On Comaximal Graphs of Near-rings

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Abstract. Let $N$ be a zero-symmetric near-ring with identity and let $\Gamma(N)$ be a graph with vertices as elements of $N$, where two different vertices $a$ and $b$ are adjacent if and only if $\langle a \rangle + \langle b \rangle = N$, where $\langle x \rangle$ is the ideal of $N$ generated by $x$. Let $\Gamma_1(N)$ be the subgraph of $\Gamma(N)$ generated by the set $\{n \in N : \langle n \rangle = N\}$ and $\Gamma_2(N)$ be the subgraph of $\Gamma(N)$ generated by the set $N\setminus v(\Gamma_1(N))$, where $v(G)$ is the set of all vertices of a graph $G$. In this paper, we completely characterize the diameter of the subgraph $\Gamma_2(N)$ of $\Gamma(N)$. In addition, it is shown that for any near-ring, $\Gamma_2(N)\setminus M(N)$ is a complete bipartite graph if and only if the number of maximal ideals of $N$ is 2, where $M(N)$ is the intersection of all maximal ideals of $N$ and $\Gamma_2(N)\setminus M(N)$ is the graph obtained by removing the elements of the set $M(N)$ from the vertices set of the graph $\Gamma_2(N)$.

1. Preliminaries

Throughout this paper $N$ is a zero-symmetric near-ring with identity. $M(N)$ denotes the intersection of all maximal ideals of $N$, $\text{Max}(N)$ denotes the set of all maximal ideals of $N$, $\langle x \rangle$ denotes the ideal of $N$ generated by $x$ and $v(G)$ denotes the set of all vertices of a graph $G$.

For any vertices $x, y$ in a graph $G$, if $x$ and $y$ are adjacent, we denote it as $x \approx y$. A graph is said to be connected if for each pair of distinct vertices $v$ and $w$, there is a finite sequence of distinct vertices $v_0 = v, v_2, \ldots, v_n = w$ such that each pair $\{v_i, v_{i+1}\}$ is an edge. Such a sequence is said to be a path and the distance, $d(v, w)$, between connected vertices $v$ and $w$ is the length of the shortest path connecting them. The diameter of a connected graph is the supremum of the distances between vertices. The degree of a vertex $v$ in $G$ is the number of edges of $G$ incident with $v$. Let $G_1$ be a subgraph of a graph $G$ and $v \in G_1$. Then $\text{deg}_{G_1}(v)$ is the number of edges of $G_1$ incident with $v$. An $r$-partite graph is one whose vertex set can be

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partitioned into $r$ subsets so that no edge has both ends in any one subset. Let $V$ be the set of vertices of a graph $G$ and $V_1 \subseteq V$. Then $G \setminus V_1$ is the graph obtained by removing the vertices of the set $V_1$ from the vertices set of the graph $G$. A complete $r$-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with part sizes $m$ and $n$ is denoted by $K_{m,n}$. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertices set $V_1$ and edges set $E_1$. The join of $G_1$ and $G_2$ is denoted by $G = G_1 \cup G_2$ with vertices set $V_1 \cup V_2$ and the set of edges is $E_1 \cup E_2 \cup \{x \approx y : x \in V_1$ and $y \in V_2\}$. Following Mason [4], an ideal $I$ of $N$ is called completely reflexive if $ab \in I$ implies $ba \in I$ for $a, b \in N$. In [2], Beck considered $\Gamma(R)$ as a graph with vertices the elements of a commutative ring $R$, where two different vertices $a$ and $b$ are adjacent if and only if $ab = 0$. He studied finitely colorable rings with this graph structure and in [1], Anderson and Naseer have made further studies of finitely colorable rings. In [6], Sharma and Bhatwadekar defined another graph structure on a commutative ring $R$ with vertices the elements of $R$ and where two distinct vertices $a$ and $b$ are adjacent if and only if $(a) + (b) = R$.

In this paper, we extend the graph structure of rings as defined by Sharma and Bhatwadekar and the results obtained by H. R. Maimani et al. [3] for commutative rings near-rings (not necessarily commutative). Let $N$ be a near-ring and let $\Gamma(N)$ be a graph with vertices the elements of $N$ and where two different vertices $a$ and $b$ are adjacent if and only if $(a) + (b) = N$.

Let $\Gamma_1(N)$ be the subgraph of $\Gamma(N)$ generated by the set $\{n \in N : \langle n \rangle = N\}$ and $\Gamma_2(N)$ be the subgraphs of $\Gamma(N)$ generated by the set $N\setminus \nu(\Gamma_1(N))$. Then clearly $\Gamma(N) = \Gamma_1(N) \cup \Gamma_2(N)$. If $N$ is a commutative ring, then the set of vertices of $\Gamma_1(N)$ consists of unit elements of $N$. Other definitions and basic concepts in near-ring theory can be found in G.Pilz [5].

2. Main results

**Theorem 2.1.** If $\{P_1, P_2, \ldots, P_n\}$ is a finite family of prime ideals of $N$ with $I \subseteq \bigcup_{i=1}^n P_i$ for any sub near-ring $I$ of $N$, then $I \subseteq P_i$ for some $i$.

**Proof.** We may assume that $I$ is not contained in the union of any collection on $n - 1$ of the $P_i$'s. If so, we can simply replace $n$ by $n - 1$. Thus for each $i$, we can find an element $a_i \in I$ with $a_i \notin P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_n$. Take $n = 2$, with $I \not\subseteq P_1$ and $I \not\subseteq P_2$. Then $a_1 \in P_1$, $a_2 \notin P_1$, and so $a_1 + a_2 \notin P_1$. Similarly, $a_1 \notin P_2$, $a_2 \in P_2$, and so $a_1 + a_2 \notin P_2$. Thus $a_1 + a_2 \notin I \subseteq P_1 \cup P_2$, contradicting $a_1, a_2 \in I$. Now assume that $n > 2$ and suppose that $I \not\subseteq P_i$ for all $i$. Observe that $\langle a_1 \rangle \langle a_2 \rangle \cdots \langle a_{n-1} \rangle \subseteq P_1 \cap P_2 \cdots \cap P_{n-1}$, but $a_n \notin P_1 \cup P_2 \cdots \cup P_{n-1}$. Now for all $i = 1, 2, \ldots, n - 1$, we have $a_i \notin P_n$, and so $\langle a_1 \rangle \langle a_2 \rangle \cdots \langle a_{n-1} \rangle \not\subseteq P_n$. Then there exists $t \in \langle a_1 \rangle \langle a_2 \rangle \cdots \langle a_{n-1} \rangle$ such that $x = t + a_n \notin P_n$. Thus $x \in I$ and $x \notin P_1 \cup P_2 \cup \cdots \cup P_n$, a contradiction. \hfill \Box

**Lemma 2.2.** Let $N$ be a near-ring. Then the following conditions hold:

(i) $\Gamma_1(N)$ is a complete graph.

(ii) $a \in M(N)$ if and only if $\deg_{\Gamma_2(N)}a = 0$.  


Proof. (i) It is clear from definition.
(ii) Let \( a \in M(N) \) and suppose \( \deg_{\Gamma_2(N)} a \neq 0 \), then there exists \( b \in \Gamma_2(N) \) such that \( (a) + (b) = N \). On the other hand there exists \( M \in \text{Max}(N) \) with \( b \in M \), and so \( M = N \), a contradiction. Conversely, assume that \( \deg_{\Gamma_2(N)} a = 0 \) and suppose that \( a \notin M(N) \). Then there exists \( M \in \text{Max}(N) \) such that \( a \notin M \), and so \( (a) + M = N \).

Case (i): If there exists \( \{a, b\} \notin M(N) \) such that \( a \notin M \), then \( (a) + (b) = N \). A contradiction. Conversely, assume that \( (a) + (b) = N \). So we have the path \( a \approx x \approx b \) and so \( d(a, b) \leq 2 \).

Case (ii): If \( \{a, b\} \subseteq M(N) \), then \( M(N) = S_a \cup S_b \), where \( S_a = \{M \in \text{Max}(N) : a \in M\} \) and \( S_b = \{M \in \text{Max}(N) : b \in M\} \). Since \( a \notin M(N) \), there exists \( x \in \Gamma_2(N) \) such that \( (a) + (x) = N \). Then \( x \notin M(N) \). Let \( M \in \text{Max}(N) \) such that \( b \notin M \). Then \( x \notin M \), and so \( (b) + (x) \notin M(N) \). Therefore by Case (i), \( d(b, x) \leq 2 \), and so \( d(a, b) \leq 3 \).

Corollary 2.3 ([3], Lemma 2.1). Let \( R \) be a commutative ring with identity. Then the following hold:

(i) \( \Gamma_1(R) \) is a complete graph.
(ii) \( a \in J(R) \) if and only if \( \deg_{\Gamma_2(R)} a = 0 \), where \( J(R) \) denotes the Jacobson radical of \( R \).

Proof. If \( R \) is commutative ring with identity, then \( J(R) = M(R) \).

Theorem 2.4. Let \( N \) be a near-ring. Then \( \Gamma_2(N) \setminus M(N) \) is connected graph and \( \text{diam} (\Gamma_2(N) \setminus M(N)) \leq 3 \).

Proof. Let \( a, b \in \Gamma_2(N) \setminus M(N) \).

Case (i): If \( \{a, b\} \notin M(N) \) such that \( \{a, b\} \notin M(N) \), then there exists \( x \in \Gamma_2(N) \setminus M(N) \) such that \( \{a, b\} + (x) = N \). Thus \( (a) + (x) = N \) and \( (b) + (x) = N \). So we have the path \( a \approx x \approx b \), and so \( d(a, b) \leq 2 \).

Case (ii): If \( \{a, b\} \subseteq M(N) \), then \( M(N) = S_a \cup S_b \), where \( S_a = \{M \in \text{Max}(N) : a \in M\} \) and \( S_b = \{M \in \text{Max}(N) : b \in M\} \). Since \( a \notin M(N) \), there exists \( x \in \Gamma_2(N) \) such that \( (a) + (x) = N \). Then \( x \notin M(N) \). Let \( M \in \text{Max}(N) \) such that \( b \notin M \). Then \( x \notin M \), and so \( (b) + (x) \notin M(N) \). Therefore by Case (i), \( d(b, x) \leq 2 \), and so \( d(a, b) \leq 3 \).

Corollary 2.5 ([3], Theorem 3.1). Let \( R \) be a commutative ring with identity. Then \( \Gamma_2(R) \setminus J(R) \) is connected graph and \( \text{diam} (\Gamma_2(R) \setminus J(R)) \leq 3 \).

Theorem 2.6. Let \( N \) be a near-ring. Then the following conditions are equivalent:

(i) \( \Gamma_2(N) \setminus M(N) \) is a complete bipartite graph.
(ii) The cardinal number of the set \( \text{Max}(N) \) is 2

Proof. i) \( \Rightarrow \) ii) Suppose that \( \Gamma_2(N) \setminus M(N) \) is a complete bipartite graph with two parts \( V_1 \) and \( V_2 \). Set \( M_1 = V_1 \cup M(N) \) and \( M_2 = V_2 \cup M(N) \). We claim that \( M_1 \) and \( M_2 \) are maximal ideals of \( N \). Let \( x, y \in M_1 \). Consider the following three cases:

Case (i): If \( x, y \in M(N) \), then \( x - y \in M_1 \).

Case (ii): If \( x \in M(N) \) and \( y \in V_1 \), then \( x - y \notin M(N) \). If \( (x - y) = N \), then \( x \notin M(N) \), a contradiction. If \( x - y \in M_2 \), then \( x - y \in V_2 \), and so \( (x - y) + (y) = N \). Thus \( (x) + (y) = N \), a contradiction. Therefore \( x - y \in V_1 \subseteq M_1 \).

Case (iii): Assume that \( x, y \in V_1 \). If \( x - y \notin M(N) \), then there is nothing to prove. Otherwise \( x - y \notin M(N) \). But by same argument of Case (ii), we have \( x - y \notin M_1 \).

Let \( x \in M_1 \) and \( n \in N \). If either \( x \in M(N) \) or \( n + x - n \in M(N) \), then \( M_1 \) is a normal subgroup of \( N \). So, we assume that \( x \notin M(N) \) and \( n + x - n \notin M(N) \). Since \( (n + x - n) \subseteq (\langle x \rangle) \), we have \( (n + x - n) \notin M(N) \). If \( n + x - n \in M_2 \), then \( n + x - n \in V_2 \), and so \( (n + x - n) + (x) = N \) which implies \( N = \langle x \rangle \), a contradiction. Therefore \( n + x - n \in V_1 \subseteq M_1 \). Let \( n \in N \) and \( x \in M_1 \). If either \( x \in M(N) \) or \( nx \in M(N) \), then \( M_1 \) is right ideal of \( N \). Otherwise \( x \notin M(N) \) and \( nx \notin M(N) \).
Also \( \langle xn \rangle \neq N \). Suppose that \( xn \in M_2 \). Then \( xn \in V_2 \), and so \( \langle xn \rangle + \langle x \rangle = N \). Thus \( \langle x \rangle = N \), a contradiction. So \( xn \in M_1 \). Let \( n, n_1 \in N \) and let \( x \in M_1 \). If either \( x \in M(N) \) or \( n(n_1 + x) - nn_1 \notin M(N) \), then \( M_1 \) is a left ideal of \( N \). Otherwise \( x \notin M(N) \) and \( n(n_1 + x) - nn_1 \notin M(N) \). Also \( \langle n(n_1 + x) - nn_1 \rangle \neq N \). Suppose that \( n(n_1 + x) - nn_1 \notin M_2 \). Then \( nn_1 \in V_2 \), and so \( \langle x \rangle + \langle n(n_1 + x) - nn_1 \rangle = N \) which implies \( N = \langle x \rangle \), a contradiction. So \( n(n_1 + x) - nn_1 \notin M_1 \). So \( M_1 \) is an ideal of \( N \). Let \( x \in N \setminus M_1 \). Then \( \langle x \rangle + \langle y \rangle = N \) for all \( y \in V_1 \) which implies \( \langle x \rangle + M_1 = N \), and so \( M_1 \) is a maximal ideal of \( N \).

With the same argument, \( M_2 \) is a maximal ideal of \( N \). Now, if \( M \in \text{Max}(N) \), then \( M \subseteq M_1 \cup M_2 \), and so \( M = M_1 \) or \( M = M_2 \) by Theorem 2.1.

\( ii) \implies i) \) Let \( \text{Max}(N) = \{M_1, M_2\} \). Thus the vertices set of \( \Gamma_2(N) \setminus M(N) \) is equal to the set \( (M_1 \setminus M_2) \cup (M_2 \setminus M_1) \). Let \( a \in M_1 \setminus M_2 \) and \( b \in M_2 \setminus M_1 \). Then \( \langle a \rangle + \langle b \rangle \notin M_1 \cup M_2 \) and so \( \langle a \rangle + \langle b \rangle = N \). □

**Corollary 2.7([3], Theorem 2.2).** Let \( R \) be a commutative ring with identity. Then the following are equivalent:

- \( i) \ \Gamma_2(R) \setminus J(R) \) is a complete bipartite graph.
- \( ii) \ \text{The cardinal number of the set Max}(R) \) is equal to \( 2 \).

**Theorem 2.8.** Let \( N \) be a near-ring and let \( n > 1 \). Then the following hold:

- \( i) \ If \ |\text{Max}(N)| = n < \infty \), then the graph \( \Gamma_2(N) \setminus M(N) \) is \( n \)-partite.
- \( ii) \ If the graph \( \Gamma_2(N) \setminus M(N) \) is \( n \)-partite, then \( |\text{Max}(N)| \leq n \). In this case if the graph \( \Gamma_2(N) \setminus M(N) \) is not \((n-1)\)-partite, then \( |\text{Max}(N)| = n \).

**Proof.** The proof is similar to that of Proposition 2.3 of [3]. □

**Theorem 2.9.** Let \( N \) be a near-ring with \( |\text{Max}(N)| \geq 2 \). Then the following hold:

- \( i) \ \Gamma_2(N) \setminus M(N) \) is a complete \( n \)-partite graph, then \( n = 2 \).
- \( ii) \ If there exists a vertex of \( \Gamma_2(N) \setminus M(N) \) which is adjacent to every other vertex, then \( N \cong \mathbb{Z}_2 \times F \), where \( \mathbb{Z}_2 = \{0, 1\} \) is the ring under addition modulo 2 and multiplication modulo 2; \( F \) is a simple near-ring.

**Proof.** (i) Let \( M_1, M_2 \) be two maximal ideals of \( N \). Since the elements of \( M_1 \setminus M(N) \) are not adjacent, and at least one element of \( M_1 \setminus M(N) \) is adjacent to \( M_2 \setminus M(N) \), so \( M_1 \setminus M(N) \) and \( M_2 \setminus M(N) \) are subsets of two distinct parts of \( \Gamma_2(N) \). Suppose \( M(N) \subseteq M_1 \cap M_2 \). Then there exists \( x \in M_1 \cap M_2 \) with \( x \notin M'(N) \), and so \( x \) belongs to \( M_1 \setminus M(N) \) and \( M_2 \setminus M(N) \), a contradiction to \( M_1 \setminus M(N) \) and \( M_2 \setminus M(N) \) are subsets of two distinct parts of \( \Gamma_2(N) \). Thus \( M(N) = M_1 \cap M_2 \) and hence \( |\text{Max}(N)| = 2 \). By Theorem 2.6, we have \( n = 2 \).

(ii) Let \( x \in \Gamma_2(N) \setminus M(N) \) such that \( x \) is adjacent to every other vertex. Clearly \( \langle x \rangle \subseteq M \) for some maximal ideal \( M \) of \( N \). Suppose \( y \neq 0 \in M(N) \). Then \( x + y \notin M(N) \) and \( \langle x + y \rangle \neq N \) which implies \( \langle x \rangle + \langle x + y \rangle = N \), and so \( M = N \), a contradiction. So \( M(N) = 0 \). Now, let \( y \in M \) with \( y \notin \{0, x\} \). Then \( N = \langle x \rangle + \langle y \rangle \subseteq M \), a contradiction. Therefore \( M = \{0, x\} = \langle x \rangle \) is a maximal ideal of \( N \). Thus for each \( s \neq 0 \in \Gamma_2(N) \), having \( \langle x \rangle + \langle s \rangle = N \) implies \( N/\langle x \rangle \cong \langle s \rangle \). Thus \( \langle s \rangle = F \) is simple and hence \( N \cong \mathbb{Z}_2 \times F \). □

**Corollary 2.10([3], Proposition 2.4).** Let \( R \) be a commutative ring with \( |\text{Max}(R)| \geq 2 \). Then the following hold:

- \( i) \ \text{If} \Gamma_2(R) \setminus J(R) \) is a complete \( n \)-partite graph, then \( n = 2 \).
Theorem 2.12. Let \( \Pi \) be a near-ring. Then \( \text{diam}(\Pi) = 1 \) if and only if \( \Pi \cong \mathbb{Z}_2 \times \mathbb{F} \), where \( \mathbb{F} \) is a field.

Proof. The proof is similar to that of Lemma 3.2 of [3].

Proof. Let \( \Pi \) be a near-ring with at least two maximal ideals and let \( \text{diam}(\Pi) \) be a completely reflexive ideal of \( \Pi \). Then \( \text{diam}(\Pi) = 2 \) if and only if one of the following holds:

(i) \( \text{Max}(\Pi) \) is a prime ideal,

(ii) \( |\text{Max}(\Pi)| = 2 \) and \( \Pi \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Proof. Let \( \text{Max}(\Pi) \) be prime and let \( a, b \in \Pi \setminus \text{Max}(\Pi) \). Then \( \langle a \rangle \langle b \rangle \not\subseteq \text{Max}(\Pi) \), and so by the same argument as in Theorem 2.4, there exists \( t \in \Pi \setminus \text{Max}(\Pi) \). If \( \text{diam}(\Pi) = 1 \), then by Lemma 2.11, \( \Pi \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). But \( \Pi \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), so \( \Pi \) is not a prime ideal, a contradiction.

Conversely, let \( \text{diam}(\Pi) = 2 \) and \( \Pi \) be nonprime. Let \( a, b \not\in \text{Max}(\Pi) \). We show that \( a \) and \( b \) are adjacent. Otherwise there exists \( t \in \Pi \setminus \text{Max}(\Pi) \) such that \( \langle a \rangle + \langle t \rangle = \langle b \rangle + \langle t \rangle = \Pi \). Then there are \( x_1, y_1, x'_1 \in \langle a \rangle \) and \( y_1, x'_1 \in \langle b \rangle \) such that \( x_1 + y_1 = x_1' + y_1 = 1 \), which implies \( x_1 x_1' + y_1 x_1' + y_1 = 1 \). Since \( x_1 x_1, y_1, x_1' \in \langle a \rangle \) and \( y_1, x_1' \in \langle b \rangle \), we have \( \langle a \rangle \langle b \rangle + \langle t \rangle = \Pi \), which implies \( \langle a \rangle \langle b \rangle \not\subseteq \text{Max}(\Pi) \), a contradiction. Therefore \( \langle a \rangle + \langle b \rangle = \Pi \), and so \( x + y = 1 \) for some \( x \in \langle a \rangle \) and \( y \in \langle b \rangle \).

Set \( S = \Pi/M(\Pi) \) and \( a_1 = x + M(\Pi) \) and \( b_1 = y + M(\Pi) \). Then \( a_1 b_1 = 0 \) and \( a_1 + b_1 = 1_S \). Since \( M(\Pi) \) is completely reflexive, we have \( \langle a_1 \rangle \langle b_1 \rangle = 0 \). If \( z \in \langle a_1 \rangle \cap \langle b_1 \rangle \), then \( z^2 \subseteq \langle a_1 \rangle \langle b_1 \rangle = 0 \). Since \( M(\Pi) \) is semiprime ideal of \( S \), we have \( z = 0 \). Thus \( \langle a_1 \rangle \cap \langle b_1 \rangle = 0 \) and hence \( S = \langle a_1 \rangle \oplus \langle b_1 \rangle \). Let \( M \) be nonzero ideal of \( \langle a_1 \rangle \) and let \( m(\neq 0) \in M \) and \( x_1(\neq 0) \in \langle b_1 \rangle \). Then by the same argument of \( a \) and \( b \), we have \( \langle m \rangle + \langle x_1 \rangle = S \) which implies \( m_1 + x_1 = 1_S \) for some \( m_1 \in \langle m \rangle \) and \( x_1' \in \langle x_1 \rangle \). Now let \( t \in \langle a_1 \rangle \). Then \( m_1 t + x_1' t = t \). Since \( x_1 t = 0 \), we have \( t = m_1 t \in M \). Thus \( \langle a_1 \rangle \) is simple. With the same argument, \( \langle b_1 \rangle \) is simple. Therefore \( |\text{Max}(S)| = 2 \), and so \( |\text{Max}(\Pi)| = 2 \).

Corollary 2.13([3], Proposition 3.3). Assume that \( R \) is not local. Then \( \text{diam}(R) = 2 \) if and only if one of the following holds:

(i) \( J(R) \) is a prime ideal.

(ii) \( |\text{Max}(R)| = 2 \) and \( R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

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