Entire Functions and Their Derivatives Share Two Finite Sets

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Abstract. In this paper, we study the uniqueness of entire functions and prove the following theorem. Let \( n (\geq 5), k \) be positive integers, and let \( S_1 = \{ z : z^n = 1 \} \), \( S_2 = \{ a_1, a_2, \cdots, a_m \} \), where \( a_1, a_2, \cdots, a_m \) are distinct nonzero constants. If two nonconstant entire functions \( f \) and \( g \) satisfy \( E_f(S_1, 2) = E_g(S_1, 2) \) and \( E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty) \), then one of the following cases must occur: (1) \( f = t g \), \( \{ a_1, a_2, \cdots, a_m \} = \{ t a_1, t a_2, \cdots, t a_m \} \), where \( t \) is a constant satisfying \( t^n = 1 \); (2) \( f(z) = d e^{cz}, g(z) = t d e^{-cz}, \{ a_1, a_2, \cdots, a_m \} = (-1)^k c^{2k} t \{ \frac{1}{a_1}, \cdots, \frac{1}{a_m} \} \), where \( t, c, d \) are nonzero constants and \( t^n = 1 \).


1. Introduction, definitions and results

Let \( f \) and \( g \) be two nonconstant meromorphic functions defined in the open complex plane \( \mathbb{C} \). If for some \( a \in \mathbb{C} \cup \{ \infty \} \), \( f \) and \( g \) have the same set of \( a \)-points with the same multiplicities then we say that \( f \) and \( g \) share the value \( a \) CM (counting multiplicities). If we do not take the multiplicities into account, \( f \) and \( g \) are said to share the value \( a \) IM (ignoring multiplicities). We assume that the reader is familiar with the notations of Nevanlinna theory that can be found, for instance, in [5] or [9].

Let \( S \) be a set of distinct elements of \( \mathbb{C} \cup \{ \infty \} \) and \( E_f(S) = \cup_{a \in S} \{ z : f(z) - a = 0 \} \), where each zero is counted according to its multiplicity. If we do not count the multiplicity the set \( \cup_{a \in S} \{ z : f(z) - a = 0 \} \) is denoted by \( \mathcal{E}_f(S) \). If \( E_f(S) = E_g(S) \) we say that \( f \) and \( g \) share the set \( S \) CM. On the other hand, if \( \mathcal{E}_f(S) = \mathcal{E}_g(S) \), we say that \( f \) and \( g \) share the set \( S \) IM. Let \( m \) be a positive integer or infinity and \( a \in \mathbb{C} \cup \{ \infty \} \). We denote by \( E_m(a, f) \) the set of all \( a \)-points of \( f \) with multiplicities not exceeding \( m \), where an \( a \)-point is counted according to its multiplicity. For a

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set $S$ of distinct elements of $C$ we define $E_m(S, f) = \cup_{a \in S} E_m(a, f)$. If for some $a \in C \cup \{\infty\}$, $E_m(a, f) = E_m(a, g)$, we say that $f$ and $g$ share the value $a$ CM. We can define $E_m(a, f)$ and $E_m(S, f)$ similarly.

In 1977, Gross [4] posed the following question.

**Question.** Can one find two finite sets $S_j (j = 1, 2)$ such that any two nonconstant entire functions $f$ and $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

Yi [10] gave a positive answer to the question. He proved.

**Theorem A**([10]). Let $f$ and $g$ be two nonconstant entire functions, $n \geq 5$ a positive integer, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a\}$, where $a \neq 0$ is a constant satisfying $a^{2n} \neq 1$. If $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

In 2001, Fang [3] investigated the question and proved the following theorems.

**Theorem B**([3]). Let $f$ and $g$ be two nonconstant entire functions, $n \geq 5$, $k$ two positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a, b, c\}$, where $a, b, c$ are nonzero finite distinct constants satisfying $a^2 \neq bc$, $b^2 \neq ac$, $c^2 \neq ab$. If $E_f(S_1) = E_g(S_1)$ and $E_f(S_2) = E_g(S_2)$, then $f \equiv g$.

**Theorem C**([3]). Let $f$ and $g$ be two nonconstant entire functions, $n \geq 5$, $k$ two positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a, b\}$, where $a, b$ are two nonzero finite distinct constants. If $E_f(S_1) = E_g(S_1)$ and $E_f(S_2) = E_g(S_2)$, then one of the following cases must occur: (1) $f \equiv g$; (2) $b = -a$, $f = e^{cz+d}$, $g = te^{-cz-d}$, where $c, d, t$ are three constants satisfying $t^n = 1$ and $(-1)^k t e^{2k} = a^2$; (3) $f = e^{cz+d}$, $g = te^{-cz-d}$, where $c, d, t$ are three constants satisfying $t^n = 1$ and $(-1)^k t e^{2k} = ab$; (4) $b = -a$, $f \equiv -g$.

**Theorem D**([3]). Let $f$ and $g$ be two nonconstant entire functions, $n \geq 5$, $k$ two positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a\}$, where $a \neq 0, \infty$. If $E_f(S_1) = E_g(S_1)$ and $E_f(S_2) = E_g(S_2)$, then one of the following cases must occur: (1) $f \equiv g$; (2) $f = e^{cz+d}$, $g = te^{-cz-d}$, where $c, d, t$ are three constants satisfying $t^n = 1$ and $(-1)^k t e^{2k} = a^2$.

In this paper, we consider the more general sets $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. To state the main results of this paper, we require the following notion of weighted sharing which was introduced by I. Lahiri [6], [7].

**Definition 1**([6]). For a complex number $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all $a$-points of $f$ where an $a$-point with multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. For a complex number $a \in C \cup \{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that $f$ and $g$ share the value $a$ with weight $k$.

The definition implies that if $f$, $g$ share a value $a$ with weight $k$ then $z_0$ is a zero of $f-a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity
two nonconstant entire functions share a value which also improves Theorem B, Theorem C and Theorem D.

Theorem 3. Let $n \geq 5$, $k$ be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. If two nonconstant entire functions $f$ and $g$ satisfy $E_f(S_1, 2) = E_g(S_1, 2)$ and $E_f(S_2, \infty) = E_g(S_2, \infty)$, then one of the following cases must occur: (1) $f = tg$, $\{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\}$, where $t$ is a constant satisfying $t^n = 1$; (2) $f(z) = d e^{cz}$, $g(z) = \frac{1}{d} e^{-cz}$, $\{a_1, a_2, \ldots, a_m\} = (-1)^k e^{2kt}\left\{\frac{1}{a_1}, \ldots, \frac{1}{a_m}\right\}$, where $t, c, d$ are nonzero constants and $t^n = 1$.

Theorem 2. Let $n \geq 5$, $k$ be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. If two nonconstant entire functions $f$ and $g$ satisfy $E_f(S_1, 1) = E_g(S_1, 1)$ and $E_f(S_2, \infty) = E_g(S_2, \infty)$, then one of the following cases must occur: (1) $f = tg$, $\{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\}$, where $t$ is a constant satisfying $t^n = 1$; (2) $f(z) = d e^{cz}$, $g(z) = \frac{1}{d} e^{-cz}$, $\{a_1, a_2, \ldots, a_m\} = (-1)^k e^{2kt}\left\{\frac{1}{a_1}, \ldots, \frac{1}{a_m}\right\}$, where $t, c, d$ are nonzero constants and $t^n = 1$.

Definition 2([6]). Let $S$ be a set of distinct elements of $C \cup \{\infty\}$ and $k$ a non-negative integer or $\infty$. We denote by $E_f(S, k)$ the set $\cup_{a \in S} E_k(a, f)$. Clearly $E_f(S) = E_f(S, \infty)$ and $E_f(S) = E_f(S, 0)$.

With the notion of weighted sharing of sets we prove the following results which improve Theorem B, Theorem C and Theorem D.

Theorem 1. Let $n \geq 5$, $k$ be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. If two nonconstant entire functions $f$ and $g$ satisfy $E_f(S_1, 2) = E_g(S_1, 2)$ and $E_f(S_2, \infty) = E_g(S_2, \infty)$, then one of the following cases must occur: (1) $f = tg$, $\{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\}$, where $t$ is a constant satisfying $t^n = 1$; (2) $f(z) = d e^{cz}$, $g(z) = \frac{1}{d} e^{-cz}$, $\{a_1, a_2, \ldots, a_m\} = (-1)^k e^{2kt}\left\{\frac{1}{a_1}, \ldots, \frac{1}{a_m}\right\}$, where $t, c, d$ are nonzero constants and $t^n = 1$.

With the notion of weighted sharing of sets we prove the following theorem which also improves Theorem B, Theorem C and Theorem D.

Theorem 4. Let $n \geq 5$, $k$ be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. If two nonconstant entire functions $f$ and $g$ satisfy $E_f(S_1, f) = E_g(S_1, g)$, $E_f(S_2, f) = E_g(S_2, g)$, $E_f(S_1, g)$ and $E_f(S_2, \infty) = E_g(S_2, \infty)$, then one of the following cases must occur: (1) $f = tg$, $\{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\}$, where $t$ is a con-
stant satisfying \( t^n = 1 \); (2) \( f(z) = de^{cz}, \ g(z) = \frac{t}{d}e^{-cz} \), \( \{a_1, a_2, \cdots, a_m\} = (-1)^k e^{2k t}\{\frac{1}{a_1}, \cdots, \frac{1}{a_m}\} \), where \( t, c, d \) are nonzero constants and \( t^n = 1 \).

2. Some lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by \( H \) the following function:

\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right). 
\]

Lemma 1([8]). Let \( f \) be a nonconstant meromorphic function, and let \( a_0, a_1, a_2, \cdots, a_n \) be finite complex numbers, \( a_n \neq 0 \). Then

\[
T(r, a_n f^n + \cdots + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f).
\]

Lemma 2([7]). Let \( H \) be defined as above. If \( F \) and \( G \) share \((1,2)\) and \( H \not\equiv 0 \), then

\[
T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G),
\]

the same inequality holds for \( T(r, G) \).

Lemma 3([2]). Let \( H \) be defined as above. If \( F \) and \( G \) share \((1,1)\) and \( H \not\equiv 0 \), then

\[
T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) + \frac{1}{2}N(r, \frac{1}{F}) + \frac{1}{2}N(r, F) + S(r, F) + S(r, G),
\]

the same inequality holds for \( T(r, G) \).

Lemma 4([11]). Let \( H \) be defined as above. If \( H \equiv 0 \) and

\[
\limsup_{r \to \infty} \frac{N(r, \frac{1}{F}) + N(r, \frac{1}{G}) + N(r, F) + N(r, G)}{T(r)} < 1, \ r \in I,
\]

where \( I \) is a set with infinite linear measure and \( T(r) = \max\{T(r, F), T(r, G)\} \), then \( FG \equiv 1 \) or \( F \equiv G \).

Lemma 5([2]). Let \( F, G \) be two nonconstant meromorphic functions such that
they share \((1,0)\), and \(H \neq 0\). Then

\[
T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + 2N(r, \frac{1}{F}) + 2N(r, F) + N(r, \frac{1}{G}) + N(r, G) + S(r, F) + S(r, G),
\]

the same inequality holds for \(T(r, G)\).

**Lemma 6\([1]\).** Let \(F, G\) be two nonconstant meromorphic functions such that \(E_4(1, F) = E_4(1, G)\) and \(E_2(1, F) = E_2(1, G)\), then one of the following cases holds
\((1)\)

\[T(r, F) + T(r, G) \leq 2 \{N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G)\} + S(r, F) + S(r, G) ; (2) F \equiv G ; (3) FG \equiv 1.\]

**Lemma 7\([5]\).** Let \(f\) be a nonconstant meromorphic function, \(n\) be a positive integer, and let \(\Psi\) be a function of the form \(\Psi = f^n + Q\), where \(Q\) is a differential polynomial of \(f\) with degree \(\leq n - 1\). If

\[N(r, f) + N\left(\frac{1}{\Psi}\right) = S(r, f),\]

then \(\Psi = (f + \alpha)^n\), where \(\alpha\) is a meromorphic function with \(T(r, \alpha) = S(r, f)\), determined by the term of degree \(n - 1\) in \(Q\).

### 3. Proof of theorem 1

Set \(F = f^n, G = g^n\). From \(E_f(S_1, 2) = E_g(S_1, 2)\), we deduce \(F\) and \(G\) share \((1, 2)\). By Lemma 1, we have

\((1)\)

\[T(r, F) = nT(r, f) + S(r, f), \quad T(r, G) = nT(r, g) + S(r, g).\]

Assume \(H \neq 0\). By Lemma 2, we have

\((2)\)

\[T(r, F) = nT(r, f) + S(r, f) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + S(r, F) + S(r, G) \leq 2T(r, f) + 2T(r, g) + S(r, f) + S(r, g).\]

Similarly, we have

\((3)\)

\[T(r, G) = nT(r, g) + S(r, f) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + S(r, F) + S(r, G) \leq 2T(r, f) + 2T(r, g) + S(r, f) + S(r, g).\]

Combining (2) and (3) together we have

\((4)\)

\[(n - 4)T(r, f) + (n - 4)T(r, g) \leq S(r, f) + S(r, g),\]
which contradicts \( n \geq 5 \). Thus \( H \equiv 0 \). By Lemma 4, we have \( FG \equiv 1 \) or \( F \equiv G \), that is \( f = tg \) or \( fg = t \) where \( t \) is a constant and \( t^n = 1 \). Next we consider the following two cases:

**Case 1.** \( f = tg \). Then \( f^{(k)} = tg^{(k)} \). By \( E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty) \), we get \( \{a_1, a_2, \cdots, a_m\} = t\{a_1, a_2, \cdots, a_m\} \).

**Case 2.** \( fg = t \). Then there exists an entire function \( h \) such that \( f = e^h \) and \( g = te^{-h} \). Therefore

\[
\alpha_k = \alpha_1^k + P(\alpha_1), \beta_k = \beta_1^k + Q(\beta_1),
\]

where \( P(\alpha_1) \) is a differential polynomial in \( \alpha_1 \) of degree \( k - 1 \), and \( Q(\beta_1) \) is a differential polynomial in \( \beta_1 \) of degree \( k - 1 \). If \( \alpha_1 \) and \( \beta_1 \) are not constants, then by Lemma 7, we have

\[
\alpha_k = \left( \alpha_1 + \frac{\gamma_1}{k} \right)^k, \beta_k = \left( \beta_1 + \frac{\gamma_2}{k} \right)^k,
\]

where \( \gamma_1, \gamma_2 \) are small functions of \( \alpha_1 \) and \( \beta_1 \), respectively. Note that \( \alpha_1 = -\beta_1 = h' \). We conclude that \( \alpha_k \beta_k \) can not be constant, which is a contradiction. Hence one of \( \alpha_1 \) and \( \beta_1 \) is constant. Thus \( h \) is a linear function. Therefore, \( f(z) = de^{cz} \) and \( g(z) = \frac{t}{d}e^{-cz} \), where \( c, d \) are nonzero constants. Now from \( E_{f^{(k)}}(S_2, \infty) = \ldots \)
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\( E_{g^n}(S_2, \infty) \), we get \( \{a_1, a_2, \ldots, a_m\} = (-1)^k c^{2k} t\{ \frac{1}{a_1}, \ldots, \frac{1}{a_m} \} \), which completes the proof of Theorem 1.

4. Proof of theorem 2

Set \( F = f^n \), \( G = g^n \). From \( E_f(S_1, 1) = E_g(S_1, 1) \), we deduce \( F \) and \( G \) share \((1, 1)\). By Lemma 1, we have

\[
T(r, F) = nT(r, f) + S(r, f), \quad T(r, G) = nT(r, g) + S(r, g).
\]

Assume \( H \neq 0 \). By Lemma 3, we have

\[
T(r, F) = nT(r, f) + S(r, f)
\leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + \frac{1}{2} N(r, \frac{1}{F}) + S(r, F) + S(r, G)
\leq \frac{5}{2} T(r, f) + 2T(r, g) + S(r, f) + S(r, g).
\]

Similarly, we have

\[
T(r, G) = nT(r, g) + S(r, g)
\leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + \frac{1}{2} N(r, \frac{1}{G}) + S(r, F) + S(r, G)
\leq 2T(r, f) + \frac{5}{2} T(r, g) + S(r, f) + S(r, g).
\]

Combining (12) and (13) together we have

\[
(n - \frac{9}{2}) T(r, f) + (n - \frac{9}{2}) T(r, g) \leq S(r, f) + S(r, g),
\]

which contradicts \( n \geq 5 \). Thus \( H \equiv 0 \). By Lemma 4, we have \( FG \equiv 1 \) or \( F \equiv G \), that is \( f = t g \) or \( fg = t \) where \( t \) is a constant and \( t^n = 1 \). Proceeding as in the proof of Theorem 1, we get the conclusion of Theorem 2. This completes the proof of Theorem 2.

5. Proof of theorem 3

Set \( F = f^n \), \( G = g^n \). From \( E_f(S_1, 0) = E_g(S_1, 0) \), we deduce \( F \) and \( G \) share \((1, 0)\). By Lemma 1, we have

\[
T(r, F) = nT(r, f) + S(r, f), \quad T(r, G) = nT(r, g) + S(r, g).
\]
Assume $H \not\equiv 0$. By Lemma 5, we have

\begin{align}
(16) \quad T(r, F) &= nT(r, f) + S(r, f) \\
&\leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 2\overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + S(r, F) + S(r, G) \\
&\leq 4T(r, f) + 3T(r, g) + S(r, f) + S(r, g).
\end{align}

Similarly, we have

\begin{align}
(17) \quad T(r, G) &= nT(r, g) + S(r, g) \\
&\leq 3T(r, f) + 4T(r, g) + S(r, f) + S(r, g).
\end{align}

Combining (16) and (17) together we have

\begin{align}
(18) \quad (n - 7)T(r, f) + (n - 7)T(r, g) &\leq S(r, f) + S(r, g),
\end{align}

which contradicts $n \geq 8$. Thus $H \equiv 0$. By Lemma 4, we have $FG \equiv 1$ or $F \equiv G$, that is $f = tg$ or $fg = t$ where $t$ is a constant and $t^n = 1$. Proceeding as in the proof of Theorem 1, we get the conclusion of Theorem 3. This completes the proof of Theorem 3.

6. Proof of theorem 4

Set $F = f^n$, $G = g^n$. By Lemma 1, we have

\begin{align}
(19) \quad T(r, F) &= nT(r, f) + S(r, f), \quad T(r, G) = nT(r, g) + S(r, g).
\end{align}

From $\mathcal{E}_4(S_1, f) = \mathcal{E}_4(S_1, g)$, $E_2(S_1, f) = E_2(S_1, g)$, we deduce $\mathcal{E}_4(1, F) = \mathcal{E}_4(1, G)$, $E_2(1, F) = E_2(1, G)$. Then $F$ and $G$ satisfy the condition of Lemma 6.

We assume Case (1) in Lemma 6 holds, that is,

\begin{align}
(20) \quad T(r, F) + T(r, G) &\leq 2\{N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G})\} + S(r, F) + S(r, G) \\
&\leq 4T(r, f) + 4T(r, g) + S(r, f) + S(r, g).
\end{align}

Combining (19) and (20) together we have

\begin{align}
(21) \quad (n - 4)T(r, f) + (n - 4)T(r, g) &\leq S(r, f) + S(r, g),
\end{align}

which contradicts $n \geq 5$. Thus by Lemma 6, we get $F \equiv G$ or $FG \equiv 1$, that is, $f = tg$ or $fg = t$ where $t$ is a constant and $t^n = 1$. Proceeding as in the proof of Theorem 1, we get the conclusion of Theorem 4. This completes the proof of Theorem 4.
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References


