**Langlands Functoriality Conjecture**

**JAE-HYUN YANG**

*Department of Mathematics, Inha University, Incheon 402-751, Korea*

e-mail: jhyang@inha.ac.kr

*To the memory of my parents*

**Abstract.** Functoriality conjecture is one of the central and influential subjects of the present day mathematics. Functoriality is the profound lifting problem formulated by Robert Langlands in the late 1960s in order to establish nonabelian class field theory. In this expository article, I describe the Langlands-Shahidi method, the local and global Langlands conjectures and the converse theorems which are powerful tools for the establishment of functoriality of some important cases, and survey the interesting results related to functoriality conjecture.

1. Introduction

Functoriality Conjecture or the Principle of Functoriality is the profound question that was raised and formulated by Robert P. Langlands in the late 1960s to establish nonabelian class field theory and its reciprocity law. Functoriality conjecture describes deep relationships among automorphic representations on different groups. This conjecture can be described in a rough form as follows: To every \( L \)-homomorphism \( \varphi : \mathcal{L}H \rightarrow \mathcal{L}G \) between the \( L \)-groups of \( H \) and \( G \) that are quasi-split reductive groups, there exists a *natural* lifting or transfer of automorphic representations of \( H \) to those of \( G \). In 1978, Gelbart and Jacquet [22] established an example of the functoriality for the symmetric square \( \text{Sym}^2 \) of \( GL(2) \) using the converse theorem on \( GL(3) \). In 2002 after 24 years, Kim and Shahidi [46] established the functoriality for the symmetric cube \( \text{Sym}^3 \) of \( GL(2) \) and thereafter Kim [43] proved the validity of the functoriality for the symmetric fourth \( \text{Sym}^4 \) of \( GL(2) \). These results have led to breakthroughs toward certain important conjectures in number theory, those of Ramanujan, Selberg and Sato-Tate conjectures. We refer to [81] for more detail on applications to the progress made toward to the conjectures just mentioned. Recently the Sato-Tate conjecture for an important class of cases related to elliptic curves has been verified by Clozel, Harris, Shepherd-Barron
Past ten years the functoriality for the tensor product $\text{GL}(2) \times \text{GL}(2) \rightarrow \text{GL}(4)$ by Ramakrishnan [71], for the tensor product $\text{GL}(2) \times \text{GL}(3) \rightarrow \text{GL}(6)$ by Kim and Shahidi [46], for the exterior square of $\text{GL}(4)$ by Kim [43] and the weak functoriality to $\text{GL}(N)$ for generic cuspidal representations of split classical groups $\text{SO}(2n+1)$, $\text{Sp}(2n)$ and $\text{SO}(2n)$ by Cogdell, Kim, Piatetski-Shapiro and Shahidi [13], [14] were established by applying appropriate converse theorems of Cogdell and Piatetski-Shapiro [15], [16] to analytical properties of certain automorphic $L$-functions arising from the Langlands-Shahidi method. In fact, the Langlands functoriality was established only for very special $L$-homomorphisms between the $L$-groups. It is natural to ask how to find a larger class of certain $L$-homomorphisms for which the functoriality is valid. It is still very difficult to answer this question.

The Arthur-Selberg trace formula has also provided some instances of Langlands functoriality (see [4], [6], [61], [62]). In a certain sense, it seems that the trace formula is a useful and powerful tool to tackle the functoriality conjecture. Nevertheless the incredible power of Langlands functoriality seems beyond present technology and knowledge. We refer the reader to [66] for Langlands’ comments on the limitations of the trace formula. Special cases of functoriality arises naturally from the conjectural theory of endoscopy (cf. [50]), in which a comparison of trace formulas would be used to characterize the internal structure of automorphic representations of a given group. I shall not deal with the trace formula, the base change and the theory of endoscopy in this article. Nowadays local and global Langlands conjectures are believed to be encompassed in the functoriality (cf. [65], [66]). Quite recently Khare, Larsen and Savin [41], [42] made a use of the functorial lifting from $\text{SO}(2n+1)$ to $\text{GL}(2n)$, from $\text{Sp}(2n)$ to $\text{GL}(2n+1)$ and the theta lifting of the exceptional group $G_2$ to $\text{Sp}(6)$ to prove that certain finite simple groups $\text{PSp}_n(F_{\ell^k})$, $G_2(F_{\ell^k})$ and $\text{SO}_{2n+1}(F_{\ell^k})$ with some mild restrictions appear as Galois groups over $\mathbb{Q}$.

This paper is organized as follows. In Section 2, we review the notion of automorphic $L$-functions and survey the Langlands-Shahidi method briefly following closely the article of Shahidi [77]. I would like to recommend to the reader two lecture notes which were very nicely written by Cogdell [12] and Kim [44] for more information on automorphic $L$-functions. In Section 3, we review the Weil-Deligne group briefly and formulate the local Langlands conjecture. We describe the recent results about the local Langlands conjecture for $\text{GL}(n)$ and $\text{SO}(2n+1)$. In Section 4, we discuss the global Langlands conjecture which is still not well formulated in the number field case. The work on the global Langlands conjecture for $\text{GL}(2)$ over a function field done by Drinfeld was extended by Lafforgue ten years ago to give a proof of the global Langlands conjecture for $\text{GL}(n)$ over a function field. We will not deal with the function field case in this article. We refer to [52] for more detail. Unfortunately there is very little known of the global Langlands conjecture in the number field case. I have an audacity to mention the Langlands hypothetical group and the hypothetical motivic Galois group following the line of Arthur’s argument in [3]. In Section 5, I formulate the Langlands functoriality conjecture in several
ways and describe the striking examples of Langlands functoriality established past ten years. I want to mention that there is a descent method of studying the opposite direction of the lift initiated by Ginzburg, Rallis and Soudry (see [25], [39]). I shall not deal with the descent method here. In the appendix, I describe a brief history of the converse theorems obtained by Hamburger, Hecke, Weil, Cogdell, Piatetski-Shapiro, Jinag and Soudry. I present the more or less exact formulations of the converse theorems. As mentioned earlier, the converse theorems play a crucial role in establishing the functoriality for the examples discussed in Section 5.

Notations: We denote by \( \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by \( \mathbb{R}^*_+ \) the multiplicative group of positive real numbers. \( \mathbb{C}^* \) (resp. \( \mathbb{R}^* \)) denotes the multiplicative group of nonzero complex (resp. real) numbers. We denote by \( \mathbb{Z} \) and \( \mathbb{Z}^+ \) the ring of integers and the set of all positive integers respectively. For a number field \( F \), we denote by \( \mathcal{O}_F \) the ring of integers and \( \mathcal{O}_F^* \) the multiplicative group of nonzero complex (resp. real) numbers. We denote by \( \mathcal{O}_k = \mathcal{O}_F^* \mathcal{O}_F \) the ring of adeles of \( F \) and the multiplicative group of ideles of \( F \) respectively. If there is no confusion, we write simply \( \mathcal{O} \) and \( \mathcal{O}^* \) instead of \( \mathcal{O}_F \) and \( \mathcal{O}_F^* \). For a field \( k \) we denote by \( \Gamma_k \) the Galois group \( \text{Gal}(\overline{k}/k) \), where \( \overline{k} \) is a separable algebraic closure of \( k \). We denote by \( G_m \) the multiplicative group in one variable. \( G_\alpha \) denotes the additive group in one variable.

2. Automorphic \( L \)-functions

Let \( G \) be a connected, quasi-split reductive group over a number field \( F \). For each place \( v \) of \( F \), we let \( F_v \) be the completion of \( F \), \( \mathfrak{o}_v \) the rings of integers of \( F_v \), \( p_v \) the maximal ideal of \( \mathfrak{o}_v \), and let \( q_v \) be the order of the residue field \( k_v = \mathfrak{o}_v/p_v \). We denote by \( \mathcal{A} = \mathcal{A}_F \) the ring of adeles of \( F \). We fix a Borel subgroup \( B \) of \( G \) over \( F \). Write \( B = TU \), where \( T \) is a maximal torus and \( U \) is the unipotent radical, both over \( F \). Let \( P \) be a parabolic subgroup of \( G \). Assume \( P \supset B \). Let \( P = MN \) be a Levi decomposition of \( P \) with Levi factor \( M \) and its unipotent radical \( N \). Then \( N \subseteq U \). For each place \( v \) of \( F \), we let \( G_v = G(F_v) \). Similarly we use \( B_v, T_v, U_v, P_v, M_v \) and \( N_v \) to denote the corresponding groups of \( F_v \)-rational points. Let \( G(\mathcal{A}), B(\mathcal{A}), \ldots, N(\mathcal{A}) \) be the corresponding adelic groups for the subgroups defined before. When \( G \) is unramified over a place \( v \) in the sense that \( G \) is quasi-split over \( F_v \) and that \( G \) is split over a finite unramified extension of \( F_v \), we let \( K_v = G(\mathfrak{o}_v) \). Otherwise we shall fix a special maximal compact subgroup \( K_v \subset G_v \). We set \( K_\mathcal{A} = \otimes_v K_v \). Then \( G(\mathcal{A}) = P(\mathcal{A})K_\mathcal{A} \).

Let \( \Pi = \otimes_v \Pi_v \) be a cuspidal automorphic representation of \( G(\mathfrak{o}_v) \). We refer to [36], [37], [59] for the notion of automorphic representations. Let \( S \) be a finite set of places including all archimedean ones such that both \( \Pi_v \) and \( G_v \) are unramified for any place \( v \notin S \). Then for each \( v \notin S \), \( \Pi_v \) determines uniquely a semi-simple conjugacy class \( c(\Pi_v) \) in the \( L \)-group \( ^L G_v \) of \( G_v \) as a group defined over \( F_v \). We refer to [9], [48] for the definition and construction of the \( L \)-group. We note that there exists a natural homomorphism \( \xi_v : \mathfrak{n}_G \longrightarrow \mathfrak{n}_G \). For a finite dimensional representation \( r \) of \( ^L G \), putting \( r_v = r \circ \xi_v \), the local Langlands \( L \)-function \( L(s, \Pi_v, r_v) \)
associated to \( \Pi_v \) and \( r_v \) is defined to be (cf. [9], [54])

\[
(2.1) \quad L(s, \Pi_v, r_v) = \det \left( I - r_v(c(\Pi_v))q_v^{-s} \right)^{-1}.
\]

We set

\[
(2.2) \quad L_S(s, \Pi, r) = \prod_{v \in S} L(s, \Pi_v, r_v).
\]

Langlands (cf. [54]) proved that \( L_S(s, \Pi, r) \) converges absolutely for sufficiently large \( \text{Re}(s) > 0 \) and defines a holomorphic function there. Furthermore he proposed the following question.

**Conjecture A** (Langlands, [54]). \( L_S(s, \Pi, r) \) has a meromorphic continuation to the whole complex plane and satisfies a standard functional equation.

F. Shahidi (cf. [77], [78]) gave a partial answer to the above conjecture using the so-called Langlands-Shahidi method. I shall describe Shahidi’s results briefly following his article [77].

Let \( P \) be a maximal parabolic subgroup of \( G \) and \( P = MN \) its Levi decomposition with its Levi factor \( M \) and its unipotent radical \( N \). Let \( A \) be the split torus in the center of \( M \). For every group \( H \) defined over \( F \), we let \( X(H)_F \) be the group of \( F \)-rational characters of \( H \). We set

\[
a = \text{Hom}(X(M)_F, \mathbb{R}).
\]

Then

\[
a^\ast = X(M)_F \otimes \mathbb{Z} \mathbb{R} \cong X(A)_F \otimes \mathbb{Z} \mathbb{R}.
\]

We set \( a_c^\ast := a^\ast \otimes_{\mathbb{R}} \mathbb{C} \). Let \( \mathfrak{z} \) be the real Lie algebra of the split torus in the center \( C(G) \) of \( G \). Then \( \mathfrak{z} \subset a \) and \( a/\mathfrak{z} \) is of dimension 1. The imbedding \( X(M)_F \hookrightarrow X(M)_F^\ast \) induces an imbedding \( a_v \hookrightarrow a \), where \( a_v = \text{Hom}(X(M)_F, \mathbb{R}) \). The Harish-Chandra homomorphism \( H_P : M(\mathbb{A}) \longrightarrow a \) is defined by

\[
\exp\langle \chi, H_P(m) \rangle = \prod_v |\chi(m_v)|_v, \quad \chi \in X(M)_F, \quad m = \otimes_v m_v \in M(\mathbb{A}).
\]

We may extend \( H_P \) to \( G(\mathbb{A}) \) by letting it trivial on \( N(\mathbb{A}) \) and \( K_\mathbb{A} \). We define \( H_{P_v} : M_v \longrightarrow a_v \) by

\[
q_v^{(\chi_v, H_{P_v}(m))} = |\chi_v(m_v)|_v, \quad \chi_v \in X(M)_{F_v}, \quad m_v \in M_v
\]

for a finite place \( v \), and define

\[
\exp\langle \chi_v, H_{P_v}(m) \rangle = |\chi_v(m_v)|_v, \quad \chi_v \in X(M)_{F_v}, \quad m_v \in M_v
\]
for an infinite place \( v \). Then we have

\[
(2.3) \quad \exp(\chi, H_P(m)) = \prod_{v < \infty} \exp \langle \chi, H_{P_v}(m) \rangle \cdot \prod_{v > \infty} q_v^{\langle \chi, H_{P_v}(m) \rangle},
\]

where \( \chi \in X(M)_F \) and \( m = \otimes_v m_v \in M(\mathcal{A}) \). We observe that the product in (2.3) is finite.

Let \( A_0 \) be the maximal \( F \)-split torus in \( T \). We denote by \( \Phi \) the set of roots of \( A_0 \). Then \( \Phi = \Phi^+ \cup \Phi^- \), where \( \Phi^+ \) is the set of roots generating \( U \) and \( \Phi^- = -\Phi^+ \).

Let \( \Delta \subset \Phi^+ \) be the set of simple roots. The unique reduced root of \( A \) in \( N \) can be identified by an element \( \alpha \in \Delta \). Let \( \rho_p \) be half the sum of roots generating \( N \). We set

\[
\hat{\alpha} = (\rho_p, \alpha)^{-1} \rho_p.
\]

Here, for any pair of roots \( \alpha \) and \( \beta \) in \( \Phi^+ \), the pairing \( \langle \alpha, \beta \rangle \) is defined as follows. Let \( \Phi^\prime \) be the set of non-restricted roots of \( T \) in \( U \). We see that the set of simple roots \( \bar{\Delta} \) in \( \Phi^\prime \) restricts to \( \Delta \). Identifying \( \alpha \) and \( \beta \) with roots in \( \Phi^\prime \), we set

\[
\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{\langle \beta, \beta \rangle},
\]

where \( (\ , \ ) \) is the standard inner product on \( \mathbb{R}^l \) with \( l = |\bar{\Delta}| \).

Let \( \pi = \otimes_v \pi_v \) be a cuspidal automorphic representation of \( M(\mathcal{A}) \). Given a \( K_\mathcal{A} \cap M(\mathcal{A}) \)-finite function \( \phi \) in the representation space \( V_\pi \) of \( \pi \), we may extend \( \phi \) to a function \( \tilde{\phi} \) on \( G(\mathcal{A}) \) (cf. [75]). Then the Eisenstein series \( E(s, \tilde{\phi}, g, P) \) is defined by

\[
(2.4) \quad E(s, \tilde{\phi}, g, P) = \sum_{\gamma \in P(F) \setminus G(F)} \tilde{\phi}(\gamma g) \exp(s\hat{\alpha} + \rho_p, H_P(\gamma g)), \quad g \in G(\mathcal{A}).
\]

The Eisenstein series \( E(s, \tilde{\phi}, g, P) \) converges absolutely for sufficiently large \( \text{Re}(s) \gg 0 \) and extends to a meromorphic function of \( s \) on \( \mathbb{C} \), with a finite number of poles in the plane \( \text{Re}(s) > 0 \), all simple and on the real axis (See [58]).

Let \( W \) be the Weyl group of \( A_0 \) in \( G \). We denote the subset of \( \Delta \) which generates \( M \) by \( \theta \). Then \( \Delta = \theta \cup \{ \alpha \} \). Then there exists a unique element \( \tilde{\omega} \in W \) such that \( \tilde{\omega}(\theta) \subset \Delta \) and \( \tilde{\omega}(\alpha) \in \Phi^- \). Fix a representative \( w \in K_\mathcal{A} \cap G(F) \) for \( \tilde{\omega} \). We shall also denote every component of \( w \) by \( w \) again.

Let

\[
I(s, \pi) = \text{Ind}_{M(\mathcal{A}) N(\mathcal{A})}^{G(\mathcal{A})} \pi \otimes \exp(s\hat{\alpha}, H_{P}(\cdot)) \otimes 1
\]

be the representation of \( G(\mathcal{A}) \) induced from \( P(\mathcal{A}) \). Then \( I(s, \pi) = \otimes_v I(s, \pi_v) \) with

\[
I(s, \pi_v) = \text{Ind}_{M_v N_v}^{G_v} \pi_v \otimes q_v^{\langle s\hat{\alpha}, H_{P_v}(\cdot) \rangle} \otimes 1,
\]

where \( q_v \) should be replaced by \( \exp \) if \( v \) is archimedean. We let \( M' \) be the subgroup of \( G \) generated by \( \tilde{\omega}(\theta) \). Then there is a parabolic subgroup \( P' \supset B \) with \( P' = M' N' \).
Here $M'$ is a Levi factor of $P'$ and $N'$ is the unipotent radical of $P'$. For $f \in I(s, \pi)$ and sufficiently large $\text{Re}(s) \gg 0$, we define

\begin{equation}
M(s, \pi) f(g) = \int_{N'(\mathbb{A})} f(w^{-1}ng) dn, \quad g \in G(\mathbb{A}).
\end{equation}

At each place $v$, for sufficiently large $\text{Re}(s) \gg 0$, we define a local intertwining operator by

\begin{equation}
A(s, \pi_v, w) f_v(g) = \int_{N'_v} f_v(w^{-1}ng) dn, \quad g \in G_v,
\end{equation}

where $f_v \in I(s, \pi_v)$. Then

\begin{equation}
M(s, \pi) = \bigotimes_v A(s, \pi_v, w).
\end{equation}

It follows from the theory of Eisenstein series that for $\text{Re}(s) \gg 0$, $M(s, \pi)$ extends to a meromorphic function of $s \in \mathbb{C}$ with only a finite number of simple poles (cf. [58]).

Let $L_M$ and $L_N$ be the Levi factor and the unipotent radical of the parabolic subgroup $L_P = L_M L_N$ of the $L$-group $L_G$. Then we have the representation $r : L_M \longrightarrow \text{End}(L_n)$ given by the adjoint action of $L_M$ on the Lie algebra $L_n$ of $L_N$. Let

\begin{equation}
\begin{array}{c}
\text{be the decomposition of } r \text{ into irreducible constituents. Each irreducible constituent } (r_i, V_i) \text{ with } 1 \leq i \leq m \text{ is characterized by }
\end{array}
\end{equation}

\begin{equation}
V_i = \{ X_{\beta} \in L_n | (\tilde{\alpha}, \beta) = i \}, \quad i = 1, \cdots, m.
\end{equation}

We refer to [55] and [77, Proposition 4.1] for more detail.

According to [53] and [55], one has

\begin{equation}
M(s, \pi) f = \left( \bigotimes_{v \in S} A(s, \pi_v, w) f_v \right) \bigotimes \left( \bigotimes_{v \notin S} \tilde{f}_v \right) \times \prod_{i=1}^{m} \frac{L_S(is, \pi, r_i)}{L_S(1 + is, \pi, r_i)},
\end{equation}

where $f = \bigotimes_v f_v$ is an element in $I(s, \pi)$ such that for each $v \notin S$, $f_v$ is the unique $K_v$-fixed vector with $f_v(e_v) = 1$, $\tilde{f}_v$ is the $K_v$-fixed vector in $I(-s, \tilde{\omega}(\pi_v))$ with $\tilde{f}_v(e_v) = 1$, and $\tilde{r}_i$ denotes the contragredient of $r_i (1 \leq i \leq m)$.

For every archimedean place $v$ of $F$, let $\varphi_v : W_{F_v} \longrightarrow L_{M_v}$ be the corresponding homomorphism (cf. [63]) attached to $\pi_v$. One has a natural homomorphism $\eta_v : L_{M_v} \longrightarrow L_M$. We put

\begin{equation}
r_{i,v} = r_i \circ \eta_v, \quad i = 1, 2, \cdots, m.
\end{equation}
Then \( r_i \circ \varphi_v = r_i \circ \eta_v \circ \varphi_v \) is a finite dimensional representation of the Weil group \( W_F \) on \( V_i \). Let \( L(s, r_i \circ \varphi_v) \) be the corresponding Artin \( L \)-function attached to \( r_i \circ \varphi_v \) (cf. [57]). We set

\[
L^S(s, \pi, r_i) = \prod_{v=\infty} L(s, r_i \circ \varphi_v) \cdot \prod_{v \notin \infty} L(s, \pi_v, r_i \circ \eta_v).
\]

Let \( \rho : M(\mathbb{A}) \rightarrow \overline{M}(\mathbb{A}) \) be the projection of \( M(\mathbb{A}) \) onto its adjoint group.

Shahidi showed the following.

**Theorem 2.1** (Shahidi [77]). Let \( \pi = \otimes_v \pi_v \) be a cuspidal automorphic representation of \( \overline{M}(\mathbb{A}) \). Then every \( L \)-function \( L^S(s, \pi, r_i \circ \rho^L) \), \( 1 \leq i \leq m \), extends to a meromorphic function of \( s \) to the whole complex plane. Moreover, if \( \pi \) is generic, then each \( L^S(s, \pi, r_i \circ \rho^L) \) satisfies a standard functional equation, that is,

\[
L^S(s, \pi, r_i \circ \rho^L) = \varepsilon_S(s, \pi, r_i \circ \rho^L) L^S(1 - s, \pi, r_i \circ \rho^L),
\]

where \( \varepsilon_S(1s, \pi, r_i \circ \rho^L) \) is the root number attached to \( \pi \) and \( r_i \circ \rho^L \), and \( \overline{\tau} \) denotes the contragredient of a representation \( \tau \).

Furthermore, for a given generic cuspidal automorphic representation \( \pi = \otimes_v \pi_v \) of \( M(\mathbb{A}) \), Shahidi defined the local \( L \)-functions \( L(s, \pi_v, r_i) \) and the local root numbers \( \varepsilon(s, \pi_v, r_i, \psi) \) with \( 1 \leq i \leq m \) at bad places \( v \) so that the completed \( L \)-function \( L(s, \pi, r_i) \) and the completed root number \( \varepsilon(s, \pi, r_i) \) defined by

\[
L(s, \pi, r_i) = \prod_{v: \text{all}} L(s, \pi_v, r_i), \quad \varepsilon(s, \pi, r_i) = \prod_{v: \text{all}} \varepsilon(s, \pi_v, r_i, \psi), \quad i = 1, \ldots, m
\]

satisfy a standard functional equation

\[
L(s, \pi, r_i) = \varepsilon(s, \pi, r_i) L(1 - s, \pi, r_i), \quad i = 1, \ldots, m.
\]

**Example 2.2** (Kim-Shahidi [45]). Let \( F \) be a number field and let \( G \) be a simply connected semisimple split group of type \( G_2 \) over \( F \). We set \( K_\infty = \prod_{v=\infty} F_v \). Let \( K_\infty \) be the standard maximal compact subgroup of \( G(K_\infty) \) and \( K_v = G(F_v) \) for a finite place \( v \). Then \( K = K_\infty \times \prod_{v=\infty} K_v \) is a maximal compact subgroup of \( G(\mathbb{A}) \). Fix a split maximal torus \( T \) in \( G \) and let \( B = TU \) be a Borel subgroup of \( G \). In what follows the roots are those of \( T \) in \( U \). Let \( \Delta = \{ \beta_1, \beta_0 \} \) be a basis of the root system \( \Phi \) with respect to \( (T, B) \) with the long simple root \( \beta_1 \) and the short one \( \beta_0 \). Then the other roots are given by

\[
\beta_2 = \beta_1 + \beta_0, \quad \beta_3 = 2\beta_1 + 3\beta_0, \quad \beta_4 = \beta_1 + 2\beta_0, \quad \beta_5 = \beta_1 + 3\beta_0.
\]

Let \( P \) be the maximal parabolic subgroup of \( G \) generated by \( \beta_1 \) with Levi decomposition \( P = MN \), where \( M \simeq GL(2) \). See [80, Lemma 2.1]. Thus one has

\[
a^* = \mathbb{R} \beta_1, \quad a = \mathbb{R} \beta_1' \quad \text{and} \quad \rho_P = \frac{5}{2} \beta_1.
\]
Let $\tilde{\alpha} = \beta_4$. Then $s\tilde{\alpha} (s \in \mathbb{C})$ corresponds to the character $|\det(m)|^s$. We note that $\mathbb{A}^* = \mathbb{A}_1^* \cdot \mathbb{R}_+^*$, where $\mathbb{A}_1^*$ is the group of ideles of norm 1. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $M(\mathbb{A}) = GL(2, \mathbb{A})$. We may and will assume that the central character $\omega_{\pi}$ of $\pi$ is trivial on $\mathbb{R}_+^*$. For a $K$-finite function $\varphi$ in the representation space of $\pi$, the Eisenstein series $E(s, \tilde{\varphi}, g) = E(s, \tilde{\varphi}, g, P)$ defined by Formula (2.4) converges absolutely for sufficiently large $\text{Re}(s) \gg 0$ and extends to a meromorphic function of $s$ on $\mathbb{C}$, with a finite number of poles in the plane $\text{Re}(s) > 0$, all simple and on the real axis. The discrete spectrum $L^2_{\text{disc}}(G(F) \backslash G(\mathbb{A}))_{(M, \pi)}$ is spanned by the residues of Eisenstein series for $\text{Re}(s) > 0$ (See [58]). We know that the poles of Eisenstein series coincide with those of its constant terms. So it is enough to consider term along $P$. For each $f \in I(s, \pi)$, the constant term of $E(s, f, g)$ along $P$ is given by

$$E_0(s, f, g) = \sum_{w \in \Omega} M(s, \pi, w)f(g), \quad \Omega = \{1, s_6s_1s_6s_1s_6\},$$

where $s_i$ is the reflection along $\beta_i$ defined by

$$s_i(\beta) = \beta - \frac{2(\beta, \beta_i)}{(\beta_i, \beta_i)}\beta_i, \quad 1 \leq i \leq 6, \beta \in \Phi.$$

Weyl group representatives are all chosen to lie in $K_\mathbb{A} \cap G(F)$. Here

$$M(s, \pi, w)f(g) = \int_{N_w^{-1}(\mathbb{A})} f(w^{-1}ng)dn = \prod_v \int_{N_w^{-1}(F_v)} f_v(w_v^{-1}n_vg_v)dn_v,$$

where $g = \otimes_v g_v \in G(\mathbb{A})$, $f = \otimes_v f_v$ is an element of $I(s, \pi)$ such that $f_v$ is the unique $K_v$-fixed function normalized by $f_v(e_v) = 1$ for almost all $v$, and

$$N_w^{-1} = \prod_{w \prec \alpha < 0} U_\alpha, \quad U_\alpha = \text{the one parameter unipotent subgroup}.$$

Let $\text{St} : GL(2, \mathbb{C}) \longrightarrow GL(2, \mathbb{C})$ be the standard representation of $GL(2, \mathbb{C})$ and $\text{Sym}^3 : GL(2, \mathbb{C}) = L^2 \longrightarrow GL(4, \mathbb{C})$ be the third symmetric power representation of $GL(2, \mathbb{C})$. Let

$$(\text{Sym}^3)^0 = \text{Sym}^3 \boxtimes (\wedge^2 \text{St})^{-1}$$

be the adjoint cube representation of $GL(2, \mathbb{C})$ (cf. [80, p. 249]). Then the adjoint representation $r$ of $L^2 = GL(2, \mathbb{C})$ on the Lie algebra $L^2 \mathfrak{n}$ of $L^2 \mathfrak{N}$ is given by

$$r = (\text{Sym}^3)^0 \oplus \wedge^2 \text{St}.$$
According to Formula (2.8), one obtain, for $w = s_6s_1s_6s_1$, the formula

$$M(s, \pi)f = \left( \bigotimes_{v \in S} M(s, \pi_v, w)f_v \right) \bigotimes_{v \notin S} \tilde{f}_v$$

$$\times L_S(s, \tilde{\pi}, (\text{Sym}^3(\pi))^0) L_S(2s, \tilde{\pi}, \wedge^2 \text{St})$$

$$\times L_S(1 + s, \tilde{\pi}, (\text{Sym}^3(\pi))^0)^{-1} L_S(1 + 2s, \tilde{\pi}, \wedge^2 \text{St})^{-1},$$

where $S$ is a finite set of places of $F$ including all the archimedean places such that $\pi_v$ is unramifies for every $v \notin S$. Here $L_S(s, \pi, \wedge^2 \text{St})$ is the partial Hecke $L$-function. Kim and Shahidi [45] proved that if $\pi$ is a non-monomial cuspidal representation of $M(\mathbb{A}) = GL(2, \mathbb{A})$ in the sense that $\pi \not\sim \pi \otimes \eta$ for any nontrivial grossencharacter $\eta$ of $F^* \backslash \mathbb{A}^*_F$, the symmetric cube $L$-function $L(s, \pi, \text{Sym}^3(\pi))$ and the adjoint cube $L$-function $L(s, \pi, (\text{Sym}^3(\pi))^0)$ are both entire and satisfy the standard functional equations

$$L(s, \pi, \text{Sym}^3(\pi)) = \varepsilon(s, \pi, \text{Sym}^3(\pi)) L(1 - s, \tilde{\pi}, \text{Sym}^3(\pi))$$

and

$$L(s, \pi, (\text{Sym}^3(\pi))^0) = \varepsilon(s, \pi, (\text{Sym}^3(\pi))^0) L(1 - s, \tilde{\pi}, (\text{Sym}^3(\pi))^0).$$

It follows from this fact that if $\pi$ is not monomial, the partial Rankin triple $L$-function $L_S(s, \pi \times \pi \times \pi)$ is entire. Ikeda [32] calculated the poles of the Rankin triple $L$-function $L_S(s, \pi \times \pi \times \pi)$ for $GL(2)$. And we have the following relations

$$L_S(s, \pi \times \pi \times \pi) = L(s, \pi, \text{Sym}^3(\pi)) (L_S(s, \pi \otimes \omega_\pi))^2$$

and

$$L(s, \pi, \text{Sym}^3(\pi)) = L_S(s, \pi \otimes \omega_\pi, (\text{Sym}^3(\pi))^0).$$

According to Formula (2.10), $L_S(s, \pi \times \pi \times \pi)$ could have double zeros at $s = 1/2$.

In [47], Kim and Shahidi studied the cuspidality of the symmetric fourth power $\text{Sym}^4(\pi)$ of a cuspidal representation $\pi$ of $GL(2, \mathbb{A})$ and the partial symmetric $m$-th power $L$-functions $L_S(s, \pi, \text{Sym}^m(\pi)) (1 \leq m \leq 9)$. For the definition of $\text{Sym}^m(\pi)$, we refer to Example 5.6 in this article. We summarize their results.

**Theorem 2.3 (Kim-Shahidi [47]).** Let $\pi$ be a cuspidal automorphic representation of $GL(2, \mathbb{A})$ with $\omega_\pi$ its central character. Then $\text{Sym}^4(\pi) \otimes \omega_\pi^{-1}$ is a cuspidal representation of $GL(5, \mathbb{A})$ except for the following three cases:

1. $\pi$ is monomial in the sense that $\pi \cong \pi \otimes \eta$ for some nontrivial Grössencharacter $\eta$ of $F$.
2. $\pi$ is not monomial and $\text{Sym}^3(\pi) \otimes \omega_\pi^{-1}$ is not cuspidal.
3. $\text{Sym}^3(\pi) \otimes \omega_\pi^{-1}$ is cuspidal and there exists a nontrivial quadratic character $\chi$ such that

$\text{Sym}^3(\pi) \otimes \omega_\pi^{-1} \cong \text{Sym}^3(\pi) \otimes \omega_\pi^{-1} \otimes \chi.$
As applications of Theorem 2.3, they obtained the following.

**Proposition 2.4** (Kim-Shahidi [47]). Let $\pi$ be a cuspidal automorphic representation of $GL(2, \mathbb{A})$ with $\omega_\pi$ its central character such that $\text{Sym}^4(\pi)$ is cuspidal. Then the following statements hold:

(a) Each partial symmetric $m$-th power $L$-functions $L_S(s, \pi, \text{Sym}^m)$ ($m = 6, 7, 8, 9$) has a meromorphic continuation and satisfies a standard functional equation.

(b) $L_S(s, \pi, \text{Sym}^3)$ and $L_S(s, \pi, \text{Sym}^7)$ are holomorphic and nonzero for $\text{Re}(s) \geq 1$.

(c) If $\omega_\pi^2 = 1$, $L_S(s, \pi, \text{Sym}^3)$ is holomorphic and nonzero for $\text{Re}(s) \geq 1$.

(d) If $\text{Sym}^4(\pi)$ is cuspidal, $L_S(s, \pi, \text{Sym}^6)$ is holomorphic and nonzero for $\text{Re}(s) \geq 1$.

(e) If $\text{Sym}^4(\pi)$ is cuspidal and $\omega_\pi^4 = 1$, $L_S(s, \pi, \text{Sym}^8)$ is holomorphic and nonzero for $\text{Re}(s) \geq 1$.

(f) If $\text{Sym}^4(\pi)$ is cuspidal, $L_S(s, \pi, \text{Sym}^9)$ has a most a simple pole or a simple zero at $s = 1$.

(g) If $\text{Sym}^4(\pi)$ is not cuspidal, $L_S(s, \pi, \text{Sym}^9)$ is holomorphic and nonzero for $\text{Re}(s) \geq 1$.

**Proposition 2.5** (Kim-Shahidi [47]). Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $GL(2, \mathbb{A})$ such that $\text{Sym}^3(\pi)$ is cuspidal. Let $\text{diag}(\alpha_v, \beta_v)$ be the Satake parameter for $\pi_v$. Then $|\alpha_v|, |\beta_v| < q_v^{1/9}$. If $\text{Sym}^4(\pi)$ is not cuspidal, then $|\alpha_v| = |\beta_v| = 1$, that is, the full Ramanujan conjecture holds.

**Proposition 2.6** (Kim-Shahidi [47]). Let $\pi = \otimes_v \pi_v$ be a nonmonomial cuspidal automorphic representation of $GL(2, \mathbb{A})$ with a trivial central character. Suppose $m \leq 9$. Then the following statements hold:

(1) Suppose $\text{Sym}^3(\pi)$ is not cuspidal. Then $L_S(s, \pi, \text{Sym}^m)$ is holomorphic and nonzero at $s = 1$, except for $m = 6, 8$; the $L$-functions $L_S(s, \pi, \text{Sym}^6)$ and $L_S(s, \pi, \text{Sym}^8)$ each have a simple pole at $s = 1$.

(2) Suppose $\text{Sym}^4(\pi)$ is cuspidal but $\text{Sym}^4(\pi)$ is not cuspidal. Then $L_S(s, \pi, \text{Sym}^m)$ is holomorphic and nonzero at $s = 1$ for $m = 1, \ldots, 7$ and $m = 9$; the $L$-function $L_S(s, \pi, \text{Sym}^9)$ has a simple pole at $s = 1$.

We are still far from solving Conjecture A. We have two known methods to study analytic properties of automorphic $L$-functions. The first is the method of constructing explicit zeta integrals that is called the Rankin-Selberg method. The second is the so-called Langlands-Shahidi method I just described briefly. In the late 1960s Langlands [55] recognized that many automorphic $L$-functions occur in the constant terms of the Eisenstein series associated to cuspidal automorphic representations of the Levi subgroups of maximal parabolic subgroups of split reductive groups through his intensively deep work on the theory of Eisenstein series. He obtained some analytic properties of certain automorphic $L$-functions using the meromorphic continuation and the functional equation of Eisenstein series. As mentioned above, he proved the meromorphic continuation of certain class of $L$-functions but
Langlands Functoriality Conjecture

3. Local Langlands conjecture

Let $k$ be a local field and let $W_k$ be its Weil group. We review the definition of the Weil group $W_k$ following the article of Tate (cf. [82]). If $k = \mathbb{C}$, then $W_{\mathbb{C}} = \mathbb{C}^\times$. If $k = \mathbb{R}$, then

$$W_{\mathbb{R}} = \mathbb{C}^\times \cup \tau \mathbb{C}^\times, \quad \tau z \tau^{-1} = \overline{z},$$

where $z \mapsto \overline{z}$ is the nontrivial element of $\text{Gal}(\mathbb{C}/\mathbb{R})$. Then $W_{\mathbb{R}}^{ab} = \mathbb{R}^\times$. Here if $Y^c$ denotes the closure of the commutator subgroup of a topological group $Y$, we set $Y^{ab} = Y/Y^c$.

Suppose $k$ is a nonarchimedean local field and $\overline{k}$ a separable algebraic closure of $k$. Let $q$ be the order of the residue field $\kappa$ of $k$. We set $\Gamma_k = \text{Gal}(\overline{k}/k)$ and $\Gamma_\kappa = \text{Gal}(\overline{\kappa}/\kappa)$. Let $\Phi_\kappa : x \mapsto x^q$ be the Frobenius automorphism in $\Gamma_\kappa$. We set $\langle \Phi_\kappa \rangle = \{ \Phi_\kappa^n \mid n \in \mathbb{Z} \}$. Let $\varphi : \Gamma_k \longrightarrow \Gamma_\kappa$ be the canonical surjective homomorphism given by $\sigma \mapsto \sigma|_\kappa$. The Weil group $W_k$ is defined to be the set $W_k = \varphi^{-1}(\langle \Phi_\kappa \rangle)$. Obviously one has an exact sequence

$$1 \longrightarrow I_k \longrightarrow W_k \longrightarrow \langle \Phi_\kappa \rangle \longrightarrow 1,$$

where $I_k = \ker \varphi$ is the inertia group of $k$. We recall that $W_k$ is topologized such that $I_k$ has the induced topology from $\Gamma_k$, $I_k$ is open in $\Gamma_k$ and multiplication by $\Phi$ is a homeomorphism. Here $\Phi$ denotes a choice of a geometric Frobenius element in $\varphi^{-1}(\Phi_\kappa) \subset \Gamma_k$. We note that we have a continuous homomorphism $W_k \longrightarrow \Gamma_k$ with dense image. According to the local class field theory, one has the isomorphism

$$k^* \cong W_k^{ab}. \quad (3.1)$$

In order to generalize the isomorphism (3.1) for $GL(1)$ to $GL(2)$, P. Deligne [18] introduced the so-called Weil-Deligne group $W'_k$. It is defined to be the group
scheme over $\mathbb{Q}$ which is the semidirect product of $W_k$ by the additive group $\mathbb{G}_a$ on which $W_k$ acts by the rule $wxw^{-1} = ||w||x$. We refer to [82, p. 19] for the definition of $||w||$. Namely, $W'_k = W_k \ltimes \mathbb{G}_a$ is the group scheme over $\mathbb{Q}$ with the multiplication

$$(w_1,a_1)(w_2,a_2) = (w_1w_2,a_1 + ||w_1||a_2), \quad w_1,w_2 \in W_k, \; a_1,a_2 \in \mathbb{G}_a.$$  

**Definition 3.1** (Deligne [19]). Let $E$ be field of characteristic 0. A representation of $W'_k$ over $E$ is a pair $\rho' = (\rho,N)$ consisting of

(a) A finite dimensional vector space $V$ over $E$ and a homomorphism $\rho : W_k \rightarrow GL(V)$ whose kernel contains an open subgroup of $I_k$, i.e., which is continuous for the discrete topology on $GL(V)$.

(b) A nilpotent endomorphism $N$ of $V$ such that $\rho(w)N\rho(w)^{-1} = ||w||N, \; w \in W_k$.

We see that a homomorphism of group schemes over $E$

$$\rho' : W_k \times E \rightarrow GL(V)$$

determines, and is determined by a pair $(\rho,N)$ as in Definition 3.1 such that

$$\rho'((w,a)) = \exp(aN)\rho(w), \quad w \in W_k, \; a \in \mathbb{G}_a.$$  

Let $\rho' = (\rho,N)$ be a representation of $W'_k$ over $E$. Define $\nu : W_k \rightarrow \mathbb{Z}$ by $||w|| = q^{-\nu(w)}, \; w \in W_k$. Then according to [19, (8.5)], there is a unique unipotent automorphism $u$ of $V$ such that

$$uN = Nu, \quad u\rho(w) = \rho(w)u, \quad w \in W_k$$

and such that

$$\exp(aN)\rho(w)u^{-\nu(w)}$$

is a semisimple automorphism of $V$ for all $a \in E$ and $w \in W_k - I_k$. Then $\rho' = (\rho u^{-\nu},N)$ is called the $\Phi$-semisimplication of $\rho'$. And $\rho'$ is called $\Phi$-semisimple if and only if $\rho' = \rho'_{ss}, \; u = 1$, i.e., the Frobeniuses acts semisimply.

Let $\rho' = (\rho,N,V)$ be a representation of $W'_k$ over $E$. We let $V_N^t := (\ker N)^t$ be the subspace of $I_k$-invariants in $\ker N$. We define a local $L$-factor by

$$(3.2) \quad Z(t,V) = \det \left(1 - t\rho(\Phi)|_{V'_k}\right)^{-1}, \quad L(s,V) = Z(q^{-s},V), \quad \text{when } E \subset \mathbb{C}.$$  

We note that if $\rho' = (\rho,N)$ is a representation of $W'_k$, then $\rho'$ is irreducible if and only if $N = 0$ and $\rho$ is irreducible.

Let $G$ be a connected reductive group over a local field. A homomorphism $\alpha : W_k \longrightarrow ^L G$ is said to be admissible [9, p. 40] if the following conditions (i)-(iii)
are satisfied
(i) $\alpha$ is a homomorphism over $\Gamma_k$, i.e., the following diagram is commutative:

$$
\begin{array}{ccc}
W_k' & \xrightarrow{\alpha} & L^G \\
\downarrow & & \downarrow \\
\Gamma_k & \xrightarrow{\phi} & G
\end{array}
$$

(ii) $\alpha$ is continuous, $\alpha(G_\alpha)$ are unipotent in $L^G$, and $\alpha$ maps semisimple elements into semisimple elements in $L^G$. Here an element $x$ is said to be semisimple if $x \in I_k$, and an element $g \in L^G$ is called semisimple if $r(g)$ is semisimple for any finite dimensional representation $r$ of $L^G$.

(iii) If $\alpha(W'_k)$ is contained in a Levi subgroup of a proper parabolic subgroup $P$ of $L^G$, then $P$ is relevant. See [9, p. 32].

Let $G_k(G)$ be the set of all admissible homomorphisms $\phi : W_k' \rightarrow L^G$ modulo inner automorphisms by elements of $L^G$. We observe that we can associate canonically to $\phi \in G_k(G)$ a character $\chi_\phi$ of the center $C(G)$ of $G$ (cf. [9, p. 43], [63]). Let $Z^0_k = C(L^G)$ be the center of $L^G$. Following [9, pp. 43-44] and [63], we can construct a character $\omega_\alpha$ of $G(k)$ associated to a cohomology class $\alpha \in H^1(W'_k, Z^0_k)$. If we write $\phi \in G_k(G)$ as $\phi = (\phi_1, \phi_2)$ with $\phi : W'_k \rightarrow L^G$ and $\phi : W'_k \rightarrow \Gamma_k$, then $\phi_1$ defines a cocycle of $W'_k$ in $L^G$, and the map $\phi \mapsto \phi_1$ yields an embedding $G_k(G) \hookrightarrow H^1(W'_k, L^G)$. Then $H^1(W'_k, L^G)$ acts on $H^1(W'_k, L^G)$ and this action leaves $G_k(G)$ stable [9, p. 40].

Let $\prod(G(k))$ be the set of all equivalence classes of irreducible admissible representations of $G(k)$. The following conjecture gives an arithmetic parametrization of irreducible admissible representations of $G(k)$.

**Local Langlands Conjecture [LLC].** Let $k$ be a local field. Let $G_k(G)$ and $\prod(G(k))$ be as above. Then there is a surjective map $\prod(G(k)) \rightarrow G_k(G)$ with finite fibres which partitions $\prod(G(k))$ into disjoint finite sets $\prod_\phi(G(k))$, simply $\prod_\phi$ called $L$-packets satisfying the following (i)-(v):

(i) If $\pi \in \prod_\phi$, then the central character $\chi_\pi$ of $\pi$ is equal to $\chi_\phi$.

(ii) If $\alpha \in H^1(W'_k, Z^0_k)$ and $\omega_\alpha$ is its associated character of $G(k)$, then

$$
\prod_{\alpha \phi} = \left\{ \pi \omega_\alpha \mid \pi \in \prod_\phi \right\}.
$$

(iii) One element of $\prod_\phi$ is square integrable modulo the center $C(G)$ of $G$ if and only if all elements are square integrable modulo the center $C(G)$ of $G$ if and only if $\phi(W'_k)$ is not contained in any proper Levi subgroup of $L^G$.

(iv) One element of $\prod_\phi$ is tempered if and only if all elements of $\prod_\phi$ are tempered if and only if $\phi(W'_k)$ is bounded.

(v) If $H$ is a connected reductive group over $k$ and $\eta : H(k) \rightarrow G(k)$ is a k-morphism with commutative kernel and cokernel, then there is a required compatibility between decompositions for $G(k)$ and $H(k)$. More precisely, $\eta$ induces a canonical map $L^\eta : L^G \rightarrow L^H$, and if we set $\phi' = L^\eta \circ \phi$ for $\phi \in G_k(G)$, then any $\pi \in \prod_{\phi'}(G(k))$, viewed as an $H(k)$-module, decomposes into a direct sum of finitely
many irreducible admissible representations belonging to \( \prod_{\phi}(H(k)) \).

**Remark 3.2.** (a) If \( k \) is archimedean, i.e., \( k = \mathbb{R} \) or \( \mathbb{C} \), [LLC] was solved by Langlands [63]. We also refer the reader to [1], [2], [49].

(b) In case \( k \) is non-archimedean, Kazhdan and Lusztig [40] had shown how to parametrize those irreducible admissible representations of \( G(k) \) having an Iwahori fixed vector in terms of admissible homomorphisms of \( W'_k \).

(c) For a local field \( k \) of positive characteristic \( p > 0 \), [LLC] was established by Laumon, Rapoport and Stuhler [67].

(d) In case \( G = GL(n) \) for a non-archimedean local field \( k \), [LLC] was established by Harris and Taylor [28], and by Henniart [31]. In both cases, the correspondence was established at the level of a correspondence between irreducible Galois representations and supercuspidal representations.

(e) Let \( k \) be a a non-archimedean local field of characteristic 0 and let \( G = SO(2n + 1) \) the split special orthogonal group over \( k \). In this case, Jiang and Soudry [38], [39] gave a parametrization of generic supercuspidal representations of \( SO(2n + 1) \) in terms of admissible homomorphisms of \( W'_k \). More precisely, there is a unique bijection of the set of conjugacy classes of all admissible, completely reducible, multiplicity-free, symplectic complex representations \( \phi : W'_k \to \mathbb{C} \) onto the set of all equivalence classes of irreducible generic supercuspidal representations of \( SO(2n + 1, k) \).

For \( \pi \in \prod_{\phi}(G) \) with \( \phi \in \mathcal{G}_k(G) \), if \( r \) is a finite dimensional complex representation of \( L \), we define the \( L \)- and \( \varepsilon \)-factors

\[
L(s, \pi, r) = L(s, r \circ \phi) \quad \text{and} \quad \varepsilon(s, \pi, r, \psi) = \varepsilon(s, r \circ \phi, \psi),
\]

where \( L(s, r \circ \phi) \) is the Artin-Weil \( L \)-function.

**Remark 3.3.** For a non-archimedean local field \( k \), Deligne [18] gave the complete formulation of [LLC] for \( GL(2) \). In [18], he utilized for the first time the Weil-Deligne group \( W'_k \), which was introduced by him in [19], in the context of \( \ell \)-adic representations, in order to obtain a correct formulation in the case of \( GL(2) \) over a non-archimedean local field.

**Remark 3.4.** The representations in the \( L \)-packet \( \prod_{\phi} \) are parametrized by the component group

\[
C_\phi := S_\phi / Z_L S_0^\phi,
\]

where \( S_\phi \) is the centralizer of the image of \( \phi \) in \( G \), \( S_0^\phi \) is the identity component of \( S_\phi \), and \( Z_L \) is the center of \( L \). We refer the reader to [5], [51] for more information on the \( L \)-packets.

**Example 3.5.** Let \( \pi \) be a spherical or unramified representation of \( G(k) \) with a non-archimedean local field \( k \). It is known that \( \pi \leftrightarrow I(\chi) \) for a unique unramified quasi-character \( \chi \) of a maximal torus \( T(k) \) of \( G(k) \). Then \( \pi \) determines a semi-simple
conjugate class \( c(\pi) = \{t_\pi\} \subset L^T \subset L^G \). Then the Langlands’ parameter \( \phi_\pi \) for \( \pi \) is
\[
\phi_\pi : k^* \longrightarrow L^T \subset L^G, \quad \phi_\pi(\tilde{\omega}) = t_\pi
\]
such that
\[
\phi_\pi(\tilde{\omega}) = t_\pi \quad \text{and} \quad \phi_\pi \text{ is trivial on } \mathcal{O}^*,
\]
where \( \tilde{\omega} \) denotes a uniformizer in \( k \).

If \( \pi = \pi(\mu_1, \ldots, \mu_n) \) is a spherical representation of \( GL_n(k) \) with unramified characters \( \mu_i (i = 1, \ldots, n) \) of \( k^* \), the semi-simple conjugacy class \( c(\pi) \) is given by
\[
c(\pi) = \{ \text{diag}(\mu_1(\tilde{\omega}), \ldots, \mu_n(\tilde{\omega}))\}
\]
and the Langlands’ parameter \( \phi_\pi \) for \( \pi \) is
\[
\phi_\pi : k^* \longrightarrow L^T \subset L^G = GL(n, \mathbb{C}), \quad \phi_\pi(\tilde{\omega}) = \text{diag}(\mu_1(\tilde{\omega}), \ldots, \mu_n(\tilde{\omega})).
\]

4. Global Langlands conjecture

Let \( k \) be a global field and \( \mathbb{A} \) its ring of adeles. This section is based on Arthur’s article [3].

As in the local case of Section 3, the global Langlands conjecture should be a nonabelian generalization of abelian global class field theory. When Deligne [19] recognized the need to introduce the Weil-Deligne group \( W'_k \) for the local Langlands correspondence for \( GL(2) \), it was realized that there seemed to be no natural global version of \( W'_k \). In fact, \( \Gamma_k, W_k \) and \( W'_k \) are too small to parameterize all automorphic representations of a reductive group. Thus in the 1970s Langlands [60] attempted to discover a hypothetical group \( L_k \) to replace the Weil-Deligne group \( W'_k \). Nowadays it is believed by experts that this group \( L_k \) should be related to the equally hypothetical motivic Galois group \( M_k \) of \( k \). The group \( L_k \) is often called the hypothetical (or conjectural) Langlands group or the automorphic Langlands group. The notion of \( L_k \) and \( M_k \) is still not clear.

The global Langlands conjecture can be written as follows.

**Global Langlands Conjecture(GLC).** Automorphic representations of \( G(\mathbb{A}) \) can be parametrized by admissible homomorphisms \( \phi : L_k \longrightarrow L^G \) required to have the following properties (1)-(4):

1. There is an \( L \)-packet \( \prod_\phi \) which consists of automorphic representations of \( G(\mathbb{A}) \) attached to \( \phi \).
2. Each \( L \)-packet \( \prod_\phi \) is a finite set.
3. Any automorphic representation of \( G(\mathbb{A}) \) belongs to \( \prod_\phi \) for a unique homomorphism \( \phi \).
4. The \( \prod_\phi \)'s are disjoint.

We first consider the case that \( k \) is a function field. Drinfeld [21] formulated a version of the global Langlands conjecture for function fields relating the irreducible
two dimensional representations of the Galois group $\Gamma_k$ with irreducible cuspidal representations of $GL(2, \mathbb{A})$, and established the global Langlands conjecture. In the early 2000s Lafforgue [52] had extended the work of Drinfeld mentioned above to $GL(n)$ to give a proof of the global Langlands conjecture for $GL(n)$ over a function field. The formulation of the global Langlands conjecture made by Drinfeld and Lafforgue is essentially the same as that in the local non-archimedean case discussed in Section 3 with a few modification. So we omit the details for the global Langlands conjecture over a function field here. We refer to [20], [21], [52] for more detail.

Next we consider the case $k$ is a number field. We first recall that according to the global class field theory, for $n = 1$, there is a canonical bijection between the continuous characters of $\Gamma_k$ and characters of finite order of the idele group $k^* \setminus \mathbb{A}^*$. We should replace $\Gamma_k$ by the Weil group $W_k$ in order to obtain all the characters of $k^* \setminus \mathbb{A}^*$. For $n \geq 2$, by analogy with the local Langlands conjecture, we need a global analogue of the Weil-Deligne group $W'_k$. However no such analogue is available at this moment. We hope that the hypothetical Langlands group $L_k$ plays a role as $W'_k$ and fits into an exact sequence

$$1 \longrightarrow L_k^c \longrightarrow L_k \longrightarrow \Gamma_k \longrightarrow 1,$$

where $L_k^c$ is a complex pro-reductive group. $L_k$ should be a locally compact group equipped with an embedding $i_v : L_{k_v} \longrightarrow L_k$ for each completion $k_v$ of $k$. Let $G$ be a connected, quasi-split reductive group over $k$. We set $G_v := G(k_v)$ for every place $v$ of $k$. Let $\mathcal{L}_k(G)$ be the set of all equivalence classes of continuous, completely reducible homomorphisms $\phi$ of $L_k$ into $^L_kG$, and $\mathcal{A}_k(G)$ the set of equivalence classes of all automorphic representations of $G(\mathbb{A})$. For each place $v$ of $k$, let $\mathcal{L}_k(G_v)$ be the set of equivalence classes of continuous, completely reducible homomorphisms $\phi_v : L_{k_v} \longrightarrow L_{G_v}$ and $\mathcal{A}_k(G_v)$ the set of continuous irreducible admissible representations of $G_v$. We would hope to have a bijection

$$\mathcal{L}_k(G) \longrightarrow \mathcal{A}_k(G), \quad \phi \mapsto \pi_\phi.$$

Moreover the set $\mathcal{L}_k^0(G)$ of equivalence classes of irreducible representations in $\mathcal{L}_k(G)$ should be in bijective correspondence with the set $\mathcal{A}_k^0(G)$ of all cuspidal automorphic representations in $\mathcal{A}_k(G)$. This would be supplemented by local bijection

$$\mathcal{L}_k(G_v) \longrightarrow \mathcal{A}_k(G_v) \quad \text{for any place $v$ of $k$.}$$

The local and global bijections should be compatible in the sense that for any $\phi : L_k \longrightarrow L_kG$, there is an automorphic representation $\pi_\phi = \otimes_v \pi_{\phi,v}$ of $G(\mathbb{A})$ with the correspondence $\phi \mapsto \pi_\phi$ such that for each place $v$ of $k$, the restriction $\phi_v = \phi \circ i_v$ of $\phi$ to $L_{k_v}$ corresponds to the local component $\pi_{\phi,v}$ of $\pi_\phi$. Of course, one expects that all of these correspondences (4.2) and (4.3) would satisfy properties similar to those in the local Langlands conjecture, e.g., the preservation of $L$- and $\varepsilon$-factors with twists, etc.
The local Langlands groups are elementary. They are given by

\[ L_{k_v} = \begin{cases} W_{k_v} & \text{if } v \text{ is archimedean}, \\ W_{k_v} \times SU(2, \mathbb{R}) & \text{if } v \text{ is nonarchimedean}, \end{cases} \]

where \( W_{k_v} \) is again the Weil group of \( k_v \). Thus the local Langlands group \( L_{k_v} \) is a split extension of \( W_{k_v} \) by compact simply connected Lie group. But the hypothetical Langlands group will be much larger. It would be an infinite fibre product of nonsplit extension

(4.4) \[ 1 \rightarrow K_c \rightarrow L_c \rightarrow W_k \rightarrow 1 \]

of the Weil group \( W_k \) by a compact, semisimple, simply connected Lie group \( K_c \). However one would have to establish something in order to show that \( L_k \) has all the desired properties.

Grothendieck’s conjectural theory of motives introduces the so-called motivic Galois group \( \mathcal{M}_k \), which is a reductive proalgebraic group over \( \mathbb{C} \) and comes with a proalgebraic projection \( \mathcal{M}_k \rightarrow \Gamma_k \). A motive of rank \( n \) is to be defined as a proalgebraic representation

\[ \mathbb{M} : \mathcal{M}_k \rightarrow GL(n, \mathbb{C}). \]

We observe that any continuous representation of \( \Gamma_k \) pulls back to \( \mathcal{M}_k \) and can be viewed as a motive in the above sense. It is conjectured that the arithmetic information in any motive \( \mathbb{M} \) is directly related to analytic information from some automorphic representations of \( G(\mathbb{A}) \). The conjectural theory of motives also applies to any completion \( k_v \) of \( k \). It produces a proalgebraic group \( \mathcal{M}_{k_v} \) over \( \mathbb{C} \) that fits into a commutative diagram

\[ \begin{array}{ccc} \mathcal{M}_{k_v} & \rightarrow & \mathcal{M}_k \\ \downarrow & & \downarrow \\ \Gamma_{k_v} & \leftarrow & \Gamma_k \end{array} \]

of proalgebraic homomorphisms.

In 1979, Langlands [60] speculated the following:

**Conjecture B** (Langlands [60, Section 2]). There is a commutative diagram

\[ \begin{array}{ccc} L_k & \phi \rightarrow & \mathcal{M}_k \\ \downarrow & & \downarrow \\ \Gamma_k & \leftarrow & \Gamma_k \end{array} \]

together with a compatible commutative diagram

\[ \begin{array}{ccc} L_{k_v} & \phi \rightarrow & \mathcal{M}_{k_v} \\ \downarrow & & \downarrow \\ \Gamma_k & \leftarrow & \Gamma_k \end{array} \]
for each completion \( k_v \) of \( k \), in which \( \phi \) and \( \phi_v \) are continuous homeomorphisms. There should be the analogue of the notion of admissibility of the maps \( \phi \) and \( \phi_v \) as in the local case (cf. [9, p. 40]).

The above conjecture implies that we can attach to any proalgebraic homomorphism \( \mu \) from \( M_k \) to \( L^G \) over \( \Gamma_k \), its associated automorphic representation of \( GL(n, k) \) with the following property: The family of semi-simple conjugacy classes \( c(M) = \{c(M_v)\} \) in \( GL(n, C) \) associated to \( c(\mathcal{M}) = \{c_v(\mathcal{M})\} \) obtained from \( \mathcal{M}_k \) and the local homomorphism \( M_{k_v} \to M_k \) at places \( v \) that are unramified for \( \mathcal{M} \). In fact, \( c_v(\mathcal{M}) \) is the image of the Frobenius class \( Fr_v \) under a different kind of \( \Gamma_k \), namely a compatible family

\[
\Gamma_k \to \prod_{\ell \in S(\mathcal{M}) \cup \{v\}} GL(n, \overline{\mathbb{Q}_\ell})
\]

of \( \ell \)-adic representations attached to \( \mathcal{M} \). Our task now is to find some natural ways to construct an explicit candidate for \( M_k \) and then to clarify the structure of \( M_k \).

It is suggested by experts [70] that \( M_k \) be a proalgebraic fibre product of certain extensions

\[
(4.5) \quad 1 \to D_c \to M_c \to T_k \to 1
\]

of a fixed group \( T_k \) by complex, semisimple, simply connected algebraic groups \( D_c \). The group \( T_k \) is an extension

\[
(4.6) \quad 1 \to S_k \to T_k \to \Gamma_k \to 1
\]

of \( \Gamma_k \) by a complex proalgebraic torus \( S_k \) (cf. [73, Chapter II], [60, Section 5], [74, Section 7]). The contribution to \( M_k \) of any \( M_c \) is required to match the contribution to \( L_k \) of a corresponding \( L_c \), in which \( K_c \) is a compact real form of \( D_c \). This construction should have to come with the following diagram

\[
\begin{array}{cccccc}
1 & \to & L_c^k & \to & L_k & \to & \Gamma_k & \to & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \to & M_c^k & \to & M_k & \to & \Gamma_k & \to & 1
\end{array}
\]

where \( M_c^k \) is a complex pro-reductive group.

Let \( \Psi_k(G) \) be the set of equivalence classes of continuous, completely reducible homomorphisms of \( M_k \) into \( L^G \), and for each place \( v \) of \( k \), let \( \Psi_{k_v}(G) \) be the set of equivalence classes of continuous, completely reducible homomorphisms of \( M_{k_v} \) into \( L^G \). Then one should have to obtain a bijective correspondence

\[
(4.7) \quad \Psi_k(G) \to A_k(G).
\]

This would be supplemented by local bijective correspondences

\[
(4.8) \quad M_{k_v}(G) \to A_{k_v}(G)
\]
for all places $v$ of $k$.

5. Langlands functoriality

As we see in section 2, Shahidi [77] gave a partially affirmative answer to Conjecture A, which is a question raised by Langlands for a larger class of automorphic $L$-functions $L(s, \pi, r)$ obtained from cuspidal automorphic representations $\pi$ of a Levi subgroup $M$ of a quasi-split reductive group and the adjoint representation of $L M$ on the real Lie algebra $L N$. This suggests trying, given a $L$-function and a quasi-split reductive group $G$, to see whether $G$ has an automorphic representation with the given $L$-function. Many instances of such questions can be regarded as special cases of the lifting problem, nowadays called the Principle of Functoriality, with respect to morphisms of $L$-groups. The motivation of this problem stems from a global side. There is also a local version for this problem. These questions were raised and also formulated by Langlands [54] in the late 1960s. Roughly speaking, the principle of functoriality describes profound relationships among automorphic forms on different groups.

Let $k$ be a local or global field, and let $H, G$ two connected reductive groups defined over $k$. A homomorphism $\sigma : L H \rightarrow L G$ is said to be an $L$-homomorphism if it satisfies the following conditions (1)-(3):

(1) $\sigma$ is a homomorphism over the absolute Galois group $\Gamma_k$, namely, the following diagram is commutative:

\[
\begin{array}{ccc}
L H & \xrightarrow{\sigma} & L G \\
\downarrow & & \downarrow \\
\Gamma_k & \xrightarrow{=} & \Gamma_k
\end{array}
\]

(2) $\sigma$ is continuous;

(3) The restriction of $\sigma$ to $L H^0$ is a complex analytic homomorphism of $L H^0$ into $L G^0$.

Let $G_k(H)$ (resp. $G_k(G)$) be the set of all admissible homomorphisms $\phi : W' \rightarrow L H$ (resp. $L G$) modulo inner automorphisms by elements of $L H^0$ (resp. $L G^0$). Suppose $G$ is quasi-split. Given a fixed $L$-homomorphism $\sigma : L H \rightarrow L G$, if $\phi$ is any element in $G_k(H)$, then the composition $\sigma \circ \phi$ is an element in $G_k(G)$. It is easily seen that the correspondence $\phi \mapsto \sigma \circ \phi$ yields the canonical map

\[G_k(\sigma) : G_k(H) \rightarrow G_k(G)\]

If $k$ is a global field and $v$ is a place of $k$, then $L G_v$ can be viewed as a subgroup of $L G$ because $\Gamma_k$ is regarded as a subgroup of $\Gamma_k$. Therefore $\sigma$ induces the $L$-homomorphism $\sigma_v : L H_v \rightarrow L G_v$ and hence a local map

\[G_k(\sigma_v) : G_k(H_v) \rightarrow G_k(G_v)\]

We refer to [9, pp. 54-58] for more detail on these stuffs.

**Langlands Functoriality Conjecture** (Langlands [54]). Let $k$ be a global field,
and let $H, G$ two connected reductive groups over $k$ with $G$ quasi-split. Suppose $\sigma : {}^LH \longrightarrow {}^LG$ is an $L$-homomorphism. Then for any automorphic representation $\pi = \bigotimes_v \pi_v$ of $H(\mathbb{A})$, there exists an automorphic representation $\Pi = \bigotimes_v \Pi_v$ of $G(\mathbb{A})$ such that

\begin{equation}
(5.1) \quad c(\Pi_v) = \sigma(c(\pi_v)), \quad v \notin S(\pi) \cup S(\Pi),
\end{equation}

where $S(\pi)$ (resp. $S(\Pi)$) denotes a finite set of ramified places of $k$ for $\pi$ (resp. $\Pi$) so that $\pi_v$ (resp. $\Pi_v$) is unramified for every place $v \notin S(\pi)$ (resp. $v \notin S(\Pi)$). We note that the condition $(5.1)$ is equivalent to the condition

\begin{equation}
(5.2) \quad L_S(s, \Pi, r) = L_S(s, \pi, r \circ \sigma), \quad S = S(\pi) \cup S(\Pi)
\end{equation}

for every finite dimensional complex representation $r$ of ${}^LG$.

**Remark 5.1.** For a nonarchimedean local field $k$, we can formulate a local version of Langlands Functoriality Conjecture replacing the word “automorphic” by “admissible” and modifying some facts of an $L$-homomorphism.

**Remark 5.2.** Suppose $k$ is a nonarchimedean local field with the ring of integers $\mathcal{O}$. Suppose $H$ and $G$ are quasi-split and there is a finite extension $E$ of $k$ such that both $H$ and $G$ split over $E$, and have an $\mathcal{O}$-structure so that both $H(\mathcal{O})$ and $G(\mathcal{O})$ are special maximal compact subgroups. Let $\pi$ be an unramified representation of $H(k)$ in the sense that $\pi$ has a nonzero $H(\mathcal{O})$-fixed vector, and let $\phi = \phi_\pi \in \mathcal{G}_k(H)$ be the unramified parameter of $\pi$. Then for any $L$-homomorphism $\sigma : {}^LH \longrightarrow {}^LG$, the parameter $\phi = \sigma \circ \phi$ is unramified and defines an $L$-packet $\prod_\mathbb{C}(G)$ which contains exactly one unramified representation $\Pi$ of $G(k)$ to be called the natural lift of $\pi$ (cf. [9, p. 55]).

If we assume that the Local Langlands Conjecture, briefly [LLC] is valid, we can reformulate the Langlands Functoriality Conjecture using [LLC] in the following way. Let $\pi = \bigotimes_v \pi_v$ be an automorphic representation of $H(\mathbb{A})$. According to [LLC], we can attach to each $\pi_v$, an element $\phi_v : W_k^v \longrightarrow {}^LH_v$ in $\mathcal{G}_{k_v}(H)$. The composition $\sigma \circ \phi_v$ is an element in $\mathcal{G}_{k_v}(G)$. By [LLC] again, one has an irreducible admissible representation $\Pi_v$ of $G_v$ attached to $\sigma \circ \phi_v$. Then $\Pi = \bigotimes_v \Pi_v$ is an irreducible admissible representation of $G(\mathbb{A})$. Therefore Langlands Functoriality Conjecture is equivalent to the statement that $\Pi$ must be an automorphic representation of $G(\mathbb{A})$.

If we assume that Global Langlands Conjecture, briefly [GLC] is valid, we can also reformulate Langlands Functoriality Conjecture using [GLC] as follows: Given an automorphic representation $\pi$ of $H(\mathbb{A})$ with its associated parameter $\phi_\pi : L_k \longrightarrow {}^LH$, there must be an $L$-packet $\prod_\mathbb{C}(G)$ attached to $\sigma \circ \phi_\pi$.

**Example 5.3.** Suppose $H = \{1\}$ and $G = GL(n)$. Clearly an automorphic representation $\pi$ of $H(\mathbb{A})$ is trivial. The choice of $\sigma$ amounts to that of an admissible homomorphism

$$\sigma : \text{Gal}(E/k) \longrightarrow GL(n, \mathbb{C}) = {}^LG$$
for a finite Galois extension $E$ of $k$. Therefore Langlands Functoriality Conjecture reduces to the following assertion.

**Strong Artin Conjecture** (Langlands [54]). Let $k$ be a number field. For an $n$-dimensional complex representation $\sigma$ of $\text{Gal}(E/k)$, there is an automorphic representation $\pi$ of $GL(n, \mathbb{A})$ such that

$$c(\pi_v) = \sigma(Fr_v), \quad v \notin S(E),$$

where $S(E)$ is a finite set of places including all the ramified places of $E$. The above conjecture was established partially but remains unsettled for the most part. We summarize the cases that have been established until now chronically.

(a) The case $n = 1$: This is the Artin reciprocity law, namely, $k^* \cong \mathbb{W}_{ab}^k$, which is the essential part of abelian class field theory. The image of $\sigma$ is of cyclic type or of dihedral type.

(b) The case where $n = 2$ and $\text{Gal}(E/k)$ is solvable: The conjecture was solved by Langlands [61] when the image of $\sigma$ in $\text{PSL}(2, \mathbb{C})$ is of tetrahedral type, that is, isomorphic to $A_4$, and by Tunnell [84] when the image of $\sigma$ in $\text{PSL}(2, \mathbb{C})$ is of octahedral type, i.e., isomorphic to $S_4$. It is a consequence of cyclic base change for $GL(2)$. These cases were used by A. Wiles [86] in his proof of Fermat’s Last Theorem.

(c) The case where $n$ is arbitrary and $\text{Gal}(E/k)$ is nilpotent: The conjecture was established by Arthur and Clozel [6] as an application of cyclic base change for $GL(n)$.

(d) The case where $n = 2$ and the image of $\sigma$ is of icosahedral type: Partial results were obtained by Taylor et al. (cf. [10]) C. Khare proved this case.

(e) The case where $n = 4$ and $\text{Gal}(E/k)$ is solvable: The conjecture was established by Ramakrishnan [72] for representations $\sigma$ that factor through the group $GO(4, \mathbb{C})$ of orthogonal similitudes.

**Example 5.4.** Let $k$ be a number field. Let $H = \text{Sp}(2n)$, $SO(2n+1)$, $SO(2n)$ be the split form, and $G = GL(N)$, where $N = 2n+1$ or $2n$. Then $L\text{Sp}(2n) = SO(2n+1, \mathbb{C})$, $L\text{SO}(2n+1) = Sp(2n, \mathbb{C})$, $L\text{SO}(2n) = SO(2n, \mathbb{C})$ and $LGL(N) = GL(N, \mathbb{C})$. As an $L$-homomorphism $\sigma: L^H \rightarrow L^G$, we take the embeddings

$$L\text{Sp}(2n) \hookrightarrow GL(2n+1, \mathbb{C}), \quad L\text{SO}(2n+1) \hookrightarrow GL(2n, \mathbb{C}), \quad L\text{SO}(2n) \hookrightarrow GL(2n, \mathbb{C}).$$

In each of these cases, the Langlands weak functorial lift for irreducible generic cuspidal automorphic representations of $H(\mathbb{A})$ was established by Cogdell, Kim, Piateski-Shapiro and Shahidi [13], [14]. Here the notion of “weak” automorphy means that an automorphic representation of $GL(n)$ exists whose automorphic $L$-function matches the desired Euler product except for a finite number of factors. The proof is based on the converse theorems for $GL(n)$ established by Cogdell and Piateski-Shapiro [15], [16]. It is still an open problem to establish the Langlands functorial lift from irreducible non-generic cuspidal automorphic representations of...
$H(\mathbb{A})$ to $G(\mathbb{A})$. Let $H = \text{GSpin}_m$ be the general spin group of semisimple rank $\left\lfloor \frac{m}{2} \right\rfloor$, i.e., a group whose derived group is $\text{Spin}_m$. Then the $L$-group of $G$ is given by

$$L\text{GSpin}_m = \begin{cases} GSO_m & \text{if } m \text{ is even;} \\ \text{GSp}_{2\left\lfloor \frac{m}{2} \right\rfloor} & \text{if } m \text{ is odd.} \end{cases}$$

In each case we have an embedding

(5.3) \quad i : L^0 H \hookrightarrow GL(N, \mathbb{C}), \quad N = m \text{ or } 2 \left\lfloor \frac{m}{2} \right\rfloor.

Asgari and Shahidi [7], [8] proved that if $\pi$ is a generic cuspidal representation of $\text{GSpin}_m$, then the functoriality is valid for the embedding (5.3).

If $H = \text{SO}(2n + 1)$, for generic cuspidal representations, Jiang and Soudry [38] proved that the Langlands functorial lift from $\text{SO}(2n + 1)$ to $\text{GL}(2n)$ is injective up to isomorphism. Using the functorial lifting from $\text{SO}(2n + 1)$ to $\text{GL}(2n)$, Khare, Larsen and Savin [41] proved that for any prime $\ell$ and any even positive integer $n$, there are infinitely many exponents $k$ for which the simple group $\text{PSp}_{2n}(\mathbb{F}_\ell^k)$ appears as a Galois group over $\mathbb{Q}$. Furthermore, in their recent paper [42] they extended their earlier work to prove that for a positive integer $t$, assuming that $t$ is even if $\ell = 3$ in the first case (1) below, the following statements (1)-(3) hold:

(1) Let $\ell$ be a prime. Then there exists an integer $k$ divisible by $t$ such that the simple group $G_2(\mathbb{F}_\ell^k)$ appears as a Galois group over $\mathbb{Q}$.

(2) Let $\ell$ be an odd prime. Then there exists an integer $k$ divisible by $t$ such that the simple finite group $\text{SO}_{2n+1}(\mathbb{F}_\ell^k)^{\text{der}}$ or the finite classical group $\text{SO}_{2n+1}(\mathbb{F}_\ell^k)$ appears as a Galois group over $\mathbb{Q}$.

(3) If $\ell \equiv 3, 5 \pmod{8}$ and $\ell$ is a prime, then there exists an integer $k$ divisible by $t$ such that the simple finite group $\text{SO}_{2n+1}(\mathbb{F}_\ell^k)^{\text{der}}$ appears as a Galois group over $\mathbb{Q}$. The construction of Galois groups in (1)-(3) is based on the functorial lift from $\text{Sp}(2n)$ to $\text{GL}(2n + 1)$, and the backward lift from $\text{GL}(2n + 1)$ to $\text{Sp}(2n)$ plus the theta lift from $G_2$ to $\text{Sp}(6)$.

**Example 5.5.** Let $k$ be a number field. For two positive integers $m$ and $n$, we let

$$H = \text{GL}(m) \times \text{GL}(n) \quad \text{and} \quad G = \text{GL}(mn).$$

Then $L^0 H = \text{GL}(m, \mathbb{C}) \times \text{GL}(n, \mathbb{C})$ and $L^0 G = \text{GL}(mn, \mathbb{C})$. We take the $L$-homomorphism

$$\sigma : \text{GL}(m) \times \text{GL}(n) \longrightarrow \text{GL}(mn, \mathbb{C})$$

given by the tensor product. Suppose $\pi = \otimes_v \pi_v$ and $\tau = \otimes_v \tau_v$ are two cuspidal automorphic representations of $\text{GL}(m, \mathbb{A})$ and $\text{GL}(n, \mathbb{A})$ respectively. By [LLC] for $\text{GL}(N)$ [28], [31], [63], one has the parametrizations

$$\phi_v : W'_k \longrightarrow \text{GL}(m, \mathbb{C}) \quad \text{and} \quad \psi'_v : W'_k \longrightarrow \text{GL}(n, \mathbb{C}).$$

Let

$$[\phi_v, \psi'_v] : W'_k \longrightarrow \text{GL}(m, \mathbb{C}) \times \text{GL}(n, \mathbb{C}) \hookrightarrow \text{GL}(mn, \mathbb{C}) = L^0 G.$$
be the admissible homomorphism of $W'_k$ into $^LH$ defined by
\[ [\phi_v, \psi_v](w) = (\phi_v(w), \psi_v(w)), \quad w \in W'_k. \]
The composition $\theta_v = \sigma \circ [\phi_v, \psi_v]$ is an admissible homomorphism of $W'_k$ into $^LG$. According to [LLC] for $GL(N)$, one has an irreducible admissible representation of $GL(mn, k_v)$ attached to $\theta_v$, denoted by $\pi_v \boxtimes \tau_v$. We set $\pi \boxtimes \tau = \bigotimes_v \pi_v \boxtimes \tau_v$.

The validity of the Langlands Functoriality Conjecture for the $L$-homomorphism $\sigma : {}^LH \longrightarrow ^LG$ implies that $\pi \boxtimes \tau$ is an automorphic representation of $GL(mn, \mathbb{A})$. Ramakrishnan [71] used the converse theorem for $GL(4)$ of Cogdell and Piatetski-Shapiro to establish the functoriality for $GL(2) \times GL(2)$. Kim and Shahidi [46] established the functoriality for $GL(2) \times GL(3)$.

Example 5.6. Let $H = GL(2)$. For a positive integer $m \geq 2$, let $G = GL(m + 1)$. Suppose $\pi = \otimes_v \pi_v$ is an automorphic representation of $H(\mathbb{A})$. According to [LLC] for $GL(n)$ [28], [31], [63], for each place $v$ of $k$, we have a semisimple conjugacy class $c(\pi_v) = \{\text{diag}(\alpha_v, \beta_v)\} \subset GL(2, \mathbb{C})$. By [LLC] for $GL(n)$ again, for each place $v$ of $k$, there is an irreducible admissible representation of $GL(m+1, k_v)$, denoted by $\text{Sym}^m(\pi_v)$ attached to the semisimple conjugacy class
\[ \{\text{diag}(\alpha_v^m, \alpha_v^{m-1}\beta_v, \cdots, \beta_v^m)\} \subset GL(m+1, \mathbb{C}). \]
We set $\text{Sym}^m(\pi) := \bigotimes_v \text{Sym}^m(\pi_v)$.

Then $\text{Sym}^m(\pi)$ is an irreducible admissible representation of $GL(m+1, \mathbb{A})$. The validity of the Langlands Functoriality Conjecture for the $L$-homomorphism $\text{Sym}^m : GL(2, \mathbb{C}) \longrightarrow GL(m+1, \mathbb{C})$ implies that $\text{Sym}^m(\pi)$ is an automorphic representation of $GL(m+1, \mathbb{A})$. As a consequence, we obtain a complete resolution of the Ramanujan-Petersson conjecture for Maass forms, the Selberg conjecture for eigenvalues and the Sato-Tate conjecture. In 1978, Gelbart and Jacquet [22] established the functoriality for $\text{Sym}^2$ using the converse theorem on $GL(3)$. In 2002, Kim and Shahidi [46] established the functoriality for $\text{Sym}^3$ using the functoriality for $GL(2) \times GL(3)$. Thereafter Kim [43] established the functoriality for $\text{Sym}^4$. The proof is based on the converse theorems for $GL(n)$ established by Cogdell and Piatetski-Shapiro [15, 16]. We refer to [47] for more results on this topic.

Example 5.7. For a positive integer $n \geq 2$, we let $H = GL(n)$ and $G = GL(N)$, $N = \frac{(n-1)n}{2}$.
Let
\[ \wedge^2 : {}^LH = GL(n, \mathbb{C}) \longrightarrow {}^LG = GL(N, \mathbb{C}) \]
be the $L$-homomorphism of $^LH$ into $^LG$ given by the exterior square. Suppose $\pi = \otimes_v \pi_v$ is a cuspidal automorphic representation of $GL(n, \mathbb{A})$. According to [LLC] for $GL(m)$, for each place $v$ of $k$, one has the admissible homomorphism

$$\phi_v : W'_k \longrightarrow L = GL(n, \mathbb{C})$$

parameterizing $\pi_v$. The composition $\psi_v = \wedge^2 \circ \phi_v$ again yields an irreducible admissible representation $\wedge^2 \pi_v$ of $GL(N, k_v)$ for every unramified representation $\pi_v$. We set

$$\wedge^2 \pi = \bigotimes_v \wedge^2 \pi_v.$$ 

Then $\wedge^2 \pi$ is an irreducible admissible representation of $GL(N, \mathbb{A})$. The validity of the Langlands functoriality implies that $\wedge^2 \pi$ is an automorphic representation of $GL(N, \mathbb{A})$. Kim [43] established the functoriality for the case $n = 4$, that is a functorial lift from $GL(4)$ to $GL(6)$.

**Remark 5.8.** As we see in Example 5.3, 5.4 and 5.5, the converse theorem for $GL(n)$ obtained by Cogdell and Piateski-Shapiro plays a crucial role in establishing the functoriality for $GL(2) \times GL(3)$, Sym$^3$ and Sym$^4$. There are widely known three methods in establishing the Langlands functoriality which are based on the theory of the Selberg-Arthur trace formula [4], [6], [61], [62], the converse theorems for $GL(n)$ [15], [16], [22] and the theta correspondence or theta lifting method (R. Howe, J. -S. Li, S. Kudla et al.).

According to the above examples and the converse theorems for $GL(n)$, we see that the importance of the Langlands Functoriality Conjecture is that automorphic $L$-functions of any automorphic representations of any group should be the $L$-functions of automorphic representations of $GL(n, \mathbb{A})$. In this sense we can say that $GL(n, \mathbb{A})$ is speculated to be the mother of all automorphic representations, and their offspring $L$-functions are already supposed to have meromorphic continuations and the standard functional equation.

**Appendix: Converse theorems**

We have seen that the converse theorems have been effectively applied to the establishment of the Langlands functoriality in certain special interesting cases (cf. Example 5.4, 5.5, 5.6 and 5.7). We understand that the converse theorems give a criterion for a given irreducible representation of $GL(n, \mathbb{A})$ to be automorphic in terms of the analytic properties of its associated automorphic $L$-functions. In this appendix, we give a brief survey of the history of the converse theorems and survey the recent results in the local converse theorems.

The first converse theorem was established by Hamburger [26] in 1921. This theorem states that any Dirichlet series satisfying the functional equation of the Riemann zeta function $\zeta(s)$ and suitable regularity conditions must be a multiple of $\zeta(s)$. More precisely, this theorem can be formulated:
Theorem A (Hamburger [26], 1921). Let two Dirichlet series \( g(s) = \sum_{n \geq 1} a_n n^{-s} \) and \( h(s) = \sum_{n \geq 1} b_n n^{-s} \) converge absolutely for \( \Re(s) > 1 \). Suppose that both \((s - 1)h(s)\) and \((s - 1)g(s)\) are entire functions of finite order. Assume we have the following functional equation:

\[
\pi^{-\frac{s}{2}} \Gamma(s/2) h(s) = \pi^{-\frac{1-s}{2}} \Gamma((1-s)/2) g(1-s).
\]

Then \( g(s) = h(s) = a_1 \zeta(s) \). Here \( \Gamma(s) \) is the usual Gamma function, and an entire function \( f(s) \) is said to be of order \( \rho \) if

\[
f(s) = O(|s|^\rho + \epsilon) \quad \text{for any} \quad \epsilon > 0.
\]

Unfortunately Hamburger’s converse theorem was not well recognized until the generalization to \( L \)-functions attached to holomorphic modular forms was done by Hecke [29] in 1936. Hecke proved his converse theorem connecting certain \( L \)-functions which satisfy a certain functional equation with holomorphic modular forms with respect to the full modular group \( SL(2, \mathbb{Z}) \). For a good understanding of Hecke’s converse theorem, we need to describe Hecke’s idea and argument roughly. Let

\[ f(\tau) = \sum_{n \geq 1} a_n e^{2\pi i n \tau} \]

be a holomorphic modular form of weight \( d \) with respect to \( SL(2, \mathbb{Z}) \). To such a function \( f \) Hecke attached an \( L \)-function \( L(s, f) \) via the Mellin transform

\[
(2\pi)^{-s} \Gamma(s) L(s, f) = \int_{0}^{\infty} f(iy) y^s \frac{dy}{y}
\]

and derived the functional equation for \( L(s, f) \). He inverted this process by taking a Dirichlet series

\[ D(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \]

and assuming that it converges absolutely in some half plane, has an analytic continuation to an entire function of finite order, and satisfies the same functional equation as \( L(s, f) \). In his masterpiece [29], he could reconstruct a holomorphic modular form from \( D(s) \) by Mellin inversion

\[
f(iy) = \sum_{n \geq 1} a_n e^{-2\pi y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} (2\pi)^{-s} \Gamma(s) D(y) y^s ds
\]

and obtain the modular transformation law for \( f(\tau) \) under \( \tau \mapsto -\tau^{-1} \) from the functional equation for \( D(s) \) under \( s \mapsto d - s \). This is Hecke’s converse theorem! You might agree that Hecke’s original idea and argument are remarkably beautiful. In 1949, in his seminal paper [68], Maass, a student of Hecke, extended Hecke’s
method to non-holomorphic forms for $SL(2,\mathbb{Z})$. In 1967, the next very important step was made by Weil in his paper [85] dedicated to C. L. Siegel (1896-1981) celebrating Siegel’s seventieth birthday. Weil showed how to work with Dirichlet series attached to holomorphic modular forms with respect to congruence subgroups of $SL(2,\mathbb{Z})$. He proved that if a Dirichlet series together with a sufficient number of twists satisfies nice analytic properties and functional equations with reasonably suitable regularity, then it stems from a holomorphic modular form with respect to a congruence subgroup of $SL(2,\mathbb{Z})$. More precisely his converse theorem can be formulated as follows.

**Theorem B (Weil [85], 1967).** Fix two positive integers $d$ and $N$. Suppose the Dirichlet series

$$D(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

satisfies the following properties:

(W1) $D(s)$ converges absolutely for sufficiently large $\text{Re}(s) \gg 0$;

(W2) For every primitive character $\chi$ of modulus $r$ with $(r, N) = 1$, the function

$$\Lambda(s, \chi) := (2\pi)^{-s} \Gamma(s) \sum_{n \geq 1} \frac{a_n \chi(n)}{n^s}$$

has an analytic continuation to an entire function of $s$ to the whole complex plane, and is bounded in vertical strips of finite width;

(W3) Every such a function $\Lambda(s, \chi)$ satisfies the functional equation

$$\Lambda(s, \chi) = w_\chi r^{-1} (r^2 N)^{\frac{d}{2}} \Lambda(d - s, \chi),$$

where

$$w_\chi = i^d \chi(N) g(\chi)^2$$

and

$$g(\chi) = \sum_{n \equiv 0 \pmod{r}} \chi(n) e^{2\pi in/r}.$$

Then the function

$$f(\tau) = \sum_{n \geq 1} a_n e^{2\pi in\tau}, \quad \tau \in \mathbb{H} = \{\tau \in \mathbb{C} | \text{Im}\tau > 0\}$$

is a holomorphic cusp form of weight $d$ with respect to the congruence subgroup $\Gamma_0(N)$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$
Weil’s proof follows closely Hecke’s idea and argument. We see that his converse theorem provides a condition for modularity of the Dirichlet series $D(s)$ under $\Gamma_0(N)$ in terms of the functional equations of Dirichlet series twisted by primitive characters. In fact, Weil’s converse theorem influenced the complete proof of the Shimura-Taniyama conjecture given by Wiles [86], Taylor et al. So we can say that the work of Weil marks the beginning of the modern era in the study of the connection between $L$-functions and automorphic forms.

In 1970 Jacquet and Langlands [33] established the converse theorem for $GL(2)$ in the adelic context of automorphic representations of $GL(2, \mathbb{A})$ based on Hecke’s original idea. In 1979 Jacquet, Piatetski-Shapiro and Shalika [34] established the converse theorem for $GL(3)$ in the adelic context. Finally in 1994, generalizing the work on the converse theorems on $GL(2)$ and $GL(3)$, Cogdell and Piatetski-Shapiro [12], [15], [16] proved the converse theorem for $GL(n)$ with arbitrary $n \geq 1$ in the context of automorphic representations. The idea and technique in the proof of Cogdell and Piatetski-Shapiro are surprisingly almost the same as Hecke’s. We now describe the converse theorems formulated and proved by them.

Let $k$ be a global field, $\mathbb{A}$ its adele ring, and let $\psi$ be a fixed nontrivial continuous additive character of $\mathbb{A}$ which is trivial on $k$. Let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of $GL(n, \mathbb{A})$, and let $\tau = \otimes_v \tau_v$ be a cuspidal automorphic representation of $GL(m, \mathbb{A})$ with $m < n$. We define formally

$$L(s, \pi \times \tau) = \prod_v L(s, \pi_v \times \tau_v) \quad \text{and} \quad \varepsilon(s, \pi \times \tau) = \prod_v \varepsilon(s, \pi_v \times \tau_v, \psi_v).$$

We say that $L(s, \pi \times \tau)$ is nice if it satisfies the following properties:

- (N1) $L(s, \pi \times \tau$ and $L(s, \tilde{\pi} \times \tilde{\tau}$ have analytic continuations to entire functions, where $\tilde{\pi}$ (resp. $\tilde{\tau}$) denotes the contragredient of $\pi$ (resp. $\tau$);
- (N2) $L(s, \pi \times \tau$ and $L(s, \tilde{\pi} \times \tilde{\tau}$ are bounded in vertical strips of finite width;
- (N3) These entire functions satisfy the standard functional equation

$$L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau) L(1 - s, \tilde{\pi} \times \tilde{\tau}).$$

**Theorem C** (Cogdell and Piatetski [15], [16], 1994). Let $\pi$ be an irreducible admissible representation of $GL(n, \mathbb{A})$ whose central character is trivial on $k^*$ and whose $L$-function $L(s, \pi)$ converges absolutely in some half plane. Assume that $L(s, \pi \times \tau)$ is nice for every cuspidal automorphic representation $\tau$ of $GL(n, \mathbb{A})$ for $1 \leq m \leq n - 2$. Then $\pi$ is a cuspidal automorphic representation of $GL(n, \mathbb{A})$.

Furthermore they proved the following theorem.

**Theorem D** (Cogdell and Piatetski [16], 1999). Let $\pi$ be an irreducible admissible representation of $GL(n, \mathbb{A})$ whose central character is trivial on $k^*$ and whose $L$-function $L(s, \pi)$ converges absolutely in some half plane. Let $S$ be a finite set of finite places. Assume that $L(s, \pi \times \tau)$ is nice for every cuspidal automorphic representation $\tau$ of $GL(m, \mathbb{A})$ for $1 \leq m \leq n - 2$, which is unramified at the places in $S$. Then $\pi$
is quasi-automorphic in the sense that there is an automorphic representation \( \pi' \) of \( GL(n, \mathbb{A}) \) such that \( \pi_v \cong \pi'_v \) for all \( v \notin S \).

The local converse theorem for \( GL(n) \) was first formulated by Piatetski-Shapiro in his unpublished Maryland notes (1976) with his idea of deducing the local converse theorem from his global converse theorem. It was proved by Henniart [30] using a local approach. The local converse theorem is a basic ingredient in the proof of [LCC] for \( GL(n) \) by Harris and Taylor [28] and by Henniart [31].

The local converse theorem for \( GL(n) \) can be formulated as follows.

**Theorem E** (Henniart [30], 1993). Let \( k \) be a nonarchimedean local field of characteristic 0. Let \( \tau \) and \( \tau' \) be irreducible admissible generic representations of \( GL(n, k) \) with the same central character. Assume the twisted local gamma factors (cf. [35]) are the same, i.e.,

\[
\gamma(s, \tau \times \rho, \psi) = \gamma(s, \tau' \times \rho, \psi)
\]

for all irreducible supercuspidal representations \( \rho \) of \( GL(m, k) \) with \( 1 \leq m \leq n - 1 \). Then \( \tau \) is isomorphic to \( \tau' \).

**Remark 1.** It is known that the twisting condition on \( m \) reduces from \( n - 1 \) to \( n - 2 \). It is expected as a conjecture of H. Jacquet [16, Conjecture 8.1] that the twisting condition on \( m \) should be reduced from \( n - 1 \) to \( \left\lfloor \frac{n}{2} \right\rfloor \).

**Remark 2.** The local converse theorem for generic representations of \( U(2, 1) \) and for \( GSp(4) \) was established by E. M. Baruch in his Ph. D. thesis (Yale Univ., 1995).

Jiang and Soudry [38] proved the local converse theorem for irreducible admissible generic representations of \( SO(2n + 1, k) \).

**Theorem F** (Jiang and Soudry [38], 2003). Let \( \sigma \) and \( \sigma' \) be irreducible admissible generic representations of \( SO(2n + 1, k) \). Assume the twisted local gamma factors are the same, i.e.,

\[
\gamma(s, \sigma \times \rho, \psi) = \gamma(s, \sigma' \times \rho, \psi)
\]

for all irreducible supercuspidal representations \( \rho \) of \( GL(m, k) \) with \( 1 \leq m \leq 2n - 1 \). Then \( \sigma \) is isomorphic to \( \sigma' \).

I shall give a brief sketch of the idea of their proof. They first reduce the proof of Theorem F to the case where both \( \sigma \) and \( \sigma' \) are supercuspidal by studying the existence of poles of twisted local gamma factors and related properties. Developing the explicit local Howe duality for irreducible admissible generic representations of \( SO(2n + 1, k) \) and the metaplectic group \( \widetilde{Sp}(2n, k) \), and using the global weak Langlands functorial lifting form \( SO(2n + 1) \) to \( GL(2n) \) (cf. Example 5.4, [13], [14]) and the local backward lifting from \( GL(2n, k) \) to \( \widetilde{Sp}(2n, k) \), they relate the local converse theorem for \( SO(2n + 1) \) with that for \( GL(2n) \) which is well known now.

As an application of Theorem F, I repeat again that Jiang and Soudry [38], [39] proved the Local Langlands Reciprocity Law for \( SO(2n + 1) \). More precisely, there exists a unique bijective correspondence between the set of conjugacy classes of all...
2n-dimensional, admissible, completely reducible, multiplicity-free, symplectic complex representations of the Weil group $W_k$ and the set of all equivalence classes of irreducible generic supercuspidal representations of $SO(2n + 1, k)$, which preserves the relevant local factors. As an application of Theorem $F$ to the global theory, they proved that the weak Langlands functorial lifting from irreducible generic cuspidal automorphic representations of $SO(2n + 1)$ to irreducible automorphic representations of $GL(2n)$ is injective up to isomorphism. It is still an open problem to establish the Langlands functorial lift from irreducible non-generic cuspidal automorphic representations of $SO(2n + 1)$ to $GL(2n)$. As another application to the global theory, they proved the rigidity theorem in the sense that if $\pi = \otimes_v \pi_v$ and $\tau = \otimes_v \tau_v$ are irreducible generic cuspidal automorphic representations of $SO(2n + 1, A)$ such that $\pi_v$ is isomorphic to $\tau_v$ for almost all local places $v$, then $\pi$ is isomorphic to $\tau$.

References


Langlands Functoriality Conjecture


Langlands Functoriality Conjecture


