Meromorphic Function Sharing Two Small Functions with Its Derivative

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Abstract. In this paper, we deal with the problem of uniqueness of meromorphic functions that share two small functions with their derivatives, and obtain the following result which improves a result of Yao and Li: Let \( f(z) \) be a nonconstant meromorphic function, \( k > 5 \) be an integer. If \( f(z) \) and \( g(z) = a_1(z)f(z) + a_2(z)f^{(k)}(z) \) share the value 0 CM, and share \( b(z) \) IM, \( N_E(r, f = 0 = f^{(k)}) = S(r) \), then \( f \equiv g \), where \( a_1(z), a_2(z) \) and \( b(z) \) are small functions of \( f(z) \).

1. Introduction and main results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We say that two meromorphic functions \( f \) and \( g \) share a finite value \( a \) IM (ignoring multiplicities) when \( f - a \) and \( g - a \) have the same zeros. If \( f - a \) and \( g - a \) have the same zeros with the same multiplicities, then we say that \( f \) and \( g \) share the value \( a \) CM (counting multiplicities).

Denote by \( N(r, f = b = g) \) the reduced counting function of the common zeros of \( f - b \) and \( g - b \) ignoring the multiplicities, and \( N_E(r, f = b = g) \) the reduced counting function of the common zeros of \( f - b \) and \( g - b \) with the same multiplicities. We say that \( f \) and \( g \) share \( b \) IM* provided that

\[
N(r, \frac{1}{f - b}) - N(r, f = b = g) = S(r, f)
\]

and

\[
N(r, \frac{1}{g - b}) - N(r, f = b = g) = S(r, f).
\]

Similarly, we say that \( f \) and \( g \) share \( b \) CM* provided that

\[
N(r, \frac{1}{f - b}) - N_E(r, f = b = g) = S(r, f)
\]

and

\[
N(r, \frac{1}{g - b}) - N_E(r, f = b = g) = S(r, f).
\]

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It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna Theory, as found in [4], [5]. In 1986, Frank-Weissenborn proved the following result.

**Theorem A([1]).** Let \( f \) be a nonconstant meromorphic function, \( a, b \) be two distinct finite complex number. If \( f \) and \( f^{(k)} \) share the value \( a, b \) CM, then \( f \equiv f^{(k)} \).

Frank asked the following question.

**Question 1.** Does the Theorem A hold if we replace the condition that \( f \) and \( f^{(k)} \) share \( b \) CM by the condition that \( f \) and \( f^{(k)} \) share \( b \) IM?

The following example given by Ping-Li shows that the answer to Question 1 is, in general, negative. Let \( a_1 \) be any finite constant, \( a_2 = a_1 + \sqrt{2}i \), \( \omega \) be a nonconstant solution of the Riccati differential equation

\[
\omega' = (\omega - a_1)(\omega - a_2)
\]

and let

\[
f = (\omega - a_1)(\omega - a_2) - \frac{1}{3}.
\]

It is easy to verify that

\[
f'' = 6\omega' f,
\]

\[
f'' + \frac{1}{6} = 6(f + \frac{1}{6})^2.
\]

Since 0 is the Picard value of \( \omega' \), then 0 must be a CM shared value of \( f \) and \( f'' \). It is easy to see that \( f \) and \( f'' \) share the value \(-\frac{1}{6}\) IM, but \( f \not\equiv f'' \).

In 1990, Yang proved the following result.

**Theorem B([3]).** Let \( f \) be a nonconstant entire function, \( k \geq 2 \) be an integer, \( a \neq 0 \) be a finite constant. If 0 is the Picard value of \( f \) and \( f^{(k)} \), and if \( f \) and \( f^{(k)} \) share a IM, then \( f = e^{Az + B}, A, B \) be two constants, where \( A^5 = 1 \), and so that \( f \equiv f^{(k)} \).

It is natural to ask what results can be obtained if \( f^{(k)} \) is replaced by a differential polynomial of \( f \), and the values 0 and \( a \) are replaced by the small functions of \( f \)? In 2006, Yao and Li proved the next result.

**Theorem C([7]).** Let \( f(z) \) be a nonconstant meromorphic function, \( a_1(z), a_2(z) \) and \( b(z) \) be small functions of \( f(z) \), and let \( g(z) = a_1(z)f + a_2(z)f' \). If \( f \) and \( g \) share the value 0 CM*, and share the function \( b(z) \) IM*, then \( f \equiv g \) or \( f \) takes one of the following two forms:

1. \( f = \frac{b}{h} \) and \( a_1b + a_2b' = -b \), where \( h \) satisfies \( \frac{h'}{h} = -\frac{1}{a_2} \).
2. \( f = \frac{a_1}{b} \) and \( a_1b + a_2b' = 0 \), where \( h \) satisfies \( \frac{h'}{h} = -\frac{2}{a_2} \).

In this paper, we obtained the following results.
Theorem 1. Let $f$ be a nonconstant meromorphic function, $k(k > 5)$ be a positive integer, $a_1(z)$, $a_2(z)$ and $b(z)$ be small functions of $f$, and let $g(z) = a_1(z)f + a_2(z)f^{(k)}$. If $f$ and $g$ share the value 0 CM, share the function $b(z)$ IM, and $N_E(r, f = 0 = f^{(k)}) = S(r)$, then $f \equiv g$.

Theorem 2. Let $f$ be a nonconstant meromorphic function, $k$ be a positive integer, $a_1(z)$, $a_2(z)$ and $b(z)$ be small functions of $f$, and let $g(z) = a_1(z)f + a_2(z)f^{(k)}$. If $f$ and $g$ share the value 0 CM, share the function $b(z)$ IM, if $N_E(r, f = 0 = f^{(k)}) = S(r)$ and $\Theta(\infty, f) > \frac{k}{2}$, then $f \equiv g$.

Corollary 1. Let $f$ be a nonconstant entire function, and $g(z) = a_1(z)f + a_2(z)f^{(k)}$. If $f$ and $g$ share the value 0 CM, and share the function $b(z)$ IM, then $f \equiv g$, where $a_1(z)$, $a_2(z)$ and $b(z)$ are defined as in Theorem 2.

Theorem 3. Let $f$ be a nonconstant meromorphic function, $a_1(z)$, $\cdots$, $a_k(z)$ ($k > 2$) and $b(z)$ be small functions of $f$, and let $L(f) = W(a_1, a_2, \cdots, a_k, f)$, where $W(a_1, a_2, \cdots, a_k, f)$ is the Wronskian of $a_1$, $\cdots$, $a_k, f$. If $f$ and $L(f)$ share $b(z)$ IM and

$$N(r, \frac{1}{f}) + N(r, \frac{1}{L}) = S(r, f),$$

then $f = L(f)$.

2. Lemmas

Lemma 1([8]). Let $f$ be a nonconstant meromorphic function, $k$ be a positive integer, then

$$N(r, \frac{1}{f^{(k)}}) < N(r, \frac{1}{f}) + kN(r, f) + S(r, f),$$

(2.1)

$$N(r, \frac{f^{(k)}}{f}) < N(r, \frac{f}{f}) + kN(r, f) + S(r, f),$$

(2.2)

$$N(r, \frac{f^{(k)}}{f}) < kN(r, f) + N(r, \frac{1}{f}) + S(r, f).$$

(2.3)

Suppose that $f$ and $g$ share the value $a$ IM, and let $z_0$ be a $a$-point of $f$ of order $p$, a $a$-point of $g$ of order $q$. We denote by $N_L(r, \frac{1}{f - a})$ the counting function of those $a$-points of $f$ where $p > q$, and we denote by $N_L(r, \frac{1}{f - a})$ the corresponding counting function that ignores the multiplicities.

Lemma 2([6]). Let $f$ be a nonconstant meromorphic function. If $f$ and $g$ share the value 1 IM, then

$$N_L(r, \frac{1}{f^{(k)} - 1}) < N(r, \frac{1}{f}) + N(r, f) + S(r, f).$$

(2.4)
Lemma 3. Let $f$ be a nonconstant meromorphic function, $a_1(z)$, $a_2(z)$ and $b(z)$ be small functions of $f$, and let $g(z) = a_1 f + a_2 f^{(k)}$, where $k$ is a positive integer. If $f$ and $g$ share the value $0$ CM, and share the function $b$ IM, $N_E(r, f = 0 = f^{(k)}) = S(r)$ and if $f \not\equiv g$, then
\[
N(r, \frac{1}{f}) + N(r, \frac{1}{g}) = S(r).
\]

Proof. Since $f$ and $g$ share $0, b$ IM, and
\[
N(r, f) = N(r, g) + S(r, f),
\]
from the second fundamental theorem, we have
\[
S(r, f) = S(r, g) = S(r).
\]
Noticing that $f$ and $g$ share the value $0$ CM and $N_E(r, f = 0 = f^{(k)}) = S(r)$, we have
\[
N(r, f = 0 = g) \leq N(r, \frac{1}{a_2}) \leq S(r)
\]
or
\[
N(r, f = 0 = g) \leq N(r, a_2) \leq S(r).
\]
So we get
\[
N(r, \frac{1}{f}) + N(r, \frac{1}{g}) = S(r).
\]

Lemma 4([2]). Let $f$ be a transcendental meromorphic function, $a_1(z)$, $a_2(z)$, \ldots, $a_k(z)$ $(k > 2)$ be linearly independent small functions of $f$, $L(f) = W(a_1, a_2, \ldots, a_k, f)$ be the Wronskian of $a_1, \ldots, a_k, f$. Then
\[
kN(r, f) \leq N(r, \frac{1}{L}) + (1 + \varepsilon)N(r, f) + S(r, f),
\]
where $\varepsilon$ is any given positive number.

3. Proof of Theorem 1

From the second fundamental theorem, Lemma 1 and Lemma 3, we have
\[
T(r, g) \leq N(r, g) + N(r, \frac{1}{g}) + N(r, \frac{1}{g - b}) + S(r)
\]
\[
\leq N(r, f) + N(r, \frac{1}{g - b}) + S(r).
\]
Since $f$ and $g$ share the small function $b(z)$ IM, we obtain

$$T(r, g) \leq \mathcal{N}(r, f) + \mathcal{N}(r, \frac{1}{f - 1}) + S(r)$$

$$\leq \mathcal{N}(r, f) + \mathcal{N}(r, \frac{1}{a_1 + a_2 f^{(k)}}) + S(r)$$

$$\leq \mathcal{N}(r, f) + T(r, \frac{f^{(k)}}{f}) + S(r)$$

$$\leq \mathcal{N}(r, f) + N(r, \frac{f^{(k)}}{f}) + S(r)$$

$$\leq (k + 1)\mathcal{N}(r, f) + S(r)$$

$$\leq N(r, g) + S(r)$$

$$\leq T(r, g) + S(r),$$

so we have

(3.1) $$T(r, g) = (k + 1)\mathcal{N}(r, f) + S(r), \quad N(r, f) = \mathcal{N}(r, f) + S(r).$$

Let $G = \frac{g}{b}$, $F = \frac{f}{b}$ and

$$H = \frac{G''}{G'} - 2 \cdot \frac{G'}{G - 1} - \frac{F''}{F'} + 2 \cdot \frac{F'}{F - 1}.$$ 

By the Lemma of logarithmic derivatives, we have $m(r, H) = S(r)$. Since $f$ and $g$ share the value $b$ IM, and share 0 CM, we know that $F$ and $G$ share the value $b$ IM*, and share 0 CM*, then

(3.2) $$N(r, H) = \mathcal{N}(r, F) + \mathcal{N}_L(r, \frac{1}{F - 1}) + \mathcal{N}_L(r, \frac{1}{G - 1}) + \mathcal{N}_0(r, \frac{1}{F'}) + \mathcal{N}_0(r, \frac{1}{G'}) + S(r),$$

where $\mathcal{N}_0(r, \frac{1}{F'})$ denotes the reduced counting function of $F'$ which are not the zeros of $F$ and $F - 1$. $\mathcal{N}_0(r, \frac{1}{G'})$ are similarly defined. From the second fundamental theorem, we have

(3.3) $$T(r, F) \leq \mathcal{N}(r, F) + \mathcal{N}(r, \frac{1}{F - 1}) - \mathcal{N}_0(r, \frac{1}{F'}) + S(r, F),$$

(3.4) $$T(r, G) \leq \mathcal{N}(r, G) + \mathcal{N}(r, \frac{1}{G - 1}) - \mathcal{N}_0(r, \frac{1}{G'}) + S(r, G).$$

If $H \neq 0$, by calculation, we know that the common simple zeros of $F - 1$ and $G - 1$ are the zeros of $H$, it follows that

(3.5) $$N^{(1)}_E(r, \frac{1}{F - 1}) \leq N(r, \frac{1}{H}) \leq T(r, H) = N(r, H) + S(r)$$
and
\[
\overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) = 2N^{(1)}_E(r, \frac{1}{F-1}) + 2\overline{N}_L(r, \frac{1}{F-1}) + 2N^{(2)}_E(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{G-1}) + 2N^{(2)}_E(r, \frac{1}{F-1}) + N(r, F - 1) + N(r, G - 1) + N_0(r, F) + N_0(r, G).
\]

Combining (3.2) – (3.6), we have
\[
T(r, F) + T(r, G) \leq 3\overline{N}(r, F) + 3\overline{N}_L(r, \frac{1}{F-1}) + 2\overline{N}_L(r, \frac{1}{G-1}) + 2N^{(2)}_E(r, \frac{1}{F-1}) + N(r, F - 1) + N(r, G - 1) + S(r)
\leq 3\overline{N}(r, F) + \overline{N}_L(r, \frac{1}{F-1}) + 2\overline{N}_L(r, \frac{1}{G-1}) + N(r, F - 1) + T(r, F) + S(r).
\]

Therefore
\[
T(r, G) \leq 3\overline{N}(r, F) + \overline{N}_L(r, \frac{1}{F-1}) + 2\overline{N}_L(r, \frac{1}{G-1}) + S(r).
\]

From Lemma 2 and Lemma 3, we get
\[
T(r, G) = (k + 1)\overline{N}(r, f) \leq 6\overline{N}(r, f) + S(r).
\]

Since \( k \geq 6 \), we get from (3.1) that \( T(r, f) = S(r, f) \), which is impossible. Hence, \( H \equiv 0 \). By integration two times, we have
\[
\frac{1}{G - 1} = \frac{A}{F - 1} + B,
\]

where \( A \neq 0 \) and \( B \) are constants. We rewrite (3.8) in the following forms
\[
F = \frac{(B - A)G + (A - B - 1)}{BG - (B + 1)},
\]
\[
G = \frac{(B + 1)F + (A - B - 1)}{BF + (A - B)}.
\]

We distinguish the following three cases.
Case 1. If $B \neq 0, -1$, then

$$N(r, 1/G - B) = N(r, F).$$

By the second fundamental theorem, Lemma 1 and the definitions of $F$ and $G$, we have

$$T(r, G) < N(r, G) + N(r, 1/G) + N(r, 1/(G - B + 1)) + S(r)$$
$$< 2N(r, f) + N(r, 1/g) + S(r)$$
$$< 2N(r, f) + S(r).$$

From the assumption and (3.1), this is impossible.

Case 2. If $B = -1$, then

$$G = \frac{A}{-F + A + 1}, \quad F = (A + 1)G - A.$$

If $A \neq -1$, then

$$N(r, 1/(G - A + 1)) = N(r, 1/F).$$

By the same reasoning as in Case 1, we get a contradiction. Thus $A = -1$, and so $FG \equiv 1, fg = b^2$. We obtain

$$N(r, f) + N(r, 1/f) = S(r).$$

It follows that

$$2T(r, \frac{f}{b}) = T(r, \frac{f^2}{b^2}) = T(r, \frac{b^2}{f^2}) + O(1)$$
$$= T(r, \frac{g}{f}) + O(1) = T(r, a_1 + a_2 \frac{f^{(k)}}{f}) + O(1)$$
$$= S(r, f).$$

This is impossible.

Case 3. If $B = 0$, by the similar discussion as the Case 2, if $A \neq 1$, we get a contradiction. Therefore $A = 1$, and so $f \equiv g$. The proof of Theorem 1 is thus completed.

4. Proof of Theorem 2

From the proof of Theorem 1, if $H \neq 0$, we obtain from (3.7) that

$$T(r, f) \leq 6N(r, f) + S(r, f).$$
This contradicts the assumption that $\Theta(\infty, f) > \frac{5}{6}$. Hence $H \equiv 0$. By the same reasoning as in the proof of Theorem 1, we have $f \equiv g$.

**Question 2.** Is it true that $f \equiv g$ if $1 < k \leq 5$?

### 5. Proof of Theorem 3

If $f \not\equiv L$, then $L \not\equiv 1$. Let $z_0$ be the common zero of $f - b$ and $L - b$, not a zero or a pole of $b$, then $\frac{L(z_0)}{f(z_0)} = 1$. Since $f$ and $L(f)$ share $b$ IM, from the lemma of logarithmic derivatives, we get

$$N(r, \frac{1}{L - b}) \leq N(r, \frac{1}{L}) + S(r, f) \leq T(r, \frac{L}{f}) \leq N(r, \frac{L}{f}) + S(r, f) \leq N(r, \frac{1}{f}) + kN(r, f) + S(r, f).$$

From the second fundamental theorem,

$$T(r, L) \leq \overline{N}(r, L) + \overline{N}(r, \frac{1}{L}) + S(r, f) \leq \overline{N}(r, f) + k\overline{N}(r, f) + S(r, f) \leq N(r, f) + kN(r, f) + S(r, f) \leq T(r, L) + S(r, f),$$

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + S(r, f) \leq \overline{N}(r, f) + k\overline{N}(r, f) + S(r, f)$$

(5.1) \quad = (k + 1)\overline{N}(r, f) + S(r, f).

So we get

(5.2) \quad \overline{N}(r, \frac{1}{L - b}) = k\overline{N}(r, f) + S(r, f),

and

(5.3) \quad N(r, f) = \overline{N}(r, f) + S(r, f).

We know that the poles of $f$ “almost all” are simple. Let

$$\alpha = \frac{L'}{L} - (k + 1)\frac{f'}{f}.$$
By the lemma of logarithmic derivatives, we get $m(r, \alpha) = S(r, f)$. Since the poles of $f$ “almost all” are simple. By calculation, we know that the simple pole are not the pole of $\alpha$. Therefore $N(r, \alpha) = S(r, f)$. Hence we have $T(r, \alpha) = S(r, f)$. We distinguish the following two cases.

Case 1. If $f$ is a rational function, since $T(r, \alpha) = S(r, f)$, then $\alpha$ must be a constant, and $L = f^{k+1}Ce^{\alpha z}$, where $C$ is a nonzero constant. If $\alpha \neq 0$, then $L$ is not a rational function, which is a contradiction. Hence $\alpha = 0$, and thus $L = Cf^{k+1}$. Since $T(r, b) = S(r, f)$, $b \neq 0$, $f$ and $L(f)$ share $b$ IM , the equation $C\omega^{k+1} - b = 0$ have $k + 1$ different roots. We select a root $\omega_0$ of this equation such that $\omega_0 \neq b$, and $f$ assumes the value $\omega_0$ which is possible. Since $k + 1 \geq 3$, and $f$ is a rational function. If $z_0$ is a zero of $f - \omega_0$, then $Cf^{k+1}(z_0) = b$. Since $f$ and $L(f)$ share $b$ IM, we have $f(z_0) = b$, therefore $\omega_0 = b$, which is a contradiction.

Case 2. If $f$ be a transcendental meromorphic function, then by Lemma 4, we know that $N(r, f) = S(r, f)$. Hence from (4.1) and (4.3), we get $T(r, f) = S(r, f)$. This is impossible.

Hence $f = L$, the proof of Theorem 3 is thus proved.

**Question 3.** If we replace $L$ by a more general differential polynomial of $f$, is it true that $f = L$?

**References**


