On Compact-covering Images of Locally Separable Metric Spaces

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Abstract. In this paper, we give the internal characterizations of compact-covering s-(resp., \( \pi \)-)images of locally separable metric spaces. As applications of these results, we obtain characterizations of compact-covering quotient s-(resp., \( \pi \)-)images of locally separable metric spaces.

1. Introduction

Finding the internal characterizations of certain images of metric spaces is a considerable interest in general topology. In the past, many nice results have been obtained [6], [11], [12], [17], [18]. Recently, many topologists were engaged in research of internal characterizations of images of locally separable metric spaces, and some noteworthy results were shown. In [12], S. Lin, C. Liu, and M. Dai gave a characterization of quotient s-images of locally separable metric spaces. After that, S. Lin, and P. Yan characterized sequence-covering s-images of locally separable metric spaces in [13]; Y. Ikeda, C. Liu and Y. Tanaka characterized quotient compact images of locally separable metric spaces in [7]; and Y. Ge characterized pseudo-sequence-covering compact images of locally separable metric spaces in [5]. In a personal communication, the first author of [12] and [13] informs that characterizations on compact-covering s-images and compact-covering \( \pi \)-images still have no answer. Thus, it is natural to rise the following question.

Question 1.1. How are compact-covering s-(resp., \( \pi \)-)images of locally separable metric spaces characterized?

In this paper, we give the internal characterizations of compact-covering s-(resp., \( \pi \)-)images of locally separable metric spaces. As applications of these results, we obtain a characterization of compact-covering quotient s-(resp., \( \pi \)-)images of locally separable metric spaces.

Throughout this paper, all spaces are assumed to be regular and \( T_1 \), all mappings are assumed continuous and onto, \( \mathbb{N} \) denotes the set of all natural numbers. Let \( f : X \to Y \) be a mapping, \( x \in X \), and \( \mathcal{P} \) be a family of subsets of \( X \), we
denote \( st(x, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : x \in P\}, \bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}, \bigcap \mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}, \) and \( f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\} \).

**Definition 1.2.** Let \( \mathcal{P} \) be a family of subsets of a space \( X \), and \( K \) be a subset of \( X \).

1. \( \mathcal{P} \) is a cover for \( K \) in \( X \), if \( K \subset \bigcup \mathcal{P} \). When \( K = X \), a cover for \( K \) in \( X \) is a cover of \( X \) [3].
2. For each \( x \in X \), \( \mathcal{P} \) is a network at \( x \) in \( X \) [15], if \( x \in \bigcap \mathcal{P} \) and if \( x \in U \) with \( U \) open in \( X \), then \( x \in P \subset U \) for some \( P \in \mathcal{P} \).
3. \( \mathcal{P} \) is a cfp-cover for \( K \) in \( X \), if for each compact subset \( H \) of \( K \), there exists a finite subfamily \( \mathcal{F} \) of \( \mathcal{P} \) such that \( H \subset \bigcup\{C_F : F \in \mathcal{F}\} \), where \( C_F \) is closed and \( C_F \subset F \) for every \( F \in \mathcal{F} \). Note that such a \( \mathcal{F} \) is a full cover in the sense of [2], when \( K = X \), a cfp-cover for \( K \) in \( X \) is a cfp-cover for \( X \) [20].
4. \( \mathcal{P} \) is a cfp-network for \( K \) in \( X \), if for each compact subset \( H \) of \( K \) satisfying \( H \subset U \) with \( U \) open in \( X \), there exists a finite subfamily \( \mathcal{F} \) of \( \mathcal{P} \) such that \( H \subset \bigcup\{C_F : F \in \mathcal{F}\} \subset \bigcup \mathcal{F} \subset U \), where \( C_F \) is closed and \( C_F \subset F \) for every \( F \in \mathcal{F} \). Note that a cfp-network \( \mathcal{P} \) for \( K \) in \( X \) is a family to have property cc for \( K \) [14], and if \( K = X \), then \( \mathcal{P} \) is a strong \( k \)-network for \( X \) in the sense of [2].
5. \( \mathcal{P} \) is point-countable [6], if every point of \( X \) meets at most countably many members of \( \mathcal{P} \).

**Definition 1.3.** Let \( f : X \to Y \) be a mapping.

1. \( f \) is a compact-covering mapping [16], if every compact subset of \( Y \) is the image of some compact subset of \( X \).
2. \( f \) is a pseudo-sequence-covering mapping [7], if every convergent sequence of \( Y \) is the image of some compact subset of \( X \).
3. \( f \) is a pseudo-open mapping [1], if \( y \in \text{int} f(U) \) whenever \( f^{-1}(y) \subset U \) with \( U \) open in \( X \).
4. \( f \) is a \( \pi \)-mapping [1], if for every \( y \in Y \) and for every neighborhood \( U \) of \( y \) in \( Y \), \( d(f^{-1}(y), X - f^{-1}(U)) > 0 \), where \( X \) is a metric space with a metric \( d \).
5. \( f \) is an \( s \)-mapping [1], if \( f^{-1}(y) \) is separable for every \( y \in Y \).

**Definition 1.4.** Let \( X \) be a space.

1. \( X \) is a sequential space [4], if a subset \( A \) of \( X \) is closed if and only if any convergent sequence in \( A \) has a limit point in \( A \).
2. \( X \) is a Fréchet space [4], if for each \( x \in \overline{A} \), there exists a sequence in \( A \) converging to \( x \).

For terms which are not defined here, please refer to [3] and [18].
2. Results

In 1960, V. Ponomarev proved that every first-countable space is precisely an open image of some Baire zero-dimension metric space [3, 4.2 D]. The Ponomarev’s method has been generalized [14], and plays a very important role in characterizations of images of metric spaces. We shall use the above method to characterize compact-covering s-images of locally separable metric spaces.

Definition 2.1. Let \( P \) be a network for a space \( X \). Assume that there exists a countable network \( P_x \subset P \) at \( x \) in \( X \) for every \( x \in X \). For every \( n \in \mathbb{N} \), put \( \Lambda_n = \Lambda \) and endowed \( \Lambda_n \) a discrete topology. Put

\[
M = \{ a = (\alpha_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{ P_{\alpha_n} : n \in \mathbb{N} \} \}
\]

forms a network at some point \( x_a \) in \( X \). Then \( M \), which is a subspace of the product space \( \prod_{n \in \mathbb{N}} \Lambda_n \), is a metric space and each point \( x_a \) is unique for every \( a \in M \). Define \( f : M \rightarrow X \) by \( f(a) = x_a \), then \( f \) is a mapping, and \((f, M, X, P)\) is a Ponomarev-system [14]. Note that under \( P \) being a point-countable network for \( X \), the Ponomarev-system \((f, M, X, P)\) exists.

It is well known that \( cfp \)-networks are preserved by compact-covering mappings. We shall strengthen this result on preservations of \( cfp \)-covers and \( cfp \)-networks for a compact subset without the assumption of the compact-covering property.

Lemma 2.2. Let \( f : X \rightarrow Y \) be a mapping.

(1) If \( P \) is a \( cfp \)-cover for a compact set \( K \) in \( X \), then \( f(P) \) is a \( cfp \)-cover for \( f(K) \) in \( Y \).

(2) If \( P \) is a \( cfp \)-network for a compact set \( K \) in \( X \), then \( f(P) \) is a \( cfp \)-network for \( f(K) \) in \( Y \).

Proof. (1). Let \( H \) be a compact subset of \( f(K) \). Then \( L = f^{-1}(H) \cap K \) is a compact subset of \( K \) satisfying \( f(L) = H \). Since \( P \) is a \( cfp \)-cover for \( K \) in \( X \), there exists a finite subfamily \( \mathcal{F} \) of \( P \) such that \( L \subset \bigcup \{ C_F : F \in \mathcal{F} \} \), where \( C_F \subset F \), and \( C_F \) is closed for every \( F \in \mathcal{F} \). Because \( L \) is compact, every \( C_F \) can be chosen compact. It implies that every \( f(C_F) \) is closed (in fact, every \( f(C_F) \) is compact), and \( f(C_F) \subset f(F) \). We get that \( H = f(L) \subset \bigcup \{ f(C_F) : F \in \mathcal{F} \} \), where \( f(F) \) is a finite subfamily of \( f(P) \). Then \( f(P) \) is a \( cfp \)-cover for \( f(K) \) in \( Y \).

(2). Similar to the proof of (1).

Now, we characterize compact-covering s-images of locally separable metric spaces as follows.

Theorem 2.3. The following are equivalent for a space \( X \).

(1) \( X \) is a compact-covering s-image of a locally separable metric space,
(2) $X$ has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ satisfying that each $X_\alpha$ has a countable network $\mathcal{P}_\alpha$, and each compact subset $K$ of $X$ has a finite compact cover $\{K_\alpha : \alpha \in \Lambda_K\}$ such that, for each $\alpha \in \Lambda_K$, $\mathcal{P}_\alpha$ is a $cfp$-network for $K_\alpha$ in $X_\alpha$.

Proof. (1) $\Rightarrow$ (2). Let $f : M \to X$ be a compact-covering $s$-mapping from a locally separable metric space $M$ onto $X$. Since $M$ is a locally separable metric space, $M = \bigoplus_{\alpha \in \Lambda} M_\alpha$, where each $M_\alpha$ is a separable metric space by [3, 4.4.F]. For each $\alpha \in \Lambda$, let $\mathcal{B}_\alpha$ be a countable base of $M_\alpha$, and put $X_\alpha = f(M_\alpha)$, $\mathcal{P}_\alpha = f(\mathcal{B}_\alpha)$. Then $\{X_\alpha : \alpha \in \Lambda\}$ is a point-countable cover for $X$, and each $\mathcal{P}_\alpha$ is a countable network for $X_\alpha$.

Let $K$ be a compact subset of $X$. Since $f$ is compact-covering, $K = f(L)$ for some compact subset $L$ of $M$. Because $L$ is a compact subset of $M$, $\Lambda_K = \{\alpha \in \Lambda : L \cap M_\alpha \neq \emptyset\}$ is finite. For each $\alpha \in \Lambda_K$, put $L_\alpha = L \cap M_\alpha$, then $L_\alpha$ is compact. Denote $K_\alpha = f(L_\alpha)$, we get that $\{K_\alpha : \alpha \in \Lambda_K\}$ is a finite compact cover for $K$.

By [2, Claim 4.2], each $\mathcal{B}_\alpha$ is a $cfp$-network for $M_\alpha$. Then $\mathcal{B}_\alpha$ is a $cfp$-network for $L_\alpha$ in $M_\alpha$. It follows from Lemma 2.2 that, for each $\alpha \in \Lambda_K$, $\mathcal{P}_\alpha$ is a $cfp$-network for $K_\alpha$ in $X_\alpha$.

(2) $\Rightarrow$ (1). For each $\alpha \in \Lambda$ and $n \in \mathbb{N}$, put $\mathcal{P}_\alpha = \{P_\beta : \beta \in \Gamma_\alpha\}$, and denote by $\Gamma_{\alpha,n}$ the countable set $\Gamma_\alpha$ endowed with the discrete topology. Put

$$M_\alpha = \{b_\alpha = (\beta_{\alpha,n}) \in \prod_{n \in \mathbb{N}} \Gamma_{\alpha,n} : \{P_{\beta_{\alpha,n}} : n \in \mathbb{N}\}\}$$

forms a network at some point $x_{b_\alpha}$ in $X_\alpha$.

Then $M_\alpha$, which is a subspace of the product space $\prod_{\alpha \in \Lambda} \Gamma_{\alpha,n}$, is a metric space and $x_{b_\alpha}$ is unique for each $b_\alpha \in M_\alpha$. Define $f_\alpha : M_\alpha \to X_\alpha$ by choosing $f_\alpha(b_\alpha) = x_{b_\alpha}$. Then the Ponomarev-system $(f_\alpha, M_\alpha, X_\alpha, \mathcal{P}_\alpha)$ exists. Put $M = \bigoplus_{\alpha \in \Lambda} M_\alpha$. Since every $\mathcal{P}_\alpha$ is countable, $M_\alpha$ is a separable metric space. Then $M$ is a locally separable metric space. Define $f : M \to X$ by choosing $f(b_\alpha) = f_\alpha(b_\alpha)$ for every $b_\alpha \in M_\alpha$. It is easy to check that $f$ is continuous and onto.

(a) $f$ is an $s$-mapping.

For each $x \in X$, since $\{X_\alpha : \alpha \in \Lambda\}$ is a point-countable cover for $X$, $\Lambda_x = \{\alpha \in \Lambda : x \in X_\alpha\}$ is countable. Note that $\Gamma_{\alpha,n}$ is countable for each $n \in \mathbb{N}$, $M_\alpha$ is a separable metric space. Then $f_\alpha^{-1}(x)$ is a separable subset of $M_\alpha$ for each $\alpha \in \Lambda_x$. Hence $f^{-1}(x) = \bigcup f_\alpha^{-1}(x) : \alpha \in \Lambda_x$ is a separable subset of $M$. It implies that $f$ is an $s$-mapping.

(b) $f$ is compact-covering.

Let $K$ be a compact subset of $X$. Then $K$ has a finite compact cover $\{K_\alpha : \alpha \in \Lambda_K\}$ such that, for each $\alpha \in \Lambda_K$, $\mathcal{P}_\alpha$ is a $cfp$-network for $K_\alpha$ in $X_\alpha$. It follows from [14, Theorem 2] that there exists a compact subset $L_\alpha$ of $M_\alpha$ satisfying $f_\alpha(L_\alpha) = K_\alpha$.

Put $L = \bigcup \{L_\alpha : \alpha \in \Lambda_K\}$, then $L$ is a compact subset of $M$ satisfying $f(L) = K$. It implies that $f$ is compact-covering. $\square$

By Theorem 2.3, we get a characterization of compact-covering quotient $s$-
images of locally separable metric spaces as follows.

**Corollary 2.4.** The following are equivalent for a space $X$.

1. $X$ is a compact-covering quotient (resp., pseudo-open) $s$-image of a locally separable metric space,
2. $X$ is a sequential (resp., Fréchet) space with a point-countable cover \( \{X_\alpha : \alpha \in \Lambda\} \) satisfying that each $X_\alpha$ has a countable network $P_\alpha$, and each compact subset $K$ of $X$ has a finite compact cover \( \{K_\alpha : \alpha \in \Lambda_K\} \) such that, for each $\alpha \in \Lambda_K$, $P_\alpha$ is a $\text{cfp}$-network for $K_\alpha$ in $X_\alpha$.

**Proof.** (1) $\Rightarrow$ (2). By Theorem 2.3, it is sufficient to prove that $X$ is a sequential (resp., Fréchet) space. This is obvious by [3, 2.4.G].

(2) $\Rightarrow$ (1). It follows from Theorem 2.3 that $X$ is a compact-covering $s$-image of a locally separable metric space under the mapping $f$. We get that $f$ is quotient (resp., pseudo-open) by [5, Remark 1.7], and [10, Lemma 2.1]. Then $X$ is a compact-covering quotient (resp., pseudo-open) $s$-image of a locally separable metric space. $\square$

**Definition 2.5.** For each $n \in \mathbb{N}$, let $P_n$ be a cover for $X$. \( \{P_n : n \in \mathbb{N}\} \) is a refinement sequence for $X$, if $P_{n+1}$ is a refinement of $P_n$ for each $n \in \mathbb{N}$. A refinement sequence for $X$ is a refinement of $X$ in the sense of [5].

**Definition 2.6.** Let \( \{P_n : n \in \mathbb{N}\} \) be a refinement sequence for $X$. \( \{P_n : n \in \mathbb{N}\} \) is a point-star network for $X$, if \( \{\text{st}(x, P_n) : n \in \mathbb{N}\} \) is a network at $x$ in $X$ for every $x \in X$. Note that a point-star network is used without the assumption of a refinement sequence in [14], and $\bigcup \{P_n : n \in \mathbb{N}\}$ is a $\sigma$-strong network for $X$ in the sense of [7].

In Section 2 of [14], S. Lin and P. Yan extended the Ponomarev-system to a sequence of covers for a space as follows.

**Definition 2.7.** Let \( \{P_n : n \in \mathbb{N}\} \) be a point-star network for a space $X$. For every $n \in \mathbb{N}$, put $P_n = \{P_\alpha : \alpha \in A_n\}$, and $A_n$ is endowed with discrete topology. Put

\[
M = \{a = (\alpha_n) \in \prod_{n \in \mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X\}.
\]

Then $M$, which is a subspace of the product space $\prod_{n \in \mathbb{N}} A_n$, is a metric space with metric $d$ described as follows.

Let $a = (\alpha_n), b = (\beta_n) \in M$. If $a = b$, then $d(a, b) = 0$. If $a \neq b$, then $d(a, b) = 1/\{\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\}\}$.

Define $f : M \rightarrow X$ by choosing $f(a) = x_a$, then $f$ is a mapping, and $(f, M, X, \{P_n\})$ is a Ponomarev-system [19]. Note that without the assumption of a refinement sequence in the notion of point-star networks, then $(f, M, X, \{P_n\})$ is a Ponomarev-system in the sense of [14].
Now, we characterize compact-covering \( \pi \)-images of locally separable metric spaces as follows.

**Theorem 2.8.** The following are equivalent for a space \( X \).

1. \( X \) is a compact-covering \( \pi \)-image of a locally separable metric space,
2. \( X \) has a cover \( \{ X_\lambda : \lambda \in \Lambda \} \), where each \( X_\lambda \) has a refinement sequence of countable covers \( \{ P_{\lambda,n} \}_{n \in \mathbb{N}} \) satisfying the following:
   
   (a) \( \{ P_n \}_{n \in \mathbb{N}} \) is a point-star network of \( X \), where \( P_n = \bigcup_{\lambda \in \Lambda} P_{\lambda,n} \) for each \( n \in \mathbb{N} \),
   
   (b) For every compact subset \( K \) of \( X \), there exists a finite subset \( \Lambda_K \) of \( \Lambda \) such that \( K \) has a finite compact cover \( \{ K_\lambda : \lambda \in \Lambda_K \} \), and for each \( \lambda \in \Lambda_K \) and \( n \in \mathbb{N} \), \( P_{\lambda,n} \) is a cfp-cover for \( K_\lambda \) in \( X_\lambda \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( f : M \to X \) be a compact-covering \( \pi \)-mapping from a locally separable metric space \( M \) with metric \( d \) onto \( X \). Since \( M \) is a locally separable metric space, \( M = \bigoplus_{\lambda \in \Lambda} M_\lambda \), where each \( M_\lambda \) is a separable metric space by [3, 4.4.F]. For each \( \lambda \in \Lambda \), denote \( f_\lambda = f|_{M_\lambda} \), \( X_\lambda = f_\lambda(M_\lambda) \), and \( M_\lambda = \overline{D_\lambda} \), where \( D_\lambda \) is a countable dense subset of \( M_\lambda \).

For each \( a \in M_\lambda \) and \( n \in \mathbb{N} \), put \( B(a,1/n) = \{ b \in M_\lambda : d(a,b) < 1/n \} \), \( B_{\lambda,n} = \{ B(a,1/n) : a \in D_\lambda \} \), and \( P_{\lambda,n} = f_\lambda(B_{\lambda,n}) \). It is clear that \( \{ P_{\lambda,n} : n \in \mathbb{N} \} \) is a refinement sequence of countable covers for \( X_\lambda \).

(a) \( \{ P_n \}_{n \in \mathbb{N}} \) is a point-star network for \( X \).

Since \( \{ P_{\lambda,n} : n \in \mathbb{N} \} \) is a refinement sequence for \( X_\lambda \) for each \( \lambda \in \Lambda \), \( \{ P_n : n \in \mathbb{N} \} \) is a refinement sequence for \( X \).

For each \( x \in U \) with \( U \) open in \( X \). Since \( f \) is a \( \pi \)-mapping, \( d(f^{-1}(x), M - f^{-1}(U)) > 2/n \) for some \( n \in \mathbb{N} \). Then, for each \( \lambda \in \Lambda \) with \( x \in X_\lambda \), we get \( d(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) > 2/n \), where \( U_\lambda = U \cap X_\lambda \). Since \( P_{\lambda,n} \) is a cover for \( X_\lambda \), there exists \( f_\lambda(B(a,1/n)) \in P_{\lambda,n} \) such that \( x \in f(B(a,1/n)) \) for some \( a \in D_\lambda \). We shall prove that \( B(a,1/n) \subset f_\lambda^{-1}(U_\lambda) \). In fact, if \( B(a,1/n) \not\subset f_\lambda^{-1}(U_\lambda) \), then there exists \( b \in B(a,1/n) \setminus f_\lambda^{-1}(U_\lambda) \). Since \( f_\lambda^{-1}(x) \cap B(a,1/n) \neq \emptyset \), there exists \( c \in f_\lambda^{-1}(x) \cap B(a,1/n) \). Then \( d(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \leq d(c,b) \leq d(c,a) + d(a,b) < 2/n \). It is a contradiction. So \( B(a,1/n) \subset f_\lambda^{-1}(U_\lambda) \), thus \( f_\lambda(B(a,1/n)) \subset U_\lambda \). Then \( st(x,P_{\lambda,n}) \subset U_\lambda \), and hence \( \bigcup \{ st(x,P_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda \} \subset U \). It implies that \( st(x,P_n) \subset U \).

Hence, \( \{ P_n \}_{n \in \mathbb{N}} \) is a point-star network for \( X \).

(b) For each compact subset \( K \) of \( X \), since \( f \) is compact-covering, \( K = f(L) \) for some compact subset \( L \) of \( M \). By compactness of \( L \), \( L_\lambda = L \cap M_\lambda \) is compact and \( \Lambda_K = \{ \lambda \in \Lambda : L_\lambda \neq \emptyset \} \) is finite. For each \( \lambda \in \Lambda_K \), put \( K_\lambda = f(L_\lambda) \), then \( \{ K_\lambda : \lambda \in \Lambda_K \} \) is a finite compact cover for \( K \). For each \( n \in \mathbb{N} \), since \( B_{\lambda,n} \) is a cfp-cover for \( L_\lambda \), \( P_{\lambda,n} \) is a cfp-cover for \( K_\lambda \) in \( X_\lambda \) by Lemma 2.2.

(2) \( \Rightarrow \) (1). For each \( \lambda \in \Lambda \), let \( x \in U_\lambda \) with \( U_\lambda \) open in \( X_\lambda \). We get that \( U_\lambda = U \cap X_\lambda \) with some \( U \) open in \( X \). Since \( st(x,P_n) \subset U \) for some \( n \in \mathbb{N} \), \( st(x,P_{\lambda,n}) \subset U_\lambda \).
It implies that \( \{ P_{\lambda,n} : n \in \mathbb{N} \} \) is a point-star network for \( X_\lambda \). Then the Ponomearev-system \( (f_\lambda, M_\lambda, X_\lambda, \{ P_{\lambda,n} \}) \) exists. Since each \( P_{\lambda,n} \) is countable, \( M_\lambda \) is a separable metric space with metric \( d_\lambda \) described as follows. For \( a = (\alpha_n), b = (\beta_n) \in M_\lambda \), if \( a = b \), then \( d_\lambda(a, b) = 0 \), and if \( a \neq b \), then \( d_\lambda(a, b) = 1/(\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\}) \).

Put \( M = \oplus_{\lambda \in \Lambda} M_\lambda \) and define \( f : M \to X \) by choosing \( f(a) = f_\lambda(a) \) for every \( a \in M_\lambda \) with some \( \lambda \in \Lambda \). Then \( f \) is a mapping and \( M \) is a locally separable metric space with metric \( d \) as follows. For \( a, b \in M \), if \( a, b \in M_\lambda \) for some \( \lambda \in \Lambda \), then \( d(a, b) = d_\lambda(a, b) \), and otherwise, \( d(a, b) = 1 \).

(a) \( f \) is a \( \pi \)-mapping.

Let \( x \in U \) with \( U \) open in \( X \), then \( st(x, P_n) \subset U \) for some \( n \in \mathbb{N} \). So, for each \( \lambda \in \Lambda \) with \( x \in X_\lambda \), we get \( st(x, P_{\lambda,n}) \subset U_\lambda \), where \( U_\lambda = U \cap X_\lambda \). It is implies that \( d_{\lambda}(f_{\lambda}^{-1}(x), M_\lambda - f_{\lambda}^{-1}(U_\lambda)) \geq 1/n \). In fact, if \( a = (\alpha_k) \in M_\lambda \) such that \( d_{\lambda}(f_{\lambda}^{-1}(x), a) < 1/n \), then there exists \( b = (\beta_k) \in f_{\lambda}^{-1}(x) \) such that \( d_{\lambda}(a, b) < 1/n \). So \( \alpha_k = \beta_k \) if \( k \leq n \). Note that \( x \in P_{\beta_n} \subset st(x, P_{\lambda,n}) \subset U_\lambda \). Then \( f_{\lambda}(a) \in P_{\alpha_n} = P_{\beta_n} \subset st(x, P_{\lambda,n}) \subset U_\lambda \). Hence \( a \in f_{\lambda}^{-1}(U_\lambda) \). It implies that \( d_{\lambda}(f_{\lambda}^{-1}(x), a) \geq 1/n \) if \( a \in M_\lambda - f_{\lambda}^{-1}(U_\lambda) \). So \( d_{\lambda}(f_{\lambda}^{-1}(x), M_\lambda - f_{\lambda}^{-1}(U_\lambda)) \geq 1/n \).

Therefore

\[
\begin{align*}
d(f^{-1}(x), M - f^{-1}(U)) &= \inf\{d(a, b) : a \in f^{-1}(x), b \in M - f^{-1}(U)\} \\
&= \min\{1, \inf\{d_\lambda(a, b) : a \in f_{\lambda}^{-1}(x), b \in M_\lambda - f_{\lambda}^{-1}(U_\lambda), \lambda \in \Lambda\}\} \\
&\geq 1/n > 0.
\end{align*}
\]

It implies that \( f \) is a \( \pi \)-mapping.

(b) \( f \) is compact-covering.

For each compact subset \( K \) of \( X \), there exists a finite subset \( \Lambda_K \) of \( \Lambda \) such that \( K \) has a finite compact cover \( \{ K_\lambda : \lambda \in \Lambda_K \} \), and for each \( \lambda \in \Lambda_K \) and \( n \in \mathbb{N} \), \( P_{\lambda,n} \) is a cfp-cover for \( K_\lambda \) in \( X_\lambda \). It follows from [14, Lemma 13] that \( K_\lambda = f_\lambda(L_\lambda) \) with some compact subset \( L_\lambda \) of \( M_\lambda \). Put \( L = \bigcup\{L_\lambda : \lambda \in \Lambda_K\} \), then \( L \) is a compact subset of \( M \) and \( f(L) = K \). It implies that \( f \) is compact-covering.

By Theorem 2.8, we get the following.

**Corollary 2.9.** The following are equivalent for a space \( X \).

1. \( X \) is a compact-covering quotient (resp., pseudo-open) \( \pi \)-image of a locally separable metric space,
2. \( X \) is a sequential (resp., Fréchet) space having a cover \( \{ X_\lambda : \lambda \in \Lambda \} \), where each \( X_\lambda \) has a refinement sequence of countable covers \( \{ P_{\lambda,n} \}_{n \in \mathbb{N}} \) satisfying the following:
   1. \( \{ P_n \}_{n \in \mathbb{N}} \) is a point-star network of \( X \), where \( P_n = \bigcup_{\lambda \in \Lambda} P_{\lambda,n} \) for every \( n \in \mathbb{N} \),
   2. For every compact subset \( K \) of \( X \), there exists a finite subset \( \Lambda_K \) of \( \Lambda \) such that \( K \) has a finite compact cover \( \{ K_\lambda : \lambda \in \Lambda_K \} \), and for each \( \lambda \in \Lambda_K \) and \( n \in \mathbb{N} \), \( P_{\lambda,n} \) is a cfp-cover for \( K_\lambda \) in \( X_\lambda \).
Proof. As in the proof of Corollary 2.4.

Finally, we give examples to illustrate theorems in the above.

**Example 2.10.** There exists a compact-covering $s$-image of a locally separable metric space which is not a compact-covering $\pi$-image of any locally separable metric space.

**Proof.** Let $X$ be a sequential fan $S_\omega$ (see [9]). Then $X$ is a Fréchet and $\aleph_0$-space. It follows from [18, Remark 8.(2)] that $X$ is a compact-covering $s$-image of a locally separable metric space. It is clear that every compact-covering mapping is a pseudo-sequence-covering mapping, and $X$ is not a pseudo-sequence-covering $\pi$-image of any metric space [8, Example 2.8]. Then $X$ is not a compact-covering $\pi$-image of any locally separable metric space.

**Example 2.11.** There exists a compact-covering $\pi$-image of a locally separable metric space which is not a compact-covering $s$-image of any locally separable metric space.

**Proof.** Let $X$ be a developable space $Y$ in [7, Example 17]. Then $X$ is a compact-covering (quotient) $\pi$-image of a locally separable metric space. Moreover, $X$ is not a quotient $s$-image of any locally separable metric space. It implies that $X$ is not a compact-covering $s$-image of any locally separable metric space.

**References**


