On Quasi-Baer and p.q.-Baer Modules

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Abstract. For an endomorphism α of R, in [1], a module $M_R$ is called α-compatible
if, for any $m \in M$ and $a \in R$, $ma = 0$ iff $m\alpha(a) = 0$, which are a generalization
of α-reduced modules. We study on the relationship between the quasi-Baerness and p.q.-
Baer property of a module $M_R$ and those of the polynomial extensions (including formal
skew power series, skew Laurent polynomials and skew Laurent series). As a consequence
we obtain a generalization of [2] and some results in [9]. In particular, we show: for an
α-compatible module $M_R$ (1) $M_R$ is p.q.-Baer module iff $M[x; \alpha]_{R[x; \alpha]}$ is p.q.-Baer module.
(2) for an automorphism α of R, $M_R$ is p.q.-Baer module iff $M[x, x^{-1}; \alpha]_{R[x, x^{-1}; \alpha]}$ is p.q.-
Baer module.

1. Introduction

Throughout this work all rings R are associative with identity and modules are
unital right $R$-modules and $\alpha : R \rightarrow R$ is an endomorphism of the ring $R$. In [7]
Clark called a ring $R$ quasi-Baer ring if the right annihilator of each right ideal of
$R$ is generated (as a right ideal) by an idempotent. Recently, Birkenmeier et al. [5]
called a ring $R$ right (resp. left) principally quasi-Baer [or simply right (resp. left)
p.q.-Baer] if the right (resp. left) annihilator of a principal right (resp. left) ideal
of $R$ is generated by an idempotent. $R$ is called p.q.-Baer if it is both right and left
p.q.-Baer. A ring is called reduced ring if it has no nonzero nilpotent elements and
$M_R$ is called α-reduced module by Lee-Zhou [9] if, for any $m \in M$ and $a \in R$,

1) $ma = 0$ implies $mR \cap Ma = 0$,
2) $ma = 0$ iff $m\alpha(a) = 0$,

where $\alpha : R \rightarrow R$ is a ring endomorphism with $\alpha(1) = 1$. The module $M_R$ is called
a reduced module if $M$ is 1$R$-reduced, where 1$R$ is the identity endomorphism of $R$.
It is clear that $R$ is a reduced ring iff $R_R$ is a reduced module.

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According to Amin [1], a module $M_R$ is called $\alpha$-compatible if $ma = 0$ if and only if $m\alpha(a) = 0$ (i.e., only the second condition is satisfied in the definition of $\alpha$-reduced modules). It is clear that, if $M_R$ is $\alpha$-compatible then, $ma = 0$ if and only if $ma^k(a) = 0$ for all $k$ and every $\alpha$-reduced modules are $\alpha$-compatible. We write $R[x], R[[x]], R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ for the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over $R$, respectively. In [9] Lee-Zhou introduced the following notation. For a module $M_R$, we consider

\[
M[x; \alpha] = \{ \sum_{i=0}^{s} m_i x^i : s \geq 0, m_i \in M \},
\]

\[
M[[x; \alpha]] = \{ \sum_{i=0}^{\infty} m_i x^i : m_i \in M \},
\]

\[
M[x, x^{-1}; \alpha] = \{ \sum_{i=-s}^{s} m_i x^i : s \geq 0, t \geq 0, m_i \in M \},
\]

\[
M[[x, x^{-1}; \alpha]] = \{ \sum_{i=-\infty}^{\infty} m_i x^i : s \geq 0, m_i \in M \}.
\]

Each of these is an Abelian group under an obvious addition operation. Moreover $M[x; \alpha]$ becomes a module over $R[x; \alpha]$ under the following scalar product operation: For $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x; \alpha]$ and $f(x) = \sum_{i=0}^{t} a_i x^i \in R[x; \alpha]$,

\[
m(x)f(x) = \sum_{k=0}^{s+t} \left( \sum_{i+j=k} m_i \alpha(a_j) \right) x^k.
\]

Similarly, $M[[x; \alpha]]$ is a module over $R[[x; \alpha]]$. The modules $M[x; \alpha]$ and $M[[x; \alpha]]$ are called the skew polynomial extension and the skew power series extension of $M$ respectively. If $\alpha \in Aut(R)$, then with a similar scalar product, $M[[x, x^{-1}; \alpha]]$ (resp. $M[x, x^{-1}; \alpha]$) becomes a module over $R[[x, x^{-1}; \alpha]]$ (resp. $R[x, x^{-1}; \alpha]$). The modules $M[x, x^{-1}; \alpha]$ and $M[[x, x^{-1}; \alpha]]$ are called the skew Laurent polynomial extension and the skew Laurent power series extension of $M$, respectively.

Following Lee-Zhou [9], a module $M_R$ is called Armendariz if, whenever $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{i=0}^{t} a_i x^i \in R[x]$, we have $m_i a_j = 0$ for all $i, j$. By [9, Lemma 1.5], every reduced module is Armendariz. In [3], we define a module $M_R$ to be quasi-Armendariz if whenever these polynomials satisfy $m(x)R[x]f(x) = 0$, we have $m_i R a_j = 0$ for all $i, j$.

For a subset $X$ of a module $M_R$, let $r_R(X) = \{ r \in R : Xr = 0 \}$. In [9] Lee-Zhou introduced quasi-Baer module as follows: $M_R$ is called quasi-Baer if, for any submodule $N$ of $M$, $r_R(N) = eR$ where $e^2 = e \in R$. Clearly $R$ is a quasi-Baer ring iff $R_R$ is quasi-Baer module; if $R$ is a quasi-Baer ring then, for any right ideal $I$ of $R$, $IR$ is a quasi-Baer module. Following [3], $M_R$ is called principally quasi-Baer (or simply $p.q.$-Baer) module if, for any $m \in M$, $r_R(mR) = eR$ where $e^2 = e \in R$. It is clear that $R$ is a right $p.q.$-Baer ring iff $R_R$ is a $p.q.$-Baer module. If $R$ is a $p.q.$-Baer ring, then for any right ideal $I$ of $R$, $IR$ is a $p.q.$-Baer module. Every submodule of a $p.q.$-Baer module is $p.q.$-Baer module. Moreover, every quasi-Baer module is $p.q.$-Baer.

Motivated by results in Lee-Zhou [9], [3] and [2] we investigate a generalization of $\alpha$-reduced modules and introduce skew quasi-Armendariz types module which are skew polynomials versions of the quasi-Armendariz modules.
2. Skew polynomial and power series modules over quasi-Baer and p.q.-Baer modules

In this section we investigate a generalization of \(\alpha\)-reduced modules and introduce skew Armendariz and skew quasi-Armendariz of power series type modules, which are skew polynomial versions of the quasi-Armendariz modules. We then extend our previous results in [2] to non \(\alpha\)-reduced \(\alpha\)-compatible modules. Assume that \(M_R\) is an \(\alpha\)-compatible module. Then we will show that:

1. \(M_R\) is p.q.-Baer module if and only if \(M[[x;\alpha]]_{R[[x;\alpha]]}\) is p.q.-Baer module.
2. \(M_R\) is quasi-Baer module if and only if \(M[[x;\alpha]]_{R[[x;\alpha]]}\) is quasi-Baer module
   and only if \(M[[x;\alpha]]_{R[[x;\alpha]]}\) is quasi-Baer module.
3. If \(M[[x;\alpha]]_{R[[x;\alpha]]}\) is p.q.-Baer module then \(M_R\) is p.q.-Baer module.

Definition 2.1. A module \(M_R\) is called,

(i) skew quasi-Armendariz, if whenever \(m(x)R[x;\alpha]f(x) = 0\) for \(m(x) = \sum_{i=0}^{s} m_i x^i \in M[[x;\alpha]]\) and \(f(x) = \sum_{j=0}^{t} a_j x^j \in R[[x;\alpha]]\), then \(m_i R_{a_j} = 0\) for all \(i,j\).

(ii) skew quasi-Armendariz of power series type, if whenever \(m(x)R[[x;\alpha]]f(x) = 0\) for \(m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]\) and \(f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x;\alpha]]\), then \(m_i R_{a_j} = 0\) for all \(i,j\).

Note that if \(M_R\) is assumed to be \(\alpha\)-reduced, then it is clear that \(M_R\) is skew quasi-Armendariz and skew quasi-Armendariz of power series type. To see that, let \(m(x)R[[x;\alpha]]f(x) = 0\) for \(m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]\) and \(f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x;\alpha]]\). Then \(m(x)Rf(x) = 0\) and so \(m(x)cf(x) = 0\) for all \(c \in R\). Hence \(0 = (\sum_{i=0}^{\infty} m_i x^i)c(\sum_{j=0}^{\infty} a_j x^j) = (\sum_{i=0}^{\infty} m_i c a_j x^i)\). Since \(M_R\) is \(\alpha\)-reduced, \(M_R\) satisfies all the hypothesis of [9, Lemma 1.5] by [9, Lemma 1.2]. Hence, we have \(m \alpha^i c a_j = 0\) and so \(m \alpha^i c a_j = 0\) for all \(i,j\), since \(M_R\) is \(\alpha\)-compatible. Then \(m_i R_{a_j} = 0\) for all \(i,j\) and therefore, \(M_R\) is skew quasi-Armendariz of power series type.

Following [8], for a module \(M_R\), \(r\text{Ann}_R(\text{sub}(M_R)) = \{r_R(U) \mid U \text{ is a submodule of } M_R\}\).

Proposition 2.2. Let \(M_R\) be an \(\alpha\)-compatible module. Then the following statements are equivalent:

1. \(M_R\) is skew quasi-Armendariz.
2. \(\psi: r\text{Ann}_R(\text{sub}(M_R)) \rightarrow r\text{Ann}_{R[[x;\alpha]]}(\text{sub}(M[[x;\alpha]]_{R[[x;\alpha]]})); \]
   \(I \mapsto I[[x;\alpha]]\) is bijective.

Proof. (1)\(\Rightarrow\)(2) Let \(I \in r\text{Ann}_R(\text{sub}(M_R))\). Then there exists a submodule \(U\) of \(M_R\) such that \(I = r_R(U)\). Then we have \(r_R(U)[x;\alpha] = r_{R[[x;\alpha]]}(U[[x;\alpha]])\) since \(M_R\) is \(\alpha\)-compatible. So \(\psi\) is well-defined. Obviously \(\psi\) is injective. Now, for a submodule \(V\) of \(M[[x;\alpha]]_{R[[x;\alpha]]}\), let \(r_{R[[x;\alpha]]}(V) \in r\text{Ann}_{R[[x;\alpha]]}(\text{sub}(M[[x;\alpha]]_{R[[x;\alpha]]})).\)

Let \(C_V\) denote the set of coefficients of elements of \(V\). Then \(C_R\) is a submodule of \(M_R\). We claim that \(\psi(r_{R(C_V R)}) = r_{R(C_V R)}[[x;\alpha]] = r_{R[[x;\alpha]]}(V)\). Let \(f(x) = a_0 + a_1 x + \cdots + a_i x^i \in r_{R(C_V R)}[[x;\alpha]]\). Then \(a_i \in r_{R(C_V R)}\) and hence \((C_V R)a_i = 0\) and in particular \(C_V a_i = 0\) for all \(i\). Since \(M_R\) is \(\alpha\)-compatible
Let \( g \) quasi-Armendariz, \( f \) and hence for all \( \psi \). Consequently, \( v \) is surjective.

(2)\(\Rightarrow\)(1) Suppose \( m(x) R[x;\alpha] f(x) = 0 \) for \( m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha] \) and \( f(x) = \sum_{j=0}^{t} a_j x^j \in R[x;\alpha] \). Then \( f(x) \in r_{R[x;\alpha]}(m(x) R[x;\alpha]) = r_{R(CR)[x;\alpha]} \), where \( C \) is denote the set of coefficients of elements of \( m(x) R[x;\alpha] \). Then \( a_j \in r_{R(CR)} \) and so \( (CR)a_j = 0 \). In particular \( m_i Ra_j = 0 \) for all \( i, j \). Therefore \( M_R \) is skew quasi-Armendariz.

**Proposition 2.3.** Let \( M_R \) be an \( \alpha \)-compatible module. Then the following statements are equivalent:

1. \( M_R \) is skew quasi-Armendariz of power series type.
2. \( \psi : \text{Ann}_{R} (\text{sub}(M_R)) \rightarrow \text{Ann}_{R[x;\alpha]}(\text{sub}(M[x;\alpha]_{R[x;\alpha]})) \);
   \( J \rightarrow J[[x;\alpha]] \) is bijective.

**Proof.** Similar to the proof of Proposition 2.2.

**Definition 2.4.** A submodule \( N \) of a left \( R \)-module \( M \) is called a pure submodule if \( L \otimes_R N \rightarrow L \otimes_R M \) is a monomorphism for every right \( R \)-module \( L \).

Following Tominaga [11], an ideal \( I \) of \( R \) is said to be left \( s \)-unital if for each \( a \in I \) there exists an \( x \in I \) such that \( xa = a \). If an ideal \( I \) of \( R \) is left \( s \)-unital, then for any finite subset \( F \) of \( I \), there exists an element \( e \in I \) such that \( ex = e \) for all \( x \in F \). By [10, Proposition 11.3.13], for an ideal \( I \), the following conditions are equivalent:

1. \( I \) is pure as a right ideal in \( R \).
2. \( R/I \) is flat as a right \( R \)-module,
3. \( I \) is left \( s \)-unital.

**Theorem 2.5.** Let \( M_R \) be an \( \alpha \)-compatible module. Then the following are equivalent:

1. \( r_{R}(mR) \) is pure as a right ideal in \( R \) for any element \( m \in M_R \).
2. \( r_{R[x;\alpha]}(m(x) R[x;\alpha]) \) is pure as a right ideal in \( R[x;\alpha] \) for any element \( m(x) \in M[x;\alpha] \). In this case \( M_R \) is skew quasi-Armendariz.

**Proof.** (1)\(\Rightarrow\)(2) Assume that condition (1) holds. First we shall prove that \( M_R \) is skew quasi-Armendariz. Suppose \( m(x) R[x;\alpha] f(x) = 0 \) for \( m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha] \) and \( f(x) = \sum_{j=0}^{t} a_j x^j \in R[x;\alpha] \). Then \( (\sum_{i=0}^{s} m_i x^i) R(\sum_{j=0}^{t} a_j x^j) = 0 \). Let \( e \) be an arbitrary element of \( R \). Then we have the following equation:

\[
0 = m_0 c_0 a_0 + \cdots + (m_s \alpha^s(c_{a_1-2}) + m_{s-1} \alpha^{s-1}(c_{a_1-1})) + m_{s-2} \alpha^{s-2}(c_{a_1}) x^{s+t-2} + \cdots + m_{a} \alpha^a(c_{a_1-1}) + m_{s-1} \alpha^{s-1}(c_{a_1}) x^{s+t-1} + m_s \alpha^s(c_{a_1}) x^{s+t}.
\]
Then $m_s\alpha^s(ca_t) = 0$ and hence $m_s\alpha_t = 0$ since $M_R$ is $\alpha$-compatible. Thus $m_sRa_t = 0$ and so $a_t \in r_R(m_sR)$. By hypothesis, $r_R(m_sR)$ is left $s$-unital, and hence there exists $e_s \in r_R(m_sR)$ such that $e_sa_t = a_t$. Replacing $c$ by $ce_s$ in Eq.(1), we obtain

\begin{equation}
0 = m_0ce_sa_0 + \cdots + (m_s\alpha^s(ca_{s-2}) + m_s\alpha^{-1}(ca_{s-1} + m_s\alpha^{-2}(ca_{s}))x^{s-1} + m_s\alpha^s(x) + m_s\alpha^{-1}(ca))x^{s+1} + m_s\alpha^s(x) + m_s\alpha^{-1}(ca))x^{s+2}.
\end{equation}

Since $e_s \in r_R(m_sR)$, $m_sRa_s = 0$ and $m_s\alpha^k(Re_s) = 0$ for all $k$ since $M_R$ is $\alpha$-compatible. Using $e_sa_t = a_t$ and $m_s\alpha^k(Re_s) = 0$, we obtain from Eq.(2)

\begin{equation}
0 = m_0ce_sa_0 + \cdots + (m_s\alpha^{-1}(ca_{s-2}) + m_s\alpha^{-2}(ca_{s}))x^{s+1} + m_s\alpha^{-1}(ca))x^{s+2}.
\end{equation}

Then we obtain $m_s\alpha^{-1}(ca_t) = 0$ and hence $m_s\alpha_t = 0$ and so $m_sRa_t = 0$ since $M_R$ is $\alpha$-compatible. Thus $a_t \in r_R(m_sR)$ and hence $a_t \in r_R(m_sR) \cap r_R(m_sR)$. Since $r_R(m_sR)$ is left $s$-unital, there exists $f \in r_R(m_sR)$ such that $fa_t = a_t$. If we put $e_{s-1} = fe_s$ then $e_{s-1}a_t = a_t$ and $e_{s-1} \in r_R(m_sR) \cap r_R(m_sR)$. Next, replacing $e$ by $ce_{s-1}$ in Eq.(1), we obtain $m_s\alpha_t = 0$ in the same way as above. Hence we have $a_t \in r_R(m_sR) \cap r_R(m_sR) \cap r_R(m_sR)$. Continuing this process, we obtain $m_sRa_t = 0$ for all $i = 0, 1, \ldots, s$. Thus we get $(\sum_{i=0}^{s} m_i x^i)R[x; \alpha](\sum_{j=0}^{t-1} a_j x^j) = 0$, since $M_R$ is $\alpha$-compatible. Using induction on $s + t$, we obtain $m_sRa_j = 0$ for all $i, j$. Thus we proved that $M_R$ is skew quasi-Armendariz. Now, let $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x; \alpha]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in r_R(x; \alpha)(m(x)R[x; \alpha])$. Then $m(x)R[x; \alpha]f(x) = 0$ and so $m_sRa_j = 0$ for all $i, j$ since $M_R$ is skew quasi-Armendariz. Since $r_R(m_sR)$ is left $s$-unital, there exists $e_i \in r_R(m_sR)$ such that $a_j = e_ia_j$ for $j = 0, 1, \ldots, t$. Put $c = e_0e_1 \cdots e_s$, then $a_j = e_ia_j$ for $j = 0, 1, \ldots, t$. Hence $ef(x) = f(x)$ and $e \in r_R(x; \alpha)(m(x)R[x; \alpha])$ since $e_i \in r_R(m_sR)$ and $M_R$ is $\alpha$-compatible. Therefore $r_R(x; \alpha)(m(x)R[x; \alpha])$ is left $s$-unital.

(2)$\Rightarrow$(1) Suppose that condition (2) holds. Let $m$ be an element of $M_R$. Since $M_R$ is $\alpha$-compatible, $r_R(mR) \subseteq r_R(x; \alpha)(mR[x; \alpha])$. Hence for any $b \in r_R(mR)$, there exists a polynomial $f(x) = \sum_{j=0}^{t} a_j x^j \in r_R(x; \alpha)(m(x)R[x; \alpha])$ such that $f(x)b = b$. Then $a_0b = b$ and $a_0 \in r_R(mR)$. This implies that $r_R(mR)$ is left $s$-unital.

**Corollary 2.6.** Let $M_R$ be an $\alpha$-compatible module. Then $M_R$ is p.q.-Baer if and only if $M[x; \alpha][x; \alpha]$ is p.q.-Baer. In this case $M_R$ is skew quasi-Armendariz.

**Proof.** Let $M_R$ be a p.q.-Baer module. Then for each $m \in M$, there exists $e^2 = e \in R$, such that $r_R(mR) = eR$. Thus $r_R(mR)$ is left $s$-unital for each $m \in M$. It follows from Theorem 2.5 that $M_R$ is skew quasi-Armendariz. If $r_R(mR) = eR$ for some $e^2 = e \in R$, then we see that $ere = re$ holds for each $r \in R$. Thus if $r_R(mR) = e_iR$ for $i = 0, 1, \ldots, n$, then we get $r_R(mR + e_1R + \cdots + e_nR) = eR$, where $e = e_0e_1 \cdots e_n$ and $e^2 = e \in R$. Now, let $m(x) = m_0 + m_1x + \cdots + m_nx^n$ and $f(x) = a_0 + a_1x + \cdots + a_kx^k$ such that $f(x) \in r_R(x; \alpha)(m(x)R[x; \alpha])$. Then $m_sR = 0$
since $M_R$ is skew quasi-Armendariz. Let $r_R(m,R) = e_iR$, for $i = 0, 1, \cdots , n$ and $e = e_0 e_1 \cdots e_n$. Then $r_{R[[x;\alpha]]}(m(x)R[[x;\alpha]]) = e_R[x;\alpha]$ and hence we learn that $M[x;\alpha]_{R[[x;\alpha]]}$ is p.q.-Baer. The proof for the converse part can be done similarly, and therefore is omitted. \hfill \Box

**Remark 2.7.** Since $\alpha$-reduced modules are $\alpha$-compatible, Corollary 2.6 extends [2, Theorem 7(1)(a)].

**Corollary 2.8.** Let $M_R$ be a module. Then $M_R$ is p.q.-Baer if and only if $M[x]_{R[[x]]}$ is p.q.-Baer. In this case $M_R$ is quasi-Armendariz.

**Corollary 2.9** ([6, Theorem 3.1]). $R$ is a right p.q.-Baer ring if and only if $R[x]$ is a right p.q.-Baer ring.

**Proposition 2.10.** Let $M_R$ be an $\alpha$-compatible module. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(1) $r_{R[[x;\alpha]]}(m(x)R[[x;\alpha]])$ is a pure as a right ideal in $R[[x;\alpha]]$ for any element $m(x) \in M[[x;\alpha]]$.

(2) $r_R(mR)$ is pure as a right ideal in $R$ for any element $m \in M_R$.

(3) $M_R$ is skew quasi-Armendariz of power series type.

**Proof.** (1) $\Rightarrow$ (2) The proof is similar to that of Theorem 2.5.

(2) $\Rightarrow$ (3) Assume that $m(x)R[[x;\alpha]]f(x) = 0$ for $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x;\alpha]]$. Then $m(x)Rf(x) = 0$ and so we have the following equation for an arbitrary $c \in R$:

$$
\sum_{k=0}^{\infty} \left( \sum_{i+j=k} m_i x^i a_j x^j \right) = \sum_{k=0}^{\infty} \left( \sum_{i+j=k} m_i \alpha^i (ca_j) x^{i+j} \right) = 0.
$$

We will show that $m_i Ra_j = 0$ for all $i,j$. We proceed by induction on $i + j$. From Eq.(3), we obtain, $m_R a_0 = 0$. This proves $i + j = 0$. Now suppose that $m_i Ra_j = 0$ for $i + j \leq n - 1$. Hence $a_j \in r_R(m_i R)$ for $j = 0, 1, \cdots , n - 1$ and $i = 0, 1, \cdots , n - 1 - j$. Since $r_R(m_i R)$ is left s-unital, there exists $e_j \in r_R(m_i R)$ such that $e_j a_j = a_j$ for $j = 0, 1, \cdots , n - 1$ and $i = 0, 1, \cdots , n - 1 - j$. From Eq.(3), we have:

$$
\sum_{i+j=k} m_i \alpha^i (ca_j) = 0 \quad \text{for all} \quad k \geq 0.
$$

If we put $f_j = e_1 e_2 \cdots e_{j(n-1-j)}$ for $j = 0, 1, \cdots , n - 1$, then $f_j a_j = a_j$ and $f_j \in r_R(m_0 R) \cap r_R(m_1 R) \cap \cdots \cap r_R(m_{n-1-j} R)$. For $k = n$ replacing $c$ by $cf_0$ in Eq.(4), we obtain $m_n a_0 = m_n cf_0 a_0 = 0$. Hence $m_n a_0 = 0$. Continuing this process (replacing $c$ by $c f_j$ in Eq.(4), for $j = 0, 1, \cdots , n - 1$ and using $\alpha$-compatibility of $M_R$), we obtain $m_i Ra_j = 0$ for $i + j = n$. Therefore $M_R$ is skew quasi-Armendariz of power series type. \hfill \Box

Since quasi-Baer modules satisfy the hypothesis of Theorem 2.5 and Proposition 2.10 we have,
Corollary 2.11 ([9, Theorem 2.13(1)]). Let $M_R$ be an $\alpha$-compatible module. Then $M_R$ is quasi-Baer iff $M[x;\alpha]_R[x;\alpha]$ is quasi-Baer iff $M[[x;\alpha]]_R[[x;\alpha]]$ is quasi-Baer.

Proof. The proof follows very similar to that of Corollary 2.6. \hfill \Box

Corollary 2.12 ([2, Theorem 7(1)(b)]). Let $M_R$ be an $\alpha$-compatible module. If $M[[x;\alpha]]_R[[x;\alpha]]$ is p.q.-Baer then $M_R$ is p.q.-Baer.

3. Skew Laurent polynomial and power series modules over quasi-Baer and p.q.-Baer modules

In this section we introduce skew quasi-Armendariz of Laurent type modules and skew quasi-Armendariz of Laurent power series type modules, which are skew Laurent polynomial version of the quasi-Armendariz modules and then study on the relationship between the quasi-Baerness and p.q.-Baer property of a module $M_R$ and those of the skew Laurent polynomials and skew Laurent series. As a consequence we obtain a generalization of [2] and some result in [9].

Definition 3.1. Let $\alpha$ be an automorphism of $R$. A module $M_R$ is called:

(i) skew quasi-Armendariz of Laurent type, if whenever $m(x)R[x,x^{-1};\alpha]f(x) = 0$ for $m(x) = \sum_{i=-\infty}^{1} m_i x^i \in M[x,x^{-1};\alpha]$ and $f(x) = \sum_{j=-\infty}^{q} a_j x^j \in R[x,x^{-1};\alpha]$, then $m_i R_\alpha = 0$ for all $i,j$.

(ii) skew quasi-Armendariz of Laurent power series type if whenever $m(x)R[[x,x^{-1};\alpha]]f(x) = 0$ for $m(x) = \sum_{i=-\infty}^{\infty} m_i x^i \in M[[x,x^{-1};\alpha]]$ and $f(x) = \sum_{j=-\infty}^{\infty} a_j x^j \in R[[x,x^{-1};\alpha]]$, then $m_i R_\alpha = 0$ for all $i,j$.

Note that if $M_R$ is assumed to be $\alpha$-reduced, then it is clear that $M_R$ is skew quasi-Armendariz of Laurent type and skew quasi-Armendariz of Laurent power series type. In a similar way as in the proof of Proposition 2.2 and Theorem 2.5, we can prove the following results.

Proposition 3.2. Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then the following statements are equivalent:

1. $M_R$ is skew quasi-Armendariz of Laurent type.
2. \( \psi : r\text{Ann}_R(\text{sub}(M_R)) \rightarrow r\text{Ann}_R[[x,x^{-1};\alpha]](\text{sub}(M[[x,x^{-1};\alpha]]_{R[[x,x^{-1};\alpha]]})); \)
   \( I \mapsto I[[x,x^{-1};\alpha]] \) is bijective.

Proposition 3.3. Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then the following statements are equivalent:

1. $M_R$ is skew quasi-Armendariz of Laurent power series type.
2. \( \psi : r\text{Ann}_R(\text{sub}(M_R)) \rightarrow r\text{Ann}_R[[x,x^{-1};\alpha]](\text{sub}(M[[x,x^{-1};\alpha]]_{R[[x,x^{-1};\alpha]]})); \)
   \( I \mapsto I[[x,x^{-1};\alpha]] \) is bijective.

Theorem 3.4. Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then the following are equivalent:

1. $r_R(mR)$ is pure as a right ideal in $R$ for any element $m \in M_R$.
2. \( r_{R[[x,x^{-1};\alpha]]}(mR[[x,x^{-1};\alpha]])) \) is a pure as a right ideal in $R[[x,x^{-1};\alpha]]$ for any
element $m(x) \in M[x, x^{-1}; \alpha]$. In this case $M_R$ is skew quasi-Armendariz of Laurent type.

**Corollary 3.5.** Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then $M_R$ is p.q.-Baer if and only if $M[x, x^{-1}; \alpha]R[[x, x^{-1}; \alpha]]$ is p.q.-Baer. In this case $M_R$ is skew quasi-Armendariz of Laurent type.

**Remark 3.6.** Since $\alpha$-reduced modules are $\alpha$-compatible modules, the Corollary 3.5 extends [2, Theorem 7(2)(a)].

**Proposition 3.7.** Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(1) $R[[x, x^{-1}; \alpha]](m(x)R[[x, x^{-1}; \alpha]])$ is pure as a right ideal in $R[[x, x^{-1}; \alpha]]$ for any element $m(x) \in M[[x, x^{-1}; \alpha]]$.

(2) $R(mR)$ is pure as a right ideal in $R$ for any element $m \in M_R$.

(3) $M_R$ is skew quasi-Armendariz of Laurent power series type.

**Corollary 3.8** ([2, Theorem 7(2)(b)]). Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. If $M[[x, x^{-1}; \alpha]]R[[x, x^{-1}; \alpha]]$ is p.q.-Baer then $M_R$ is p.q.-Baer.

Since quasi-Baer modules satisfy the hypothesis of Theorem 3.4 and Proposition 3.7 we have:

**Corollary 3.9** ([9, Theorem 2.13(2)]). Let $\alpha$ be an automorphism of a ring $R$ and $M_R$ be an $\alpha$-compatible module. Then $M_R$ is quasi-Baer iff $M[[x, x^{-1}; \alpha]]R[[x, x^{-1}; \alpha]]$ is quasi-Baer.

**Corollary 3.10** ([9, Corollary 2.14]). $M_R$ is quasi-Baer iff $M[[x]]R[[x]]$ is quasi-Baer iff $M[[x]][R[[x]]$ is quasi-Baer iff $M[[x, x^{-1}]]R[[x, x^{-1}]]$ is quasi-Baer.

**Corollary 3.11** ([4, Theorem 1.8]). $R$ is quasi-Baer iff $R[[x]]$ is quasi-Baer iff $R[[x, x^{-1}]]$ is quasi-Baer.

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**References**


