Coefficient Estimates in a Class of Strongly Starlike Functions

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Abstract. In this paper we consider some coefficient estimates in the subclass $SL^\ast$ of strongly starlike functions defined by a certain geometric condition.

1. Introduction

Let $H$ denote the class of analytic functions in the unit disc $U = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $A$ denote the subclass of $H$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. Everywhere in this paper $z \in U$ unless we make a note. We say that an analytic function $f$ is subordinate to an analytic function $g$, and write $f(z) \prec g(z)$, if and only if there exists a function $\omega$, analytic in $U$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$. In particular, if $g$ is univalent in $U$, we have the following equivalence

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subseteq g(U).$$

Let us denote $Q(f, z) = zf'(z)$. The class $SS^\ast(\beta)$ of strongly starlike functions of order $\beta$

$$SS^\ast(\beta) := \{f \in A : |\Arg Q(f, z)| < \beta\pi/2\}, \quad 0 < \beta \leq 1$$

was introduced in [5] and [1]. For $\beta = 1$ this class becomes the well known class $S^\ast$ of starlike functions. In this paper we consider the class $SL^\ast$:

$$(1) \quad SL^\ast := \{f \in A : |Q^2(f, z) - 1| < 1\}.$$

It is easy to see that $f \in SL^\ast$ if and only if $Q(f, z) < q_0(z) = \sqrt{1 + z}$, $q_0(0) = 1$. We observe that $L := \{w \in \mathbb{C} : \Re w > 0, |w^2 - 1| < 1\}$ is the interior of the right half of the lemniscate of Bernoulli $\gamma_1 : (x^2 + y^2)^2 - 2(x^2 - y^2) = 0$, see Figure 1. Moreover $L \subset \{w : |\Arg w| < \pi/4\}$, thus $SL^\ast \subset SS^\ast(1/2) \subset S^\ast$. The class $SL^\ast$ was introduced in [4] and there the authors give also the following representation formula.
\[ \gamma_1 : (x^2 + y^2)^2 - 2(x^2 - y^2) = 0 \]
\[ \text{Re } D = \text{Re } E = \frac{\sqrt{3}}{2} \]
\[ \text{Im } D = -\text{Im } E = \frac{1}{2} \]

**Theorem A([4]).** The function \( f \) belongs to the class \( \mathcal{SL}^* \) if and only if there exists an analytic function \( q \in \mathcal{H} \), \( q(0) = 0 \), \( q(z) \prec q_0(z) = \sqrt{1 + z} \), \( q_0(0) = 1 \) such that

\[ f(z) = z \exp \int_0^z \frac{q(t) - 1}{t} \, dt. \quad (2) \]

Let \( q_1(z) = \frac{3 + 2z}{3 + z} \), \( q_2(z) = \frac{5 + 3z}{5 + z} \), \( q_3(z) = \frac{8 + 4z}{8 + z} \). Because \( q_i(z) \prec q_0(z) \), \( i = 1, 2, 3 \), then by (2) we obtain that the functions \( f_1(z) = z + \frac{z^2}{3} \), \( f_2(z) = z(1 + \frac{z}{5})^2 \), \( f_3(z) = z(1 + \frac{z}{8})^3 \) are in \( \mathcal{SL}^* \). If we take \( q_0(z) = \sqrt{1 + z} \), \( q_0(0) = 1 \) then we obtain from (2) the function \( f_0 \)

\[ f_0(z) := \frac{4z \exp(2\sqrt{1 + z} - 2)}{(1 + \sqrt{1 + z})^2} = z + \frac{1}{2}z^2 + \frac{1}{16}z^3 + \frac{1}{96}z^4 - \frac{1}{128}z^5 + \cdots . \]

Rønning considered in [3] an analogously defined class connected with a parabolic region:

\[ \mathcal{S}_p^* := \{ f \in \mathcal{A} : \text{Re}[Q(f, z)] > |Q(f, z) - 1| \}. \]

Kanas and Wiśniowska introduced in [2] the concept of a \( k \)-starlike functions

\[ k - \mathcal{ST} := \{ f \in \mathcal{A} : \text{Re}[Q(f, z)] > k|Q(f, z) - 1| \}, \quad k \geq 0. \]

In this way they obtained a continuous passage from starlike functions \((k = 0)\) to the class \( \mathcal{S}_p^* \) \((k = 1)\). Moreover for \( 0 < k < 1 \) the quantity \( Q(f, z) \) takes its values...
in a convex domain on the right of a hyperbola while for $k > 1$ inside an ellipse. Let us consider the conic region $P(k) = \{ w \in \mathbb{C} : \text{Re} w > k|w - 1| \}$ connected with the class $k - ST$ described above. For $k > 1$ the curve $\partial P(k)$ is the ellipse $\gamma_2 : x^2 = k^2(x - 1)^2 + k^2y^2$. For $k \geq 2 + \sqrt{2}$ this ellipse lies entirely inside $\mathbb{C}$. Therefore $k - ST \subset \mathcal{S}L^*$, for $k \geq 2 + \sqrt{2}$.

2. Main results

Theorem 1. If the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to the class $\mathcal{S}L^*$, then

$$\sum_{k=2}^{\infty} (k^2 - 2)|a_k|^2 \leq 1.$$  

Proof. If $f \in \mathcal{S}L^*$, then $Q(f, z) \prec q_0(z) = \sqrt{1 + z}$. Hence $Q(f, z) = \sqrt{1 + \omega(z)}$, where $\omega$ satisfies $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$. Therefore $f^2(z) = (zf'(z))^2 - f^2(z)\omega(z)$ and using this we can obtain

$$2\pi \sum_{k=1}^{\infty} |a_k|^2 r^{2k} = \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \geq \int_0^{2\pi} |\omega(re^{i\theta})||f^2(re^{i\theta})| d\theta = \int_0^{2\pi} |(re^{i\theta}f'(re^{i\theta}))^2 - f^2(re^{i\theta})| d\theta \geq \int_0^{2\pi} |re^{i\theta}f'(re^{i\theta})|^2 - |f(re^{i\theta})|^2 d\theta = 2\pi \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k} - 2\pi \sum_{k=1}^{\infty} |a_k|^2 r^{2k}$$

for $0 < r < 1$. The extremes in this sequence of inequalities give

$$2 \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \geq \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k}, \quad 0 < r < 1.$$

Eventually, if we let $r \to 1^-$ then we obtain (4). \qed

Corollary 1. If the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to the class $\mathcal{S}L^*$, then $|a_k| \leq \sqrt{\frac{1}{k^2 - 2}}$ for $k \geq 2$.  

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Theorem 2. If the function \( f(z) = \sum_{k=1}^{\infty} a_k z^k \) belongs to the class \( \mathcal{SL}^* \), then

\[ |a_2| \leq 1/2, \quad |a_3| \leq 1/4, \quad |a_4| \leq 1/6. \]

Those estimations are sharp.

Proof. If \( f(z) = \sum_{k=1}^{\infty} a_k z^k \) belongs to the class \( \mathcal{SL}^* \) then \( [zf'(z)]^2 = f^2(z)[\omega(z) - 1] \), where \( \omega \) satisfies \( \omega(0) = 0, \quad |\omega(z)| < 1 \) for \( |z| < 1 \). Let us denote

\[ [zf'(z)]^2 = \sum_{k=2}^{\infty} A_k z^k, \quad f^2(z) = \sum_{k=2}^{\infty} B_k z^k, \quad \omega(z) = \sum_{k=1}^{\infty} C_k z^k. \]

Then we have

\[ A_k = \sum_{l=1}^{k-1} l(k-l)a_l a_{k-l}, \quad B_k = \sum_{l=1}^{k-1} a_l a_{k-l} \]

and

\[ \sum_{k=2}^{\infty} (A_k - B_k) z^k = \left[ \sum_{k=1}^{\infty} C_k z^k \right] \left[ \sum_{k=2}^{\infty} B_k z^k \right]. \]

Thus

\[ A_2 = a_1 = 1, \quad A_3 = 4a_1 a_2 = 4a_2, \quad A_4 = 6a_3 + 4a_2^2, \quad A_5 = 8a_1 a_4 + 12a_2 a_3 \]

and

\[ B_2 = a_1 = 1, \quad B_3 = 2a_2, \quad B_4 = 2a_3 + a_2^2, \quad B_5 = 2a_1 a_4 + 2a_2 a_3. \]

Equating the second, third and fourth coefficients of both sides of (8) we obtain

(i) \( A_3 - B_3 = C_1 B_2 \),

(ii) \( A_4 - B_4 = C_1 B_3 + C_2 B_2 \),

(iii) \( A_5 - B_5 = C_1 B_4 + C_2 B_3 + C_3 B_2 \).

So by (9), (10) we have

(j) \( a_2 = \frac{1}{2} C_1 \),

(jj) \( a_3 = \frac{1}{16} C_1^2 + \frac{1}{4} C_2 \),

(jjj) \( a_4 = \frac{1}{96} C_1^3 + \frac{1}{24} C_1 C_2 + \frac{1}{6} C_3 \).
It is well known that $|C_k| \leq 1$, $\sum_{k=1}^{\infty} |C_k|^2 \leq 1$ therefore we obtain (5). For the proof of sharpness let us consider $q(z) = \sqrt{1 + z^n}$. Using the representation formula (2) we obtain the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ such that $[zf'(z)]^2 = f^2(z)[z^n - 1]$ and with the notation (6) we have

$$\sum_{k=2}^{\infty} (A_k - B_k)z^k = \sum_{k=2}^{\infty} B_kz^{k+n}.$$ 

So $A_k = B_k$ for $k \leq n + 1$. This gives $a_1 = 1, a_2 = \cdots = a_n = 0$. While $A_{n+2} - B_{n+2} = B_2$ gives

$$\sum_{l=1}^{n+1} [(l(n+2) - l) - 1]a_l a_{n+2-l} = 1$$

thus $2n a_{n+1} = 1$. Therefore there exists a function $f$ in the class $\mathcal{SL}^*$ such that $f(z) = z + \frac{1}{2n} z^{n+1} + \cdots$. \hfill $\Box$

**Conjecture.** Let $f \in \mathcal{SL}^*$ and $f(z) = \sum_{k=1}^{\infty} a_k z^k$. Then $|a_{n+1}| \leq \frac{1}{2n}$.

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**References**


