) is the best dominant.

Proof. Define the functions \( p(z) \) by

\[
p(z) := \frac{z^2(f \ast g)'(z)}{([f \ast g](z))^2}.
\]

Then clearly \( p(z) \) is analytic in \( \Delta \). Also by a simple computation, we find from (3.4) that

\[
\frac{zp'(z)}{p(z)} = \frac{(z(f \ast g)(z))''}{(f \ast g)(z)'} - \frac{2z(f \ast g)'(z)}{(f \ast g)(z)}.
\]

Also we find that

\[
p(z) := \alpha + \beta z^2 \frac{(f \ast g)'(z)}{([f \ast g](z))^2} + \gamma \left( \frac{z^2(f \ast g)'(z)}{([f \ast g](z))^2} \right)^2 + \delta \left[ \frac{(z(f \ast g)(z))^\prime}{(f \ast g)(z)} - \frac{2z(f \ast g)'(z)}{(f \ast g)(z)} \right] = \alpha + \beta p(z) + \gamma p^2(z) + \delta \frac{zp'(z)}{p(z)}.
\]

In view of (3.5) the subordination (3.3) becomes

\[
\alpha + \beta p(z) + \gamma p^2(z) + \delta \frac{zp'(z)}{p(z)} \prec \alpha + \beta q(z) + \gamma q^2(z) + \delta \frac{zq'(z)}{q(z)}
\]

and this can be rewritten as (2.1), where \( \theta(w) := \alpha + \beta w + \gamma w^2 \) and \( \phi(w) = \frac{\delta}{w} \).

Note that \( \theta(w) \) and \( \phi(w) \) are analytic in \( \mathbb{C} \setminus \{0\} \). Since \( \delta \neq 0 \), we have \( \phi(w) \neq 0 \).

Let the functions \( Q(z) \) and \( h(z) \) defined as

\[
Q(z) := zq'(z)\phi(q(z)) = \delta \frac{zq'(z)}{q(z)}
\]
On Sufficient Conditions

\[ h(z) := \alpha + \beta q(z) + \gamma q^2(z) + \delta \frac{zq'(z)}{q(z)}. \]

In light of hypothesis of Theorem 2.1, we see that \( Q(z) \) is starlike and

\[ \Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ \frac{\beta q(z) + 2\gamma q^2(z) - \frac{zq'(z)}{\delta} + (1 + \frac{zq''(z)}{q'(z)})}{Q(z)} \right\}. \]

Hence the result follows as an application of Theorem 2.1.

By taking \( \alpha = \beta = \gamma = 0 \) and \( \delta = 1 \) in Theorem we get the following result of Ravichandran et al.[10].

**Corollary 3.2.** If \( f(z) \in A \) and

\[ \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} = \frac{zq'(z)}{q(z)}, \]

then

\[ \frac{z^2 f'(z)}{f^2(z)} < q(z). \]

**Theorem 3.3.** Let \( q(z) \) be convex univalent in \( \Delta \) with \( q(0) = 1 \) and satisfies

\[ (3.6) \quad \Re \left\{ \frac{\beta q(z) + 2\gamma q(z)^2}{\delta} \right\} > 0. \]

If \( f \in A, 0 \neq \frac{z^2(f * g)'(z)}{[(f * g)(z)]^2} \in \mathcal{H}[1, 1] \cap \mathcal{Q} \) and \( \psi(z) \) as defined by (3.2) is univalent in \( \Delta \), then

\[ (3.7) \quad \alpha + \beta q(z) + \gamma q^2(z) + \delta \frac{zq'(z)}{q(z)} \prec \psi(z) \]

implies

\[ q(z) < \frac{z^2(f * g)'(z)}{[(f * g)(z)]^2} \]

and \( q(z) \) is best subordinant.

**Proof.** In view of (3.5) the superordination (3.7) becomes

\[ \alpha + \beta q(z) + \gamma q^2(z) + \delta \frac{zq'(z)}{q(z)} \prec \alpha + \beta p(z) + \gamma p^2(z) + \delta \frac{zp'(z)}{p(z)} \]

and this can be written as (2.2), where \( \theta(w) = \alpha + \beta w + \gamma w^2 \) and \( \phi(w) = \frac{\delta}{w} \). Note that \( \theta(w) \) and \( \phi(w) \) are analytic in \( \mathbb{C} \setminus \{0\} \). In light of hypothesis of Theorem 2.2, we see that

\[ \Re \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} = \Re \left\{ \frac{\beta q(z) + 2\gamma q(z)^2}{\delta} \right\}. \]
Hence the result follows as an application of Theorem 2.2. □

By combining Theorem 3.1 and Theorem 3.3 we get the following sandwich result.

**Theorem 3.4.** Let $q_1(z)$ and $q_2(z)$ be convex univalent functions defined on $\Delta$ with $q_1(0) = q_2(0) = 1$ where $q_1(z)$ satisfies (3.6) and $q_2(z)$ satisfies (3.1). Let $f \in \mathcal{A}$ and $0 \neq \frac{(f*g)(z)}{|(f*g)(z)|^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ and $\psi(z)$ as defined by (3.2) is univalent in $\Delta$, then

$$\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta \frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \frac{z^2(f*g)'(z)}{[(f*g)(z)]^2} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.

By taking $g(z) = z + \sum_{n=2}^{\infty} \Gamma_n z^n$ in Theorem 3.4, where $\Gamma_n$ is as defined in (1.2), we get the following result involving Dziok-Srivatsava operator.

**Corollary 3.5.** Let $q_1(z)$ and $q_2(z)$ be convex univalent functions defined on $\Delta$ with $q_1(0) = q_2(0) = 1$ where $q_1(z)$ satisfies (3.6) and $q_2(z)$ satisfies (3.1). Let $f \in \mathcal{A}$ and $0 \neq \frac{z[H^{l,m}(\alpha_1)f(z)]''}{|H^{l,m}(\alpha_1)f(z)|^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ and

$$\psi(z) = \alpha + \beta \frac{z^2[H^{l,m}(\alpha_1)f(z)]''}{|H^{l,m}(\alpha_1)f(z)|^2} + \gamma \left[ \frac{z^2[H^{l,m}(\alpha_1)f(z)]''}{|H^{l,m}(\alpha_1)f(z)|^2} \right]^2 + \delta \left[ \frac{(zH^{l,m}(\alpha_1)f(z))''}{(H^{l,m}(\alpha_1)f(z))''} - \frac{2z(H^{l,m}(\alpha_1)f(z))''}{H^{l,m}(\alpha_1)f(z)} \right]$$

is univalent in $\Delta$ then

$$\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta \frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \frac{z^2[H^{l,m}(\alpha_1)f(z)]''}{|H^{l,m}(\alpha_1)f(z)|^2} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.

By taking $l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1$ and $\beta_1 = \epsilon$ in Corollary 3.5 we get the following result involving Carlson-Shaffer linear operator.

**Corollary 3.6.** Let $q_1(z)$ and $q_2(z)$ be convex univalent functions defined on $\Delta$
with \( q_1(0) = q_2(0) = 1 \), where \( q_1(z) \) satisfies (3.6) and \( q_2(z) \) satisfies (3.1). Let \( f \in \mathcal{A} \) and \( 0 \neq \frac{z^2(L(a,c)f(z))'}{(L(a,c)f(z))^2} \in \mathcal{H}[1,1] \cap Q \) and

\[
\psi(z) := \alpha + \beta\frac{z^2(L(a,c)f(z))'}{(L(a,c)f(z))^2} + \gamma \left[ \frac{z^2(L(a,c)f(z))'}{(L(a,c)f(z))^2} \right]^2 + \delta \left[ \frac{(zL(a,c)f(z))''}{(L(a,c)f(z))'} - \frac{2z(L(a,c)f(z))'}{L(a,c)f(z)} \right]
\]

is univalent in \( \Delta \) then

\[
\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta \frac{zq_2'(z)}{q_2(z)}
\]

implies

\[
q_1(z) \prec \frac{z^2(L(a,c)f(z))'}{(L(a,c)f(z))^2} \prec q_2(z),
\]

where \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and best dominant.

By fixing \( g(z) = \frac{z}{1-z} \), in Theorem 3.4 we get the following result involving Ruscheweyh derivative.

**Corollary 3.7** Let \( q_1(z) \) and \( q_2(z) \) be convex univalent functions defined on \( \Delta \) with \( q_1(0) = q_2(0) = 1 \) where \( q_1(z) \) satisfies (3.6) and \( q_2(z) \) satisfies (3.1). Let \( f \in \mathcal{A} \) and \( 0 \neq \frac{z^2(D^\lambda f(z))'}{(D^\lambda f(z))^2} \in \mathcal{H}[1,1] \cap Q \) and

\[
\psi(z) := \alpha + \beta\frac{z^2(D^\lambda f(z))'}{(D^\lambda f(z))^2} + \gamma \left[ \frac{z^2(D^\lambda f(z))'}{(D^\lambda f(z))^2} \right]^2 + \delta \left[ \frac{(zD^\lambda f(z))''}{(D^\lambda f(z))'} - \frac{2z(D^\lambda f(z))'}{D^\lambda f(z)} \right]
\]

is univalent in \( \Delta \), then

\[
\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta \frac{zq_2'(z)}{q_2(z)}
\]

implies

\[
q_1(z) \prec \frac{z^2(D^\lambda f(z))'}{(D^\lambda f(z))^2} \prec q_2(z),
\]

where \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and best dominant.

By fixing \( g(z) = z + \sum_{n=2}^{\infty} \left( \frac{\lambda+\alpha}{\lambda+\beta} \right)^m z^n \), we get the following result involving Multiplier transformation.

**Corollary 3.8**. Let \( q_1(z) \) and \( q_2(z) \) be convex univalent functions defined on \( \Delta \)
with \( q_1(0) = q_2(0) = 1 \), where \( q_1(z) \) satisfies (3.6) and \( q_2(z) \) satisfies (3.1). Let \( f \in 
abla \) and \( 0 \neq \frac{z^2(I(m, \lambda)f(z))'}{[I(m, \lambda)f(z)]^2} \in \mathcal{H}[1, 1] \cap \mathbb{Q} \) and

\[
\psi(z) := \alpha + \beta \frac{z^2(I(m, \lambda)f(z))'}{[I(m, \lambda)f(z)]^2} + \gamma \left[ \frac{z^2(I(m, \lambda)f(z))'}{[I(m, \lambda)f(z)]^2} \right]^2 + \delta \left[ \frac{(zI(m, \lambda)f(z))''}{I(m, \lambda)f(z)} - \frac{2z(I(m, \lambda)f(z))'}{I(m, \lambda)f(z)} \right]
\]

is univalent in \( \Delta \), then

\[
\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta \frac{zq_1'(z)}{q_1(z)} < \psi(z) < \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta \frac{zq_2'(z)}{q_2(z)}
\]

implies

\[
q_1(z) < \frac{z^2(I(m, \lambda)f(z))'}{[I(m, \lambda)f(z)]^2} \prec q_2(z),
\]

where \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and best dominant.

By taking \( \lambda = 0 \) in the Corollary 3.8 we get the following result involving Sălăgean derivative.

**Corollary 3.9.** Let \( q_1(z) \) and \( q_2(z) \) be convex univalent functions defined on \( \Delta \) with \( q_1(0) = q_2(0) = 1 \), where \( q_1(z) \) satisfies (3.6) and \( q_2(z) \) satisfies (3.1). Let \( f \in 
abla \) and \( 0 \neq \frac{z^2D^{m+1}f(z)}{[D^mf(z)]^2} \in \mathcal{H}[1, 1] \cap \mathbb{Q} \) and

\[
\psi(z) := \alpha + \beta \frac{z^2D^{m+1}f(z)}{[D^mf(z)]^2} + \gamma \left[ \frac{z^2D^{m+1}f(z)}{[D^mf(z)]^2} \right]^2 + \delta \left[ \frac{z(zD^mf(z))''}{D^{m+1}f(z)} - \frac{2D^{m+1}f(z)}{D^mf(z)} \right]
\]

is univalent in \( \Delta \), then

\[
\alpha + \beta q_1(z) + \gamma q_1^2(z) + \delta \frac{zq_1'(z)}{q_1(z)} < \psi(z) < \alpha + \beta q_2(z) + \gamma q_2^2(z) + \delta \frac{zq_2'(z)}{q_2(z)}
\]

implies

\[
q_1(z) < \frac{zD^{m+1}f(z)}{[D^mf(z)]^2} \prec q_2(z),
\]

where \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and best dominant.

By taking \( g(z) = \frac{z}{1-z} \), \( \alpha = 0 \), \( \beta = 1 \) and \( \gamma = 0 \) we get the following result.

**Corollary 3.10.** Let \( q_1(z) \) and \( q_2(z) \) be convex univalent functions defined on \( \Delta \) with \( q_1(0) = q_2(0) = 1 \) where \( q_1(z) \) satisfies

\[
\Re \left\{ \frac{q_1(z)}{\delta} \right\} > 0
\]
and $q_2(z)$ satisfies
\[
\Re \left\{ \frac{q_2(z)}{\delta} - \frac{z q_2'(z)}{q_2(z)} \right\} > 0.
\]

Let $f \in \mathcal{A}$ and $0 \neq \frac{z^2 f'(z)}{(f(z))^2} \in H[1, 1] \cap \mathcal{Q}$ and
\[
\psi(z) := \frac{z^2 f'(z)}{(f(z))^2} + \delta \left[ \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right]
\]
is univalent in $\Delta$ then
\[
q_1(z) + \delta \frac{z q_1'(z)}{q_1(z)} \prec \psi(z) \prec q_2(z) + \delta \frac{z q_2'(z)}{q_2(z)}
\]
implies
\[
q_1(z) \prec \frac{z^2 f'(z)}{(f(z))^2} \prec q_2(z),
\]
where $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.

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