Multivalent Harmonic Uniformly Starlike Functions

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Abstract. In this paper, we investigate a generalized family of complex-valued harmonic functions that are multivalent, sense-preserving, and are associated with \( k \)-uniformly harmonic functions in the unit disk. The results obtained here include a number of known and new results as their special cases.

1. Introduction

A harmonic function \( f \) defined in a simply connected complex domain \( D \subseteq \mathbb{C} \) can be expressed by \( f(z) = h(z) + g(z) \), \( z \in D \). We call \( h \) the analytic part and \( g \) the co-analytic part of \( f \). If the co-analytic part of \( f \) is zero, then \( f \) reduces to the analytic case. The mapping \( z \rightarrow f(z) \) is sense-preserving and locally one-to-one in \( D \) if and only if the Jacobian of \( f \) is positive, that is, if and only if

\[
J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0, \quad z \in D.
\]

Denote by \( H \) the family of functions \( f = h + g \) which are harmonic, sense-preserving and univalent in the open unit disk \( \Delta = \{ z : |z| < 1 \} \) with

\[
h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.
\]

The class \( H \) was defined and studied by Clunie and Sheil-Small [10]. Also, see

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excellent monograph entitled, ‘Harmonic mapping in the plane’ by Duren [11]. For a fixed positive integer \( m \geq 1 \), let \( H(m) \) denote the family of all multivalent harmonic functions \( f = h + \overline{\gamma} \) which are sense-preserving in \( \Delta \) and are of the form

\[
(1.2) \quad h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \quad g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1.
\]

Recent interest in the study of multivalent harmonic functions in the plane prompted the publication of several articles, such as [3], [4], [5], [9] and [15]. Note that \( H(1) \equiv H \). We say that \( f \in H(m) \) is a multivalent harmonic starlike of order \( \beta, 0 \leq \beta < 1 \) if \( f \) satisfies the condition

\[
\frac{\partial}{\partial \theta} (\arg(f(re^{i\theta}))) \geq m\beta
\]

for each \( z = re^{i\theta}, 0 \leq \theta < 2\pi \) and \( 0 \leq r < 1 \). Denote this class of multivalent harmonic starlike functions of order \( \beta \) by \( S^*_H(m, \beta) \). The classes \( S^*_H(1, \beta) \) and \( S^*_H(m, \beta) \) were studied in [3], [5] and [15].

Let \( G_H(k, m, \beta, t) \) be the family of functions \( f \) in \( H(m) \) satisfying the inequality

\[
(1.3) \quad \Re \left( \frac{zf'(z)}{z'[1-t]z^m + tf(z)} \right) \geq k \left| \frac{zf'(z)}{z'[1-t]z^m + tf(z)} - m \right| + m\beta,
\]

for some \( k, (0 \leq k < \infty), m(m \geq 1), \beta(0 \leq \beta < 1), t(0 \leq t \leq 1), z \in \Delta \) and where

\[
z' = \frac{\partial}{\partial \theta}(z = re^{i\theta}), \quad f'(z) = \frac{\partial}{\partial \theta}f(re^{i\theta}) = i(zh'(z) - \overline{g'(z)}).
\]

Using the fact that \( \Re w > k|w - m| + m\beta \leftrightarrow \Re[(ke^{i\theta} + 1)w - kme^{i\theta}] \geq m\beta \), it follows from the condition (1.3) that \( f \) is in \( G_H(k, m, \beta, t) \) if and only if

\[
(1.4) \quad \Re \left[ \frac{(ke^{i\theta} + 1)(zh'(z) - \overline{g'(z)})}{(1-t)z^m + t(h(z) + g(z))} - kme^{i\theta} \right] \geq m\beta.
\]

The set \( G_H(k, m, \beta, t) \) is a comprehensive family that contains several previously studied subclasses of \( H(m) \) or \( H \). For example,

\[
G_H(0, m, \beta, 1) \equiv S^*_H(m, \beta); [3], [15] \\
G_H(0, m, 0, 1) \equiv S^*_H(m, 0); [5] \\
G_H(0, 1, \beta, 1) \equiv S^*_H(1, \beta) \equiv S^*_H(\beta); [16] \\
G_H(0, 1, 0, 1) \equiv S^*_H(0) \equiv S^*; [24], [25]
\]
Finally, we define the family $S, H, S \equiv G$ and generalizations for various subclasses of $S$. We also examine their convolution and convex combination properties. We remark the results so obtained for these general families can be viewed as extensions and generalizations for various subclasses of $S$, $H$, $S(m)$, and $H(m)$ as listed previously in this section.

\[ G_H(0, m, \beta, 0) \equiv R_H(m, \beta) := \left\{ f \in H(m) : \text{Re} \left( \frac{f'(z)}{zf(z)} \right) \geq m\beta, \ 0 \leq \beta < 1 \right\}; \quad [4] \]
\[ G_H(0, 1, \beta, 0) \equiv R_H(1, \beta) \equiv R_H(\beta); \quad [2] \]
\[ G_H(1, m, \beta, 1) \equiv G_H(m, \beta) := \left\{ f \in H(m) : \text{Re} \left( (1 + e^{i\alpha}) \frac{zf'(z) - me^{i\alpha}}{zf(z)} \right) \geq m\beta \right\}; \quad [17] \]
\[ G_H(1, 1, \beta, 1) \equiv G_H(1, \beta) \equiv G_H(\beta); \quad [23] \]
\[ G_H(k, 1, \beta, t) \equiv G_H(k, \beta, t). \quad [1] \]

Let $S(m)$ be the well known family of functions $h$ in $H(m)$ that are analytic and univalent in $\Delta$ and are of the form $h(z) = z^m + \sum_{n=2}^{\infty} a_n z^{n-1}, z \in \Delta$. We observe that $S(1) \subset S, S(m) \subset H(m)$, and $G_s(k, m, \beta, t) \subset G_H(k, m, \beta, t)$. Also the family $G_s(k, m, \beta, t)$ contains several previously studied subclasses of analytic functions in $\Delta$. For example

\[ G_s(0, 1, \beta, t) := \left\{ h \in S : \text{Re} \left( \frac{zh'(z)}{(1-t)z + th(z)} \right) > \beta \right\}; \quad [7], [20] \]
\[ G_s(1, 1, \beta, 1) := \left\{ h \in S : \text{Re} \left( \frac{zh'(z)}{h(z)} \right) \geq \left| \frac{zh'(z)}{h(z)} - 1 \right| + \beta \right\}; \quad [8] \]
\[ G_s(k, 1, 0, 1) \equiv G_ST := \left\{ h \in S : \text{Re} \left( \frac{zh'(z)}{h(z)} \right) \geq k \left| \frac{zh'(z)}{h(z)} - 1 \right| \right\}; \quad [18] \]
\[ G_s(1, 1, 0, 1) \equiv G_ST \equiv 1 - ST; \quad [19], [21], [22]. \]

Finally, we define the family

\[ G_{TH}(k, m, \beta, t) := TH(m) \cap G_H(k, m, \beta, t), \]

where $TH(m), m \geq 1$ denote the class of functions $f = h + \overline{g}$ in $H(m)$ so that $h$ and $g$ are of the form

\[ h(z) = z^m - \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \quad g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, z \in \Delta. \quad (1.5) \]

The class $TH(m)$ was first studied in [5].

In this paper, we investigate coefficient conditions, extreme points, and distortion bounds for functions in the families $G_H(k, m, \beta, t)$ and $G_{TH}(k, m, \beta, t), m \geq 1$. We also examine their convolution and convex combination properties. We remark that the results so obtained for these general families can be viewed as extensions and generalizations for various subclasses of $S$, $H$, $S(m)$, and $H(m)$ as listed previously in this section.
2. Main results

We first prove sufficient coefficient conditions for harmonic functions in \( \mathcal{G}_H(k, m, \beta, t) \). These conditions are shown to be necessary for the functions in \( \mathcal{G}_T(k, m, \beta, t) \).

**Theorem 1.** Let \( f = h + \tilde{g} \) be so that \( h \) and \( g \) are given by (1.2). If

\[
\sum_{n=2}^{\infty} \frac{(n + m - 1)(k + 1) - tm(k + \beta)}{m(1 - \beta) + 1 - |m(1 - \beta) - 1|} |a_{n+m-1}| \\
+ \sum_{n=1}^{\infty} \frac{(n + m - 1)(k + 1) + tm(k + \beta)}{m(1 - \beta) + 1 - |m(1 - \beta) - 1|} |b_{n+m-1}| \leq \frac{1}{2},
\]

when \( k \geq 0, m \geq 1, 0 \leq \beta < 1 \) and \( 0 \leq t \leq 1 \), then \( f \in \mathcal{G}_H(k, m, \beta, t) \).

**Proof.** Suppose that (2.1) holds. It suffices to prove that \( \text{Re}\{A(z)/B(z)\} > 0 \), where

\[
A(z) = (ke^{i\theta} + 1)(zh'(z) - \tilde{g}z(z)) - m(ke^{i\theta} + \beta)((1 - t)z^m + th(z) + \tilde{t}g(z)), \\
B(z) = (1 - t)z^m + th(z) + \tilde{t}g(z).
\]

Using the fact that \( \text{Re}\omega \geq 0 \) if and only if \( |1 + \omega| \geq |1 - \omega| \) it suffices to show that

\[
|A(z) + B(z)| - |A(z) - B(z)| \geq 0.
\]

Substituting for \( A(z) \) and \( B(z) \) in (2.2), we obtain

\[
\frac{|A(z) + B(z)|}{|A(z) - B(z)|} = \left| \frac{(m(1 - \beta) + 1)z^m}{(m(1 - \beta) + 1)z^m} \right| \\
+ \sum_{n=2}^{\infty} \left| \frac{((n + m - 1) - m\beta t + t) + ke^{i\theta}((n + m - 1) - m\beta t + t)}{(m(1 - \beta) + 1 - |m(1 - \beta) - 1|)z^{n+m-1}} \right| |a_{n+m-1}| \\
- \sum_{n=1}^{\infty} \left| \frac{((n + m - 1) + m\beta t - t) + ke^{i\theta}((n + m - 1) + m\beta t - t)}{(m(1 - \beta) + 1 - |m(1 - \beta) - 1|)z^{n+m-1}} \right| |b_{n+m-1}| \\
- \sum_{n=2}^{\infty} \left| \frac{((n + m - 1) - m\beta t - t) + ke^{i\theta}((n + m - 1) - m\beta t - t)}{(m(1 - \beta) + 1 - |m(1 - \beta) - 1|)z^{n+m-1}} \right| |a_{n+m-1}| \\
- \sum_{n=1}^{\infty} \left| \frac{((n + m - 1) + m\beta t + t) + ke^{i\theta}((n + m - 1) + m\beta t + t)}{(m(1 - \beta) + 1 - |m(1 - \beta) - 1|)z^{n+m-1}} \right| |b_{n+m-1}|.
\]
The functions
\[
\sum_{n=1}^{\infty} 2\left|(n + m - 1)(k + 1) - tm(k + \beta)\right|\left[a_{n+m-1}\right]z^{n-1}
\]
are sharp.

Corollary 1. Let \( f = h + g \) be so that \( h \) and \( g \) are given by (1.2). Also, let \( m \geq 1/(1 - \beta), 0 \leq \beta < 1 \) and \( 0 \leq t \leq 1 \). If the condition
\[
\sum_{n=1}^{\infty} (n + m - 1)(k + 1) + tm(k + \beta)\left|a_{n+m-1}\right| \leq (1 - \beta)
\]
is satisfied, then \( f \in G_{H}(k, m, \beta, t) \).

Corollary 2. Let \( f = h + \overline{g} \) be so that \( h \) and \( g \) are given by (1.2). Also, suppose \( 1 \leq m \leq 1/(1 - \beta), 0 \leq \beta < 1 \) and \( 0 \leq t \leq 1 \). If the condition
\[
\sum_{n=1}^{\infty} (n + m - 1)(k + 1) + tm(k + \beta)\left|b_{n+m-1}\right| \leq (1 - \beta)
\]
is satisfied, then \( f \in G_{H}(k, m, \beta, t) \).
holds, then \( f \in G_H(k, m, \beta, t) \).

**Theorem 2.** Let \( f = h + \overline{g} \) be so that \( h \) and \( g \) are given by (1.5). Also, let \( k \geq 0, 0 \leq t \leq 1 \) and \( 0 \leq \beta < 1 \). Furthermore,

(i) if \( 1 \leq m \leq 1/(1 - \beta) \), then \( f \in G_H(k, m, \beta, t) \) if and only if

\[
\sum_{n=2}^{\infty} [(n + m - 1)(k + 1) - tm(k + \beta)] |a_{n+m-1}| + \sum_{n=1}^{\infty} [(n + m - 1)(k + 1) + tm(k + \beta)] |b_{n+m-1}| \leq m(1 - \beta);
\]

(ii) if \( m(1 - \beta) \geq 1 \), then \( f \in G_H(k, m, \beta, t) \) if and only if

\[
\sum_{n=2}^{\infty} [(n + m - 1)(k + 1) - tm(k + \beta)] |a_{n+m-1}| + \sum_{n=1}^{\infty} [(n + m - 1)(k + 1) + tm(k + \beta)] |b_{n+m-1}| \leq 1.
\]

**Proof.** In view of Corollary 1 and Corollary 2, it suffices to show that \( f \in G_H(k, m, \beta, t) \) if the condition (2.4) does not hold. We note that a necessary and sufficient condition for \( f = h + \overline{g} \), given by (1.5), to be in \( G_H(k, m, \beta, t) \) is that the coefficient condition (1.4) to be satisfied. Equivalently, we must have

\[
Re \left\{ \frac{(ke^{i\theta} + 1)(zh'(z) - zg'(\overline{z})) - m(ke^{i\theta} + \beta)((1 - t)z^m + th(z) + tg(\overline{z}))}{(1 - t)z^m + th(z) + tg(\overline{z})} \right\} \geq 0.
\]

Upon choosing the value of \( z \) on the positive real axis and using \( Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1 \), where \( 0 \leq |z| = r < 1 \), the above inequality reduces to

\[
\{m(1 - \beta) - \sum_{n=2}^{\infty} [(n + m - 1)(k + 1) - mt(k + \beta)] |a_{n+m-1}|r^{n-1} - \sum_{n=1}^{\infty} [(n + m - 1)(k + 1) + mt(k + \beta)] |b_{n+m-1}|r^{n-1}\}
\]

\[
	imes \{1 - \sum_{n=2}^{\infty} |a_{n+m-1}|r^{n-1} + t \sum_{n=1}^{\infty} |b_{n+m-1}|r^{n-1} \} - 1 \geq 0.
\]

If condition (2.5) does not hold then the numerator of (2.6) is negative for \( r \) sufficiently close to 1 because of conditions (i) or (ii). Thus there exits \( z_0 = r_0 > 1 \), for which the left side of (2.6) is negative. This contradicts the required condition for \( f \in G_H(k, m, \beta, t) \). Using definition (1.3), and according to the arguments given in [2] and [16], we obtain distortion bounds for the functions in \( G_H(k, m, \beta, t) \) in
Theorem 3 and extreme points of the closed convex hulls of $G_{\text{clco}}(k, m, \beta, t)$, denoted by $\text{clco}G_{\text{clco}}(k, m, \beta, t)$, in Theorem 4. The proofs of Theorems 3 and 4 are similar to the corresponding results in [2] and [16] and so are omitted. \qed

**Theorem 3.** If $f \in G_{\text{clco}}(k, m, \beta, t)$, then for $|z| = r < 1$

$$|f(z)| \leq \begin{cases} (1 + |b_m|)r^m + \left(\frac{m(1-\beta)}{(m+1)(k+1)-tm(k+\beta)} - \frac{m(k+1)+tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m|\right)r^{m+1}, \\
(1 + |b_m|)r^m + \left(\frac{1}{(m+1)(k+1)-tm(k+\beta)} - \frac{m(k+1)+tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m|\right)r^{m+1}, \end{cases}$$

if $m(1-\beta) \leq 1$

$$|f(z)| \geq \begin{cases} (1 - |b_m|)r^m - \left(\frac{m(1-\beta)}{(m+1)(k+1)-tm(k+\beta)} - \frac{m(k+1)+tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m|\right)r^{m+1}, \\
(1 - |b_m|)r^m - \left(\frac{1}{(m+1)(k+1)-tm(k+\beta)} - \frac{m(k+1)+tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m|\right)r^{m+1}, \end{cases}$$

if $m(1-\beta) \geq 1$

**Corollary 3.** If $f \in G_{\text{clco}}(k, m, \beta, t)$, then

$$\left\{ \omega : |\omega| < \frac{1 - \left[ \frac{m(1-\beta)}{(m+1)(k+1)-tm(k+\beta)} - \frac{m(k+1)+tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m| \right]}{1 - \left[ \frac{(k+1)-2tm(k+\beta)}{(k+1)-2tm(k+\beta)} - \frac{m(k+1)+tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m| \right]} \right\} \subset f(\Delta).$$

**Theorem 4.** A function $f$ is in $\text{clco}G_{\text{clco}}(k, m, \beta, t)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} (X_{n+m-1}h_{n+m-1} + Y_{n+m-1}g_{n+m-1})$$

where

$$h_m(z) = z^m,$$

$$h_{n+m-1}(z) = \begin{cases} z^m - \frac{m(1-\beta)}{(n+m-1)(k+1)-tm(k+\beta)}z^{n+m-1}(n = 2, 3, 4, \ldots), \\
z^m - \frac{1}{(n+m-1)(k+1)-tm(k+\beta)}z^{n+m-1}(n = 2, 3, 4, \ldots), \end{cases}$$

if $m(1-\beta) \leq 1$

if $m(1-\beta) \geq 1$
\[
\begin{align*}
g_{n+m-1}(z) &= \begin{cases} 
  z^m + \frac{m(1 - \beta)}{(n + m - 1)(k + 1) + tm(k + \beta)}(\overline{z})^{n+m-1} & \text{if } m(1 - \beta) \leq 1 \\
  z^m + \frac{1}{(n + m - 1)(k + 1) + tm(k + \beta)}(\overline{z})^{n+m-1} & \text{if } m(1 - \beta) \geq 1 
\end{cases} (n = 1, 2, 3, \ldots), \\
&= \begin{cases} 
  z^m \cdot \sum_{n=1}^{\infty} \frac{(X_{n+m} + Y_{n+m}) - 1}{1 - \beta} & \text{if } m(1 - \beta) \leq 1 \\
  z^m \cdot \sum_{n=1}^{\infty} \frac{(X_{n+m} + Y_{n+m}) - 1}{1 + \beta} & \text{if } m(1 - \beta) \geq 1 
\end{cases}. 
\end{align*}
\]

In particular, the extreme points of \(G_{TH}(k, m, \beta, t)\) are \(\{h_{n+m-1}\}\) and \(\{g_{n+m-1}\}\).

In the next two theorems, we prove that the class \(G_{TH}(k, m, \beta, t)\) is invariant under convolution and convex combinations of its members. We first recall that for harmonic functions

\[f(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}|z^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}|(\overline{z})^{n+m-1}\]

and

\[F(z) = z^m - \sum_{n=2}^{\infty} |A_{n+m-1}|z^{n+m-1} + \sum_{n=1}^{\infty} |B_{n+m-1}|(\overline{z})^{n+m-1}\]

in \(TH(m)\), the convolution of \(f\) and \(F\) is defined as

\[(f * F)(z) = f(z) * F(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}A_{n+m-1}|z^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}B_{n+m-1}|(\overline{z})^{n+m-1}.
\]

Using this definition, we first show that \(G_{TH}(k, m, \beta, t)\) is closed under convolution.

**Theorem 5.** For \(0 \leq \alpha \leq \beta < 1\), let \(f \in G_{TH}(k, m, \beta, t)\) and \(F \in G_{TH}(k, m, \alpha, t)\), then

\[f * F \in G_{TH}(k, m, \beta, t) \subset G_{TH}(k, m, \alpha, t)\]

**Proof.** Let \(f, F \in G_{TH}(k, m, \beta, t)\) be given by (2.7) and (2.8), respectively. Note that the coefficients of \(f\) and \(F\) must satisfy the conditions similar to the inequality (2.4). For \(F \in G_{TH}(k, m, \alpha, t)\) we observe that \(|A_{n+m-1}| \leq 1\) and \(|B_{n+m-1}| \leq 1\).
Since
\begin{align*}
\sum_{n=2}^{\infty} \frac{(n+m-1)(k+1) - tm(k+\beta)}{m(1-\beta) + 1 - |m(1-\beta) - 1|} |a_{n+m-1}| |A_{n+m-1}| \\
+ \sum_{n=1}^{\infty} \frac{(n+m-1)(k+1) + tm(k+\beta)}{m(1-\beta) + 1 - |m(1-\beta) - 1|} |b_{n+m-1}| |B_{n+m-1}| \\
\leq \sum_{n=2}^{\infty} \frac{(n+m-1)(k+1) - tm(k+\beta)}{m(1-\beta) + 1 - |m(1-\beta) - 1|} |a_{n+m-1}| \\
+ \sum_{n=1}^{\infty} \frac{(n+m-1)(k+1) + tm(k+\beta)}{m(1-\beta) + 1 - |m(1-\beta) - 1|} |b_{n+m-1}|.
\end{align*}

The right hand side of the above inequality is bounded by 1 because \( f \in G_{\mathcal{F}}(k, m, \beta, t) \). Therefore the result follows.
Finally, we determine the convex combination of the members of \( G_{\mathcal{F}}(k, m, \beta, t) \).

**Theorem 6.** The class \( G_{\mathcal{F}}(k, m, \beta, t) \) is closed under convex combination.

**Proof.** For \( i = 1, 2, 3, \cdots \) suppose \( f_i \in G_{\mathcal{F}}(k, m, \beta, t) \), where \( f_i \) are given by
\[
f_i(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}| (\overline{z})^{n+m-1}.
\]

For \( \sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1 \), the convex combination of \( f_i \) may be written as
\[
\sum_{i=1}^{\infty} t_i f_i(z) = z^m - \sum_{n=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{n+m-1}| \right) z^{n+m-1} + \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{n+m-1}| \right) (\overline{z})^{n+m-1}.
\]

Since
\begin{align*}
\sum_{n=2}^{\infty} \frac{(n+m-1)(k+1) - tm(k+\beta)}{m(1-\beta) + 1 - |m(1-\beta) - 1|} |a_{n+m-1}| \\
+ \sum_{n=1}^{\infty} \frac{(n+m-1)(k+1) + tm(k+\beta)}{m(1-\beta) + 1 - |m(1-\beta) - 1|} |b_{n+m-1}| \\
\leq \begin{cases} 
  m(1-\beta), & \text{if } m(1-\beta) \geq 1, \\
  1, & \text{if } m(1-\beta) \leq 1,
\end{cases}
\end{align*}

it follows from the above equation
\[
\sum_{n=2}^\infty [(n + m - 1)(k + 1) - tm(k + \beta)] \sum_{i=1}^\infty t_i |a_{n+m-1}|
\]
\[
+ \sum_{n=1}^\infty [(n + m - 1)(k + 1) + tm(k + \beta)] \sum_{i=1}^\infty t_i |b_{n+m-1}|
\]
\[
= \sum_{i=1}^\infty t_i \left\{ \sum_{n=2}^\infty [(n + m - 1)(k + 1) - tm(k + \beta)] |a_{n+m-1}|
\]
\[
+ \sum_{n=1}^\infty [(n + m - 1)(k + 1) + tm(k + \beta)] |b_{n+m-1}| \right\}
\]
\[
\leq \begin{cases} 
m(1-\beta) \sum_{i=1}^\infty t_i = m(1-\beta), & \text{if } m(1-\beta) \leq 1, \\
\sum_{i=1}^\infty t_i = 1, & \text{if } m(1-\beta) \geq 1,
\end{cases}
\]
and so \( \sum_{i=1}^\infty t_i f_i(z) \in G_{PF}(k, m, \beta, t) \). \qedhere

\section*{References}


