On Approximation by Post-Widder and Stancu Operators Preserving $x^2$

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Abstract. In the papers [5]-[7] was examined approximation of functions by the modified Szász-Mrakyan operators and other positive linear operators preserving $e_2(x) = x^2$. In this paper we introduce the Post-Widder and Stancu operators preserving $x^2$ in polynomial weighted spaces. We show that these operators have better approximation properties than classical Post-Widder and Stancu operators.

1. Introduction

1.1. The Post-Widder operators

\begin{align*}
(1) \quad P_n(f; x) &\equiv P_n(f(t); x) := \int_0^\infty f(t) p_n(x, t) dt, \quad x \in I, \ n \in N, \\
(2) \quad p_n(x, t) &:= \left(\frac{n/x}{n+1}\right)^n t^{n-1} \left\{ -\frac{nt}{x} \right\}, \\
I &= (0, \infty), \ N = \{1, 2, \cdots\}, \text{ were examined in many papers and monographs (e.g. [4]) for real-valued functions } f \text{ bounded on } I. \text{ It is known ([4], Chapter 9) that } P_n \text{ are well defined also for functions } e_k(x) = x^k, \ k \in N_0 = N \cup \{0\}, \text{ and} \\
(3) \quad P_n(e_0; x) &= 1, \quad P_n(e_1; x) = x, \quad P_n(e_2; x) = \frac{n+1}{n} x^2 \\
\text{and generally} \quad (4) \quad P_n(e_k; x) &= \frac{n(n+1) \cdots (n+k-1)}{n^k} x^k, \quad k \in N,
\end{align*}

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for $x \in I$ and $n \in N$. Denoting by
\begin{equation}
\varphi_x(t) := t - x \quad \text{for } t \in I \text{ and a fixed } x \in I,
\end{equation}
we have
\begin{equation}
P_n \left( \varphi_x^2(t); x \right) = \frac{x^2}{n} \quad \text{for } x \in I, \ n \in N.
\end{equation}

From the results given in [4], Chapter 9, we can deduce that for every function $f$ continuous and bounded on $I$ there holds
\begin{equation}
\left| P_n(f; x) - f(x) \right| \leq M \omega \left( f; \frac{x}{\sqrt{n}} \right), \quad x \in I, \ n \in N,
\end{equation}
where $\omega(f; \cdot)$ is the modulus of continuity of $f$ and $M = \text{const.} > 0$ independent on $x$ and $n$.

1.2. The Stancu operators

\begin{equation}
L_n(f; x) \equiv L_n(f(t); x) := \int_0^\infty f(t) s_n(x, t) dt, \quad x \in I, \ n \in N,
\end{equation}
where
\begin{equation}
s_n(x, t) := \frac{t^{n-1}}{B(nx, n+1) (1+t)^{nx+n+1}},
\end{equation}
with the Euler beta function
\begin{equation}
B(a, b) := \int_0^1 t^{a-1}(1-t)^{b-1} dt \equiv \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt, \quad a, b > 0,
\end{equation}
were introduced in [10] for real-valued functions $f$ bounded and locally integrable on $I = (0, \infty)$. The Stancu operators $L_n$ are also well defined for functions $e_k(x) = x^k$, $k \in N_0$, (see [10], [1], [2]) and
\begin{equation}
L_n(e_0; x) = 1, \quad L_n(e_1; x) = x, \quad \text{for } n \in N,
L_n(e_2; x) = x^2 + \frac{x(x+1)}{n-1} \quad \text{for } n \geq 2,
\end{equation}
and generally
\begin{equation}
L_n(e_k; x) = \frac{nx(nx+1)\cdots(nx+k-1)}{n(n-1)\cdots(n-k+1)}, \quad x \in I, \ n \geq k \geq 2.
\end{equation}

In [10] was proved that for every function $f$ continuous and bounded on $I$ there holds the following inequality
\begin{equation}
\left| L_n(f; x) - f(x) \right| \leq \left( 1 + \sqrt{x(x+1)} \right) \omega \left( f; \frac{1}{\sqrt{n-1}} \right)
\end{equation}
for $x \in I$ and $n \geq 2$, where $\omega(f; \cdot)$ is the modulus of continuity of $f$.

1.3. In papers [8] and [9] were examined approximation properties certain modified Post-Widder and Stancu operators for differentiable functions in polynomial weighted spaces. In [5] were investigated modified Szász-Mirakyan operators $D^*_n$ preserving the function $e_2(x) = x^2$ and was proved that these operators have better approximation properties than classical Szász-Mirakyan operators. The similar results were given for certain positive linear operators in the papers [6] and [7].

1.4. The purpose of this note is to investigate modified Post-Widder and Stancu operators $P^*_n$ and $L^*_n$ preserving $e_2(x) = x^2$ in polynomial weighted spaces. These operators have better approximation properties than $P_n$ and $L_n$ given by (1) and (8). The definition and some properties of operators $P^*_n$ and $L^*_n$ will be given in Section 2. The main theorems will be given in Section 3.

1.5. First we give definition of polynomial weighted space $C_r$.

Similarly to [3] let $r \in N_0$,

$$w_0(x) := 1, \quad w_r(x) := (1 + x^r)^{-1} \quad \text{if} \quad r \geq 1, \quad x \in I,$$

and let $C_r \equiv C_r(I)$ be the set of all real-valued functions $f$ defined on $I$, for which $w_r f$ is uniformly continuous and bounded on $I$ and the norm is given by

$$\|f\|_r := \|f(\cdot)\|_r := \sup_{x \in I} w_r(x) |f(x)|.$$

It is obvious that if $q < r$, then $C_q \subset C_r$ and $\|f\|_r \leq \|f\|_q$ for $f \in C_q$. For $f \in C_r$, $r \in N_0$, we shall consider the modulus of continuity

$$\omega(f; C_r; t) := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_r, \quad t \geq 0,$$

where $\Delta_h f(x) = f(x + h) - f(x)$.

In this paper we shall apply the following inequalities

$$(w_r(x))^2 \leq w_{2r}(x), \quad (w_r(x))^{-2} \leq 4(w_{2r}(x))^{-1},$$

for $x \in I$ and $r \in N_0$, which immediately result from (13).

We shall denote by $M_i(r)$, $i \in N$, suitable positive constants depending only on indicated parameter $r$.

2. The definition and elementary properties of $P^*_n$ and $L^*_n$

2.1. We introduce for $f \in C_r$, $r \in N_0$, the following modified Post-Widder operators $P^*_n$

$$P^*_n(f; x) := \int_0^\infty f(t) p_n(u_n(x), t) dt = P_n(f; u_n(x)), \quad x \in I, \quad n \in N,$$
where $P_n(f)$ and $p_n$ are given by (1) and (2) and

\[ u_n(x) := \sqrt{\frac{n}{n+1} x}, \]

and modified Stancu operators

\[ L_n^*(f; x) := \int_0^\infty f(t) s_n(v_n(x), t) dt = L_n(f; v_n(x)) \]

for $x \in I$ and $n \geq r \geq 2$ or $n \geq 2$ if $r = 0, 1$, where $L_n(f)$ and $s_n$ are given by (8) and (9) and

\[ v_n(x) := -\frac{1 + \sqrt{1 + 4n(n-1)x^2}}{2n}. \]

2.2. The formulas (18) and (20) imply that

\[ 0 < u_n(x) < x, \quad 0 \leq v_n(x) \leq x \] for $x \in I$, $n \in N$.

From (17)-(20) and (1)-(4) and (8)-(11) we immediately obtain the following

**Lemma 1.** Let $e_k(x) = x^k$ for $k \in N_0$ and $x \in I$. Then for all $x \in I$ and $n \in N$ we have

\[ P_n^*(e_0; x) = 1, \quad P_n^*(e_1; x) = u_n(x), \quad P_n^*(e_2; x) = x^2 \]

and

\[ P_n^*(e_k; x) = \frac{n(n+1) \cdots (n+k-1)u_n^k(x)}{n^k} \quad \text{if } k \geq 3. \]

Moreover, for $x \in I$ and $n \geq 2$ we have

\[ L_n^*(e_0; x) = 1, \quad L_n^*(e_1; x) = v_n(x), \quad L_n^*(e_2; x) = x^2 \]

and generally

\[ L_n^*(e_k; x) = \frac{n u_n(x)(nu_n(x) + 1) \cdots (nu_n(x) + k - 1)}{n(n-1) \cdots (n-k+1)} \quad \text{for } n \geq k \geq 2. \]

The formulas (22) and (23) show that $P_n^*$ and $L_n^*$ preserve the functions $e_0$ and $e_2$.

**Lemma 2.** For function $\varphi_x$ given by (5) there hold the following analogies of (6):

\[ P_n^*(\varphi_x^2(t); x) = 2x(x - u_n(x)) \leq \frac{x^2}{n} \quad \text{for } x \in I, \quad n \in N, \]

and

\[ L_n^*(\varphi_x^2(t); x) = 2x(x - v_n(x)) \leq \frac{x(x+1)}{n-1} \quad \text{for } x \in I, \quad n \geq 2. \]
Proof. We shall prove only (25) because the proof of (24) is analogous. By linearity of $L^*$ and (5) and (23) we have

$$L_n^*(\varphi^2(t); x) = L_n^*(e_2; x) - 2x L_n^*(e_1; x) + x^2 L_n^*(e_0; x)$$

$$= 2x(x - v_n(x)) \quad \text{for } x > 0, \ n \geq 2.$$ 

Next, by (20) we get

$$0 < x - v_n(x) = \frac{2nx + 1 - \sqrt{1 + 4n(n-1)x^2}}{2n} \leq \frac{2x(x+1)}{2nx + 1 + 2(n-1)x} \leq \frac{x + 1}{2(n-1)x} \quad \text{for } x > 0, \ n \geq 2.$$ 

This completes the proof of (25). \hfill \Box

Lemma 3. Let $r \in N_0$ and let $w_r$ be the weighted function given by (13). Then for $n \in N$ the following inequalities

(26) $\|P_n^*(1/w_r)\|_r \leq 1, \quad \|L_n^*(1/w_r)\|_r \leq 1 \quad \text{if } r = 0, 1,$

(27) $\|P_n^*(1/w_r)\|_r \leq 1 + (r-1)!, \quad \text{if } r \geq 2,$

and

(28) $\|L_n^*(1/w_r)\|_r \leq 1 + 2^{r-1}(1 + r^{r-1}) \quad \text{for } n \geq r \geq 2,$

hold. Moreover, for every $f \in C_r$ we have

(29) $\|P_n^*(f)\|_r \leq \|f\|_r \|P_n^*(1/w_r)\|_r, \quad n \in N,$

(30) $\|L_n^*(f)\|_r \leq \|f\|_r \|L_n^*(1/w_r)\|_r, \quad n \geq r.$

The formulas (17)-(20) and inequalities (29) and (30) show that $P_n^*$, $n \in N$, and $L_n^*$ with $n \geq r$ are positive linear operators acting from the space $C_r$ to $C_r$, $r \in N_0$.

Proof. Similarly to Lemma 2 we shall consider only operators $L_n^*$. The inequality (26) is obvious by (13), (23), (21) and (14). If $r \geq 2$, then by linearity of $L_n^*$ and
(13), Lemma 1 and (21) we get

\[ L^*_n(1/w^*_r; x) = L^*_n(e_0; x) + L^*_n(e_r; x) \]

\[ \leq 1 + \frac{nx(nx + 1) \cdots (nx + r - 1)}{n(n - 1) \cdots (n - r + 1)} \]

\[ \leq 1 + \frac{n^{-1}x(x + 1/n) \cdots (x + (r - 1)/n)}{(n - 1)(n - 2) \cdots (n - r + 1)} \]

\[ \leq 1 + \frac{2^{r-1}((n-r+1)^{r-1} + r^{r-1})(x+1)^r}{(n-r+1)^{r-1}} \]

\[ \leq 1 + 2^{2r-1}(1 + r^{-1})(1 + x^r) \]

for \( x \in I \) and \( n \geq r \). This inequality and (14) imply (28).

The inequality (30) immediately follows from (19) and (14).

Applying the Hölder inequality and Lemma 2, Lemma 3 and (16), we easily obtain the following

**Lemma 4.** Let \( r \in N_0 \) and let \( \varphi_x \) be given by (5). Then there exist \( M_i(r) = \text{const.} > 0 \), \( i = 1, 2 \), such that for \( x \in I \) and \( n \in N \)

\[ w^*_r(x)P^*_n \left( \frac{\varphi(t)}{w^*_r(t)}; x \right) \leq M_1(r) \sqrt{2x(x - u_n(x))} \]

and

\[ w^*_r(x)L^*_n \left( \frac{\varphi(t)}{w^*_r(t)}; x \right) \leq M_2(r) \sqrt{2x(x - v_n(x))}, \ \text{for} \ n \geq 2r. \]

### 3. Theorems

**3.1.** Denote by \( C^1_r \equiv C^1_r(I) \), with a fixed \( r \in N_0 \), the set of all functions \( f \in C_r \) which the first derivative belonging also to \( C_r \).

**Theorem 1.** Let \( r \in N_0 \). Then there exist \( M_i(r) = \text{const.} > 0 \), \( i = 3, 4 \), such that for every \( f \in C^1_r \), \( x \in I \) and \( n \in N \) the following inequalities

\[ w^*_r(x)|P^*_n(f; x) - f(x)| \leq M_3(r) \| f' \|_{r} \sqrt{2x(x - u_n(x))} \]

and

\[ w^*_r(x)|L^*_n(f; x) - f(x)| \leq M_4(r) \| f' \|_{r} \sqrt{2x(x - v_n(x))}, \ \text{for} \ n \geq 2r, \]

hold.

**Proof.** From (17), (18) and Lemma 1 we deduce that

\[ |P^*_n(f(t); x) - f(x)| = |P^*_n(f(t) - f(x); x)| \leq P^*_n \left( \int_x^t f'(y)dy; x \right) \]
for every \( f \in C^1_r \), \( x \in I \) and \( n \in N \). Next by (13) and (14) we have
\[
\left| \int_x^t f'(y) dy \right| \leq \| f' \| \| \int_x^t \frac{dy}{w_r(y)} \| \leq \| f' \| r \left( \frac{1}{w_r(t)} + \frac{1}{w_r(x)} \right) |t - x|, \quad x, t \in I.
\]
Consequently, we get
\[
w_r(x)P_n^*(f(t); x) - f(x) \leq \| f' \| r \left\{ P_n^* \left( \frac{|\varphi_x(t)|}{w_r(t)}; x \right) + P_n^* \left( \frac{|\varphi_x(t)|}{w_0(t)}; x \right) \right\},
\]
for \( x \in I \), \( n \in N \), where \( \varphi_x \) is defined by (5). Now using (31), we obtain the desired estimation (33).

Similarly, applying (32), we obtain (34).

**Theorem 2.** Let \( r \in N_0 \). Then there exist \( M_i(r) = \text{const.} > 0 \), \( i = 5, 6, \) such that for every \( f \in C_r \), \( x \in I \) and \( n \in N \) we have
\[
w_r(x)P_n^*(f(t); x) - f(x) \leq M_5(r) \omega(f; C_r; \sqrt{2|x - u_n(x)|})
\]
and
\[
w_r(x)L_n^*(f; x) - f(x) \leq M_6(r) \omega(f; C_r; \sqrt{2|x - v_n(x)|}), \quad n \geq 2r,
\]
where \( \omega(f; C_r) \) is the modulus of continuity of \( f \) defined by (15).

**Proof.** Because the proofs of (35) and (36) are analogous, we shall prove only (35). We shall use the Steklov function \( f_h \) of \( f \in C_r \), i.e.
\[
f_h(x) := \frac{1}{h} \int_0^h f(x + t) dt, \quad x, h > 0.
\]
From (37) and (15) it follows that
\[
\| f_h - f \| r \leq \omega(f; C_r; h),
\]
\[
\| f_h \| r \leq h^{-1} \omega(f; C_r; h),
\]
for every \( f \in C_r \) and \( h > 0 \). These inequalities show that if \( f \in C_r \) with a fixed \( r \in N_0 \), then \( f_h \in C^1_r \) for every \( h > 0 \). Hence for \( f \in C_r \) and \( h > 0 \) we can write
\[
P_n^*(f(t); x) - f(x) = P_n^*(f(t) - f_h(t); x) + P_n^*(f_h(t); x) - f_h(x)
\]
\[
+ f_h(x) - f(x) \quad \text{for } x \in I, \quad n \in N.
\]
By (29), (26), (27) and (38) we see that there exists \( M_7(r) = \text{constant} > 0 \) such that
\[
w_r(x)P_n^*(f(t) - f_h(t); x) \leq M_7(r) \| f - f_h \| r
\]
\[
\leq M_7(r) \omega(f; C_r; h).
\]
Applying Theorem 1 for $f_h$ and (39), we get

\begin{equation}
|w_r(x)| P_n^*(f_h(t); x) - f_h(x)| \leq M_3(r) \|f_h\|_r \sqrt{2x(x - u_n(x))} \\
\leq M_3(r) h^{-1} \omega(f; C_r; h) \sqrt{2x(x - u_n(x))},
\end{equation}

using (41), (42) and (38), we deduce from (40)

\begin{equation}
|w_r(x)| P_n^*(f; x) - f(x)| \leq M_8(r) \omega(f; C_r; h) \times \left\{1 + h^{-1} \sqrt{2x(x - u_n(x))}\right\}
\end{equation}

for $x > 0$, $h > 0$ and $n \in N$. Now, for given $x$ and $n$ setting $h = \sqrt{2x(x - u_n(x))}$ to (43), we obtain desired inequality (35) and we complete the proof. □

From Theorem 2 and Lemma 2 results the following

**Corollary.** For every $f \in C_r$, $r \in N_0$, we have $\lim_{n \to \infty} P_n^*(f; x) = f(x)$, $x \in I$, and this convergence is uniform on every interval $[a, b]$, $a > 0$.

The above statement is also true for Stancu operators $L_n^*_r$.

### 3.2. Considering the Stancu operators $L_n$ in polynomial weighted spaces $C_r$ and using methods of proofs of Theorem 1 and Theorem 2, we can obtain the following estimation

\begin{equation}
|w_r(x)| L_n(f; x) - f(x)| \leq M_9(r) \omega(f; C_r; \sqrt{x(x+1) \over n-1}),
\end{equation}

for every $f \in C_r$, $r \in N_0$, $x > 0$ and $n \geq 2r + 2$.

The inequalities (44), (36) and (12) show that the Stancu operators $L_n^*$ have better approximation properties than $L_n$ for functions $f \in C_r$, $r \in N_0$, and $n \geq 2r + 2$. Moreover, by (20) and Lemma 2 we get for arguments of moduli of continuity of $f$ given in (36) and (44)

$$
\sqrt{\frac{x(x+1)}{n-1}} - \sqrt{2x(x - v_n(x))} = \frac{\sqrt{x(x+1)}}{n-1} - \frac{\sqrt{4x^2(x+1)}}{\sqrt{2nx + 1 + \sqrt{1 + 4n(n-1)x^2}}} = \frac{\sqrt{x(x+1)}}{n-1} \left(1 - \frac{\sqrt{4(n-1)x}}{\sqrt{2nx + 1 + \sqrt{1 + 4n(n-1)x^2}}}\right)
$$
On Approximation by Post-Widder and Stancu Operators Preserving $x^2$

\[
\sqrt{n-1} \frac{x(x+1)}{\sqrt{1+\frac{4n(n-1)x^2}{2n^2+1} - \frac{2(n-1)x + 2x + 1}{\sqrt{1+\frac{4n(n-1)x^2}{2n^2+1} + 4(n-1)x}}}} \\
\times \frac{1}{\sqrt{2nx+1 + \sqrt{1+4n(n-1)x^2 + 4(n-1)x}}} \\
> \sqrt{n-1} \frac{x(x+1)}{2nx+1 + \sqrt{4n^2+2 + \sqrt{4(n-1)x}}} \\
> \sqrt{n-1} \frac{x(x+1)}{2n^2+2} > \frac{x(x+1)}{4n+1},
\]

for all $x > 0$ and $n \geq 2r + 2$.

Analogously, estimations (7), (35) and (24) show that $P_n^*, n \in N$, have better approximation properties than $P_n$ for functions $f \in C_r$ (see [8]). Moreover, by (7), (35) and (18) we can obtain

\[
\frac{x}{\sqrt{n}} - \sqrt{2x(x-u_n(x))} \geq \frac{x}{4(n+1)\sqrt{n}} \quad \text{for } x > 0 \text{ and } n \in N.
\]

References