Hypersurfaces of an almost \( r \)-paracontact Riemannian Manifold Endowed with a Quarter Symmetric Non-metric Connection

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Abstract. We define a quarter symmetric non-metric connection in an almost \( r \)-paracontact Riemannian manifold and we consider invariant, non-invariant and anti-invariant hypersurfaces of an almost \( r \)-paracontact Riemannian manifold endowed with a quarter symmetric non-metric connection.

1. Introduction

Let \( \nabla \) be a linear connection in an \( n \)-dimensional differentiable manifold \( M \). The torsion tensor \( T \) and the curvature tensor \( R \) of \( \nabla \) are given respectively by

\[
T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y],
\]

\[
R(X,Y)Z \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

The connection \( \nabla \) is symmetric if its torsion tensor \( T \) vanishes, otherwise it is non-symmetric. The connection \( \nabla \) is a metric connection if there is a Riemannian metric \( g \) in \( M \) such that \( \nabla g = 0 \), otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

In [8], S. Golab introduced the idea of a quarter symmetric linear connection in a differentiable manifold. A linear connection is said to be a quarter-symmetric connection if its torsion tensor \( T \) is of the form

\[
T(X,Y) = u(Y)\varphi X - u(X)\varphi Y,
\]

where \( u \) is a 1-form and \( \varphi \) is a \((1,1)\)-tensor field. In [8], [11] some properties of some kinds of quarter symmetric non-metric connections were studied.

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A. Bucki and A. Miernowski defined an almost r-paracontact structures and studied some properties of invariant hypersurfaces of an almost r-paracontact structures in [5] and [6] respectively. A. Bucki also studied almost r-paracontact structures of P-Sasakian type in [3]. I. Mihai and K. Matsumoto studied submanifolds of an almost r-paracontact Riemannian manifold of P-Sasakian type in [10]. Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter-symmetric metric connection were studied by first and third author and J. B. Jun in [2]. Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric metric connection were studied by J. B. Jun and the first author in [9]. Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric non-metric connection were studied by first and second author in [1].

Motivated by the studies of the above authors, in this paper, we study quarter symmetric non-metric connection in an almost r-paracontact Riemannian manifold. We consider invariant, non-invariant and anti-invariant hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric non-metric connection.

The paper is organized as follows: In Section 2, we give a brief introduction about an almost r-paracontact Riemannian manifold. In Section 3, we show that the induced connection on a hypersurface of an almost r-paracontact Riemannian manifold with a quarter symmetric non-metric connection with respect to the unit normal is also a quarter symmetric non-metric connection. We find the characteristic properties of invariant, non-invariant and anti-invariant hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric non-metric connection.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold with a positive definite metric g. If there exist a tensor field \( \varphi \) of type (1,1), r vector fields \( \xi_1, \xi_2, \cdots, \xi_r \) \( (n > r) \), and r 1-forms \( \eta_1, \eta_2, \cdots, \eta_r \) such that

\[
\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad \alpha, \beta \in (r) = \{1, 2, 3, \cdots, r\},
\]

\[
\varphi^2(X) = X - \eta^\alpha(X)\xi_\alpha,
\]

\[
\eta^\alpha(X) = g(X, \xi_\alpha), \quad \alpha \in (r),
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha} \eta^\alpha(X)\eta^\alpha(Y),
\]

where \( X \) and \( Y \) are vector fields on \( M \), then the structure \( \sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)} \) is said to be an almost r-paracontact Riemannian structure and \( M \) is an almost r-paracontact Riemannian manifold [5]. From (2.1)-(2.4), we have

\[
\varphi(\xi_\alpha) = 0, \quad \alpha \in (r),
\]

\[
\eta^\alpha \circ \varphi = 0, \quad \alpha \in (r),
\]
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\[ \Psi(X, Y) \overset{\text{def}}{=} g(\varphi X, Y) = g(X, \varphi Y). \]

For Riemannian connection \( \hat{\nabla} \) on \( M \), the tensor \( N \) is given by

\[ N(X, Y) = \left( \hat{\nabla}_{\varphi Y} \varphi \right) X - \left( \hat{\nabla}_{X \varphi} \varphi \right) Y - \left( \hat{\nabla}_{\varphi X} \varphi \right) Y + \left( \hat{\nabla}_{Y \varphi} \varphi \right) X + \eta^\alpha(Y) \hat{\nabla}_{Y} \xi_\alpha. \]

An almost \( r \)-paracontact Riemannian manifold \( M \) with structure \( \sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)} \) is said to be of para-contact type if

\[ 2\Psi(X, Y) = \left( \hat{\nabla}_X \eta^\alpha \right) Y + \left( \hat{\nabla}_Y \eta^\alpha \right) X, \quad \text{for all } \alpha \in (r). \]

If all \( \eta^\alpha \) are closed, then the last equation reduces to

\[ \Psi(X, Y) = \left( \hat{\nabla}_X \eta^\alpha \right) Y, \quad \text{for all } \alpha \in (r) \]

and \( M \) satisfying this condition is called an almost \( r \)-paracontact Riemannian manifold of \( s \)-paracontact type [3]. An almost \( r \)-paracontact Riemannian manifold \( M \) with a structure \( \sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)} \) is said to be \( P \)-Sasakian if it satisfies (2.6) and

\[ \left( \hat{\nabla}_Z \Psi \right)(X,Y) = -\sum_{\alpha} \eta^\alpha(X) \left[ g(Y, Z) - \sum_{\beta} \eta^\beta(Y) \eta^\beta(Z) \right] - \sum_{\alpha} \eta^\alpha(Y) \left[ g(X, Z) - \sum_{\beta} \eta^\beta(Y) \eta^\beta(Z) \right] \]

for all vector fields \( X, Y \) and \( Z \) on \( M \) [3]. The conditions (2.9) and (2.10) are equivalent to

\[ \varphi X = \hat{\nabla}_X \xi_\alpha, \quad \text{for all } \alpha \in (r) \]

and

\[ \left( \hat{\nabla}_Y \varphi \right) X = -\sum_{\alpha} \eta^\alpha(X) \left[ Y - \sum_{\beta} \eta^\beta(Y) \xi_\alpha \right] - \left[ g(X, Y) - \sum_{\alpha} \eta^\alpha(X) \eta^\alpha(Y) \right] \sum_{\beta} \xi_\beta. \]
respectively.

We define a quarter symmetric non-metric connection $\nabla$ on $M$ by

$$\nabla_X Y = \nabla^* X Y + \eta^\alpha(Y)\varphi X,$$

for any $\alpha \in (r)$. Using (2.13) we get

$$(\nabla_Y \varphi) X = -\sum_\alpha \eta^\alpha(X) [Y - \eta^\alpha(Y)\xi_\alpha]$$

and

$$\nabla_X \xi_\alpha = 2\varphi X.$$ 

3. Hypersurfaces of almost $r$-paracontact Riemannian manifold with a quarter-symmetric non-metric connection

Let $\tilde{M}^{n+1}$ be an almost $r$-paracontact Riemannian manifold with a positive definite metric $g$ and $M^n$ be a hypersurface immersed in $\tilde{M}^{n+1}$ by immersion $f : M^n \rightarrow \tilde{M}^{n+1}$. If $B$ denote the differential of $f$ then any vector field $\overline{X} \in \chi(M^n)$ implies $B\overline{X} \in \chi(M^{n+1})$. We denote the object belonging to $M^n$ by the mark of hyphen placed over them, e.g, $\overline{\varphi}, \overline{X}, \overline{\eta}, \overline{\xi}$ etc.

Let $N$ be the unit normal field to $M^n$. Then the induced metric $\overline{g}$ on $M^n$ is defined by

$$\overline{g}(\overline{X}, \overline{Y}) = g(\overline{X}, \overline{Y}).$$

Then we have [7]

$$g(\overline{X}, N) = 0 \quad \text{and} \quad g(N, N) = 1.$$ 

Equation of Gauss with respect to Riemannian connection $\overline{\nabla}$ is given by

$$\overline{\nabla}_{\overline{X}} \overline{Y} = \overline{\nabla}^*_X \overline{Y} + h(\overline{X}, \overline{Y}) N.$$

If $\overline{\nabla}$ is the induced connection on hypersurface from $\overline{\nabla}$ with respect to unit normal $N$, then Gauss equation is given by

$$\overline{\nabla}^*_{\overline{X}} \overline{Y} = \overline{\nabla}_{\overline{X}} \overline{Y} + h(\overline{X}, \overline{Y}) N.$$
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where, \( h \) is second fundamental tensor satisfying

\[(3.5) \quad h(X, Y) = h(Y, X) = g(H(X), Y)\]

and \( H \) is the shape operator of \( M^n \) in \( \bar{M}^{n+1} \). If \( \nabla \) is the induced connection on hypersurface from the quarter symmetric non-metric connection \( \nabla \) with respect to unit normal \( N \), then we have

\[(3.6) \quad \nabla_X Y = \nabla_X Y + m(X, Y)N,\]

where \( m \) is a tensor field of type \((0, 2)\) on hypersurface \( M^n \). From (2.13), using \( \varphi \bar{X} = \varphi X + b(X)N \) we obtain

\[(3.7) \quad \nabla_X \bar{Y} = \hat{\nabla}_X \bar{Y} + \eta^\alpha(\bar{Y})(\varphi \bar{X} + b(X)N).\]

From equations (3.4), (3.6) and (3.7), we get

\[\nabla_X Y + m(X, Y)N = \nabla_X Y + h(X, Y)N + \eta^\alpha(\bar{Y})\varphi \bar{X} + \eta^\alpha(\bar{Y})b(X)N.\]

By taking tangential and normal parts from both the sides, we obtain

\[(3.8) \quad \nabla_X \bar{Y} = \hat{\nabla}_X \bar{Y} + \eta^\alpha(\bar{Y})\varphi \bar{X}\]

and

\[(3.9) \quad m(X, Y) = h(X, Y) + \eta^\alpha(\bar{Y})b(X).\]

Thus we get the following theorem:

**Theorem 3.1.** The connection induced on a hypersurface of an almost \( r \)-paracontact Riemannian manifold with a quarter-symmetric non-metric connection with respect to the unit normal is also a quarter-symmetric non-metric connection.

From (3.6) and (3.9), we have

\[(3.10) \quad \nabla_X Y = \nabla_X Y + h(X, Y)N + \eta^\alpha(\bar{Y})b(X),\]

which is Gauss equation for a quarter symmetric non-metric connection. The equation of Weingarten with respect to the Riemannian connection \( \hat{\nabla} \) is given by

\[(3.11) \quad \hat{\nabla}_X N = -H \bar{X}\]

for every \( \bar{X} \) in \( M^n \). From equation (2.13), we have

\[(3.12) \quad \nabla_X N = \hat{\nabla}_X N + a_\alpha \varphi \bar{X} + a_\alpha b(X)N,\]
where
\begin{equation}
\eta^\alpha(N) = a_\alpha = m(\xi_\alpha).
\end{equation}

From (3.11) and (3.12), we have
\begin{equation}
\nabla_X N = - M X,
\end{equation}
where $M X = H X - a_\alpha \varphi X - a_\alpha b(X) N$, which is Weingarten equation with respect to quarter symmetric non-metric connection.

Now, suppose that $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is an almost $r$-paracontact Riemannian structure on $\tilde{M}^{n+1}$, then every vector field $X$ on $\tilde{M}^{n+1}$ is decomposed as
\begin{equation}
X = \overline{X} + l(X) N,
\end{equation}
where $l$ is a 1-form on $\tilde{M}^{n+1}$ and for any vector field $\overline{X}$ on $M^n$ and normal $N$, we have
\begin{align}
\varphi \overline{X} &= \overline{\varphi X} + b(\overline{X}) N, \\
\varphi N &= \overline{N} + KN,
\end{align}
where $\overline{\varphi}$ is a tensor field of type $(1,1)$ on hypersurface $M^n$, $b$ is a 1-form on $M^n$ and $K$ is a scalar function on $M^n$. For each $\alpha \in (r)$, we have
\begin{equation}
\xi_\alpha = \overline{\xi_\alpha} + a_\alpha N,
\end{equation}
where $a_\alpha = m(\xi_\alpha) = \eta^\alpha(N), \alpha \in (r)$. Now, we define $\overline{\eta}^\alpha$ by
\begin{equation}
\overline{\eta}^\alpha(\overline{X}) = \eta^\alpha(\overline{X}), \quad \alpha \in (r).
\end{equation}
Making use of (3.16), (3.17), (3.18) and (3.13), from (2.1)-(2.5), we obtain
\begin{align}
\varphi^2 \overline{X} + b(\overline{X}) N &= \overline{X} - \overline{\eta}^\alpha(\overline{X}) \overline{\xi_\alpha}, \\
b(\overline{\varphi \overline{X}}) + K b(\overline{X}) &= - a_\alpha \overline{\eta}^\alpha(\overline{X}), \\
\overline{\varphi N} + KN &= - \sum_{\alpha} a_\alpha \overline{\xi_\alpha}, \\
b(\overline{N}) + K^2 &= 1 - \sum_{\alpha} (a_\alpha)^2, \\
\varphi(\overline{\xi_\alpha}) + a_\alpha \overline{N} &= 0, \\
Ka_\alpha + b(\overline{\xi_\alpha}) &= 0, \\
(\overline{\eta}^\alpha \circ \varphi)(\overline{X}) + b(\overline{X}) a_\alpha &= 0, \\
\overline{\eta}^\alpha(\overline{\xi_\beta}) + a_\alpha a_\beta &= \delta_\beta^\alpha, \\
\overline{\eta}^\alpha(\overline{X}) &= \overline{\eta}(\overline{X}, \overline{\xi_\alpha}).
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(3.29) \[ g(\varphi X, \varphi Y) - b(X)b(Y) = g(X, Y) - \sum \alpha \eta^\alpha (X) \eta^\alpha (Y), \]

and

(3.30) \[ \Psi(X, Y) = g(\varphi X, Y) = g(X, \varphi Y) = \Psi(X, Y). \]

where $\alpha, \beta \in (r)$. Using (3.1), (3.2), (3.17), (3.18) and (2.7), we have

\[ g(\varphi X, N) = g(\varphi X, N) - b(X) = g(X, \varphi N) - b(X) = 0. \]

Hence we get

(3.31) \[ g(X, N) = b(X), \]

(see [4]). Differentiating (3.16) and (3.17) along $M^n$ and making use of (3.10), (3.29) and (3.1) we get

\[ (\nabla_Y \varphi) X = (\nabla_Y \varphi) X - h(X, Y) N + \eta^\alpha (\varphi X)b(Y) - b(X) [H(Y) - a_\alpha \varphi Y] \]

+ \[ h(\varphi X, Y) + (\nabla_Y b)X - Kh(X, Y) + a_\alpha b(X)b(Y) \] \[ N, \]

(3.32)

and

\[ (\nabla_Y \varphi) N = \nabla_Y \varphi X N - a_\alpha Y + a_\alpha a_\alpha \varphi Y - (a_\alpha N - \eta^\alpha (N)) \]

+ \[ K (a_\alpha \varphi Y - H(Y)) + [h(Y, N) + Y(K) + bH(Y)]. \]

(3.33)

From (3.18) and (3.13), we have

(3.34) \[ \nabla_X \xi_\alpha = \nabla_X \xi_\alpha - a_\alpha H(Y) + (a_\alpha)^2 \varphi Y + \eta^\alpha (\xi_\alpha)b(Y) \]

+ \[ [(a_\alpha)^2 b(Y) + \varphi(a_\alpha) + h(Y, \xi_\alpha)] N \]

and

(3.35) \[ (\nabla_Y \eta^\alpha) X = (\nabla_Y \eta^\alpha) X - h(Y, X)a_\alpha. \]

From identity

\[ (\nabla_Z \Psi) (X, Y) = g((\nabla_Z \varphi) X, Y), \]

using (3.30), (3.31) and (3.32) we have

\[ (\nabla_Z \Psi) (X, Y) = (\nabla_Z \varphi) (X, Y) - h(X, Z)b(Y) \]

- \[ b(X)h(Z, Y) + a_\alpha b(X)\eta(Y, Z). \]

(3.36)
Theorem 3.2. If $M^n$ is an invariant hypersurface immersed in an almost $r$-paracontact Riemannian manifold $M^{n+1}$ endowed with a quarter symmetric non-metric connection with structure $\sum = (\varphi, \xi^{\alpha}, \eta^{\alpha}, g)_{\alpha \in (r)}$, then either

(i) All $\xi_{\alpha}$ are tangent to $M^n$ and $M^n$ admits an almost $r$-paracontact Riemannian structure $\sum_1 = (\varphi, \xi^{\alpha}, \eta^{\alpha}, g)_{\alpha \in (r)}$, $(n-r > 2)$ or

(ii) One of $\xi_{\alpha}$ (say $\xi_1$) is normal to $M^n$ and remaining $\xi_{\alpha}$ are tangent to $M^n$ and $M^n$ admits an almost $(r-1)$-paracontact Riemannian structure $\sum_2 = (\varphi, \xi^{\alpha}, \eta^{\alpha}, g)_{\alpha \in (r)}$, $(n-r > 1)$.

Proof. The proof is similar to the proof of Theorem 3.3 in [4].

Corollary 3.3. If $M^n$ is a hypersurface immersed in an almost $r$-paracontact Riemannian manifold $M^{n+1}$ with a structure $\sum = (\varphi, \xi^{\alpha}, \eta^{\alpha}, g)_{\alpha \in (r)}$ endowed with a quarter symmetric non-metric connection, then the following statements are equivalent.

(i) $M^n$ is invariant,
(ii) The normal field $N$ is an eigenvector of $\varphi$,
(iii) All $\xi_{\alpha}$ are tangent to $M^n$ if and only if $M^n$ admits an almost $r$-paracontact Riemannian structure $\sum_1$, or one of $\xi_{\alpha}$ is normal and $(r-1)$ remaining $\xi_i$ are tangent to $M^n$ if and only if $M^n$ admits an almost $(r-1)$ paracontact Riemannian structure $\sum_2$.

Theorem 3.4. If $M^n$ is an invariant hypersurface immersed in an almost $r$-paracontact Riemannian manifold $M^{n+1}$ of P-Sasakian type endowed with a quarter symmetric non-metric connection then the induced almost $r$-paracontact Riemannian structure $\sum_1$ or $(r-1)$ paracontact Riemannian structure $\sum_2$ are also of P-Sasakian type.

Proof. The computations are similar to the proof of Theorem 3.1 in [4].

Lemma 3.5([4]). \( \nabla_X (\text{trace} \varphi) = \text{trace} (\nabla_X \varphi) \).

Theorem 3.6. Let $M^n$ be a non-invariant hypersurface of an almost $r$-paracontact Riemannian manifold $M^{n+1}$ with a structure $\sum = (\varphi, \xi^{\alpha}, \eta^{\alpha}, g)_{\alpha \in (r)}$ satisfying $\nabla \varphi = 0$ along $M^n$ then $M^n$ is totally geodesic if and only if

\[
(\nabla^g \varphi) X + b(X)a_\alpha \varphi Y + \eta^\alpha (\varphi X) b(Y) = 0.
\]

Proof. From (3.32) we have

\[
(\nabla^g \varphi) X - b(Y, X) \nabla Y - b(X) (H(Y) - a_\alpha \varphi Y) = 0
\]

and

\[
h(\varphi X, Y) + (\nabla^g b)(Y) X - Kh(Y, X) + a_\alpha b(X) b(Y) = 0.
\]

If $M^n$ is totally geodesic, then $h = 0$ and $H = 0$. So from (3.37), we get

\[
(\nabla^g \varphi) X + b(X)a_\alpha \varphi Y + \eta^\alpha (\varphi X) b(Y) = 0.
\]
Conversely, if \((\nabla_Y \mathcal{F}) X + b(X) a_\alpha \mathcal{F} + \mathcal{F} \circ (\mathcal{F} X) b(\mathcal{F}) = 0\), then
\[
(3.39) \quad h(\mathcal{F}, X) N + b(X) H(\mathcal{F}) = 0.
\]
Making use of (3.31) and (3.5) we have
\[
(3.40) \quad b(\mathcal{F}) h(\mathcal{F}, Z) + b(Z) h(X, Z) = 0.
\]
Using (3.39), we get from (3.5)
\[
(3.41) \quad b(\mathcal{F}) h(X, Z) = b(Z) h(X, Z).
\]
So from (3.40) and (3.41) we get \(b(Z) h(X, Y) = 0\). This gives us \(h = 0\) since \(b \neq 0\). Using \(h = 0\) in (3.40), we get \(H = 0\). Thus, \(h = 0\) and \(H = 0\). Hence \(M^n\) is totally geodesic. This completes the proof of the theorem.

We have also the following:

**Theorem 3.7.** Let \(M^n\) be a non-invariant hypersurface of an almost \(r\)-paracontact Riemannian manifold \(M^{n+1}\) with a quarter symmetric non-metric connection and satisfying \(\nabla \mathcal{F} = 0\) along \(M^n\). If \(\text{trace} \mathcal{F} = \text{constant}\), then
\[
(3.41) \quad h(X, N) = \frac{1}{2} a_\alpha \sum_a b(e_a) \mathcal{F}(e_a, X).
\]

**Proof.** From (3.37) we have
\[
\bar{g} \left( (\nabla_X \mathcal{F}) X, X \right) = 2 h(X, Y) b(X) - a_\alpha b(X) g(X, Y)
\]
and using \(N = \sum_a b(e_a) e_a\)
\[
\nabla_X \text{trace} \mathcal{F} = 2 h(X, N) - a_\alpha \sum_a b(e_a) \mathcal{F}(e_a, X).
\]
Using Lemma 3.5, we get (3.41), where \(N = \sum a b(e_a) e_a\). Thus our theorem is proved.

Let \(M^n\) be an almost \(r\)-paracontact Riemannian manifold of \(S\)-paracontact type, then from (2.11), (3.16) and (3.34), we get
\[
(3.42) \quad \mathcal{F} X = \frac{1}{2} \left[ \nabla_X \xi_\alpha - a_\alpha H(X) + (a_\alpha)^2 \mathcal{F} X + \eta^\alpha (\xi_\alpha) b(X) \right], \quad \alpha \in (r)
\]
\[
(3.43) \quad b(X) = \frac{1}{2} \left[ X(a_\alpha) + h(X, \xi_\alpha) + (a_\alpha)^2 b(X) \right], \quad \alpha \in (r).
\]
Making use of (3.43), if \(M^n\) is totally geodesic then \(a_\alpha = 0\) and \(h = 0\). Hence \(b = 0\), that is, \(M^n\) is invariant.
So we have the following Proposition:

**Proposition 3.8.** If $M^n$ is totally geodesic hypersurface of an almost r-paracontact Riemannian manifold $M^{n+1}$ with a quarter symmetric non-metric connection of S-paracontact type with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ and all $\xi_\alpha$ are tangent to $M^n$, then $M^n$ is invariant.

**Theorem 3.9.** If $M^n$ is an anti-invariant hypersurface of an almost r-paracontact Riemannian manifold $M^{n+1}$ with a quarter symmetric non-metric connection of S-paracontact type with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ then $\nabla_X \xi_\alpha + b(X) = 0$.

**Proof.** If $M^n$ is anti-invariant then $\overline{\varphi} = 0, a_\alpha = 0$ and from (3.42), we have

$$\nabla_X \xi_\alpha + b(X) = 0.$$ 

This completes the proof of the theorem. \hfill \Box

Now, let $M^n$ be an almost r-paracontact Riemannian manifold of P-Sasakian type. Then from (2.14) and (3.32), we have

$$(\nabla_{\overline{\varphi}} X) - h(X,Y)\nabla - b(X)H(Y) + a_\alpha b(X)\varphi Y + \eta^\alpha (\varphi X) b(Y)$$

$$+ [h(\varphi X,Y) + (\nabla b)X - Kh(X,Y) + a_\alpha b(X)b(Y)] N$$

$$= - \sum \eta^\alpha (X) [Y - \eta^\alpha (Y) \xi_\alpha] - \left[ g(X,Y) - \sum \eta^\alpha (X) \eta^\alpha (Y) \right] \sum \xi_\beta.$$ 

From above equation we have

$$\sum \eta^\alpha (X) [Y - \eta^\alpha (Y) \xi_\alpha] - \left[ g(X,Y) - \sum \eta^\alpha (X) \eta^\alpha (Y) \right] \sum \xi_\beta.$$ 

(3.44) \hspace{1cm} $$(\nabla_{\overline{\varphi}} X) - h(X,Y)\nabla - b(X)H(Y) + a_\alpha b(X)\varphi Y + \eta^\alpha (\varphi X) b(Y)$$

$$= - \sum \eta^\alpha (X) [Y - \eta^\alpha (Y) \xi_\alpha] - \left[ g(X,Y) - \sum \eta^\alpha (X) \eta^\alpha (Y) \right] \sum \xi_\beta.$$ 

**Theorem 3.10.** Let $\tilde{M}^{n+1}$ be an almost r-paracontact Riemannian manifold of P-Sasakian type with a quarter symmetric non-metric connection with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ and let $M^n$ be a hypersurface immersed in $\tilde{M}^{n+1}$ such that none of $\xi_\alpha$ is tangent to $\tilde{M}^n$. Then $M^n$ is totally geodesic if and only if.

$$\nabla_{\overline{\varphi}} X + a_\alpha b(X)\varphi Y = - \sum \eta^\alpha (X) [Y - \eta^\alpha (Y) \xi_\alpha]$$

$$- \eta^\alpha (\varphi X) b(Y) - \left[ g(X,Y) - \sum \eta^\alpha (X) \eta^\alpha (Y) \right] \sum \xi_\beta.$$ 

(3.45) \hspace{1cm} $$(\nabla_{\overline{\varphi}} X + a_\alpha b(X)\varphi Y = - \sum \eta^\alpha (X) [Y - \eta^\alpha (Y) \xi_\alpha]$$

$$- \eta^\alpha (\varphi X) b(Y) - \left[ g(X,Y) - \sum \eta^\alpha (X) \eta^\alpha (Y) \right] \sum \xi_\beta.$$ 

**Proof.** If (3.45) is satisfied then from (3.45), we get $b(\overline{\varphi})h(X,Y) = 0$. Since $b \neq 0$ hence $h(X,Y) = 0$. Conversely, let $M^n$ be totally geodesic, that
is, \( h(\mathbf{X}, \mathbf{Y}) = 0 \), \( H = 0 \), then (3.45) is satisfied. From (3.43), \( b(\mathbf{X}) = \frac{1}{2} \left[ \mathbf{X}(a_\alpha) + h(\mathbf{Y}, \xi_\alpha) + (a_\alpha)^2 b(\mathbf{X}) \right] \). If \( a_\alpha = h = 0 \) then \( b = 0 \), which is a contradiction. Hence all \( \xi_\alpha \) are not tangent to \( M^n \). So we get the result as required. □

References


