On the Hilbert Type Integral Inequalities with Some Parameters and Its Reverse

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Abstract. This paper deals with some new generalizations of the Hardy-Hilbert type integral inequalities with some parameters. We also consider the equivalent inequalities and the reverse forms.

1. Introduction

If \( f(x), g(x) \geq 0 \), such that \( 0 < \int_0^\infty f^2(x) \, dx < \infty \) and \( 0 < \int_0^\infty g^2(x) \, dx < \infty \) then

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \pi \left( \int_0^\infty f^2(x) \, dx \right)^{\frac{1}{2}} \left( \int_0^\infty g^2(x) \, dx \right)^{\frac{1}{2}},
\]

where the constant factor \( \pi \) is the best possible constant (see [1]). Inequality (1.1) had been extended by Hardy-Riesz as:

If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(x) \geq 0 \), such that \( 0 < \int_0^\infty f^p(x) \, dx < \infty \) and \( 0 < \int_0^\infty g^q(x) \, dx < \infty \), then we have the following Hardy-Hilbert’s integral inequality:

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \frac{\pi}{\sin(\frac{\pi}{2})} \left( \int_0^\infty f^p(x) \, dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(x) \, dx \right)^{\frac{1}{q}},
\]

where the constant factor \( \frac{\pi}{\sin(\frac{\pi}{2})} \) is the best possible constant (see [2]). This inequality play an important role in mathematical analysis and its applications (see [3]). In [4] and [5], Yang gave some new generalizations of (1.2) by introducing a parameter \( \lambda > 0 \), and Yang et al. [6] gave an extension of the above results by introducing the index of conjugate parameter \((r, s)\) \( (r > 1, \frac{1}{r} + \frac{1}{s} = 1) \) as follows:

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If \( f(x), g(x) \geq 0 \) and \( 0 < \int_0^\infty x^{p(1 - \frac{1}{2}) - 1} f^p(x) dx < \infty, \ 0 < \int_0^\infty x^{q(1 - \frac{1}{2}) - 1} g^q(x) dx < \infty, \) then
\[
(1.3) \quad \int_0^\infty \int_0^\infty f(x)g(y) \frac{dx}{(x+y)^{r}} < B\left(\frac{1}{r}, \frac{1}{r}\right) \left( \int_0^\infty x^{p(1 - \frac{1}{2}) - 1} f^p(x) dx \right)^{\frac{1}{r}} \left( \int_0^\infty x^{q(1 - \frac{1}{2}) - 1} g^q(x) dx \right)^{\frac{1}{r}},
\]
where the constant factor \( B\left(\frac{1}{r}, \frac{1}{r}\right) \) is the best possible constant. In particular, for \( \lambda = 1, r = p, \) inequality (1.3) reduces to (1.2); for \( \lambda = 4, r = s = 2, \) inequality (1.3) reduces to:
\[
(1.4) \quad \int_0^\infty \int_0^\infty f(x)g(y) \frac{dx}{(x+y)^{s}} < \frac{1}{6} \left( \int_0^\infty x^{p(1 - \frac{1}{2}) - 1} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_0^\infty x^{q(1 - \frac{1}{2}) - 1} g^q(x) dx \right)^{\frac{1}{q}}.
\]
Recently, Xie et al.\cite{8} gave a new Hilbert type integral inequality with some parameters and its reverse as follows:

If \( p > 1, \ \frac{1}{p} + \frac{1}{4} = 1, \ a, b > 0, \ a \neq b, \ f(x), g(x) \geq 0, \) such that
\[
0 < \int_0^\infty \frac{1}{x^{p-1}} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_0^\infty \frac{1}{x^{q-1}} g^q(x) dx < \infty,
\]
then
\[
(1.5) \quad \int_0^\infty \int_0^\infty f(x)g(y) \frac{dx}{(x+y)^{\frac{1}{2}}} < K \left( \int_0^\infty \frac{1}{x^{p-1}} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_0^\infty \frac{1}{x^{q-1}} g^q(x) dx \right)^{\frac{1}{q}},
\]
where the constant factor \( K = \frac{a+b}{(b-a)^2} \left[ \frac{1}{b-a} \ln\left(\frac{b}{a}\right) - \frac{2}{a+b} \right] \) is the best possible constant. If \( 0 < p < 1, \) then
\[
(1.6) \quad \int_0^\infty \int_0^\infty f(x)g(y) \frac{dx}{(x+y)^{\frac{1}{2}}} < K \left( \int_0^\infty \frac{1}{x^{p-1}} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_0^\infty \frac{1}{x^{q-1}} g^q(x) dx \right)^{\frac{1}{q}},
\]
where \( K \) the constant factor is the best possible.

In this paper, by introducing some parameters and estimating the weight function, we prove Hilbert type integral inequality with a best constant factor similar to (1.4) and (1.5). The equivalent inequalities and the reverse forms are considered.

\section{Main Results}

In order to obtain our results, we need the following lemmas.

\textbf{Lemma 2.1.} If \( a, b > 0, \ a \neq b, \ a > 0, \) the weight function \( \omega(x) \) and \( \omega(y) \) defined by
\[
(2.1) \quad \omega(x) = \int_0^\infty \frac{x^{2a}y^{2a-1}}{(x^{a} + ay^{a})^2} \frac{dy}{x^{a} + by^{a}}, \ x \in (0, \infty),
\]
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\[ \int_0^\infty \frac{x^{2\alpha-1} y^{2\alpha}}{(x^\alpha + ay^\alpha)^2 (x^\alpha + by^\alpha)^2} \, dx, \quad y \in (0, \infty), \]
then we have

\[ \tilde{\omega}(x) = \omega(y) = K := \frac{1}{\alpha} \left( \frac{a+b}{(b-a)^2} \left[ \frac{1}{b-a} \ln \left( \frac{b}{a} \right) - \frac{2}{a+b} \right] \right). \]

Proof. For fixed \( x \), setting \( u = \frac{y^\alpha}{x^\alpha} \) in (2.1), we obtain

\[ \tilde{\omega}(x) = \frac{1}{\alpha} \int_0^\infty \frac{u}{(1+au)^2 (1+bu)^2} \, du \]

= \( \frac{1}{\alpha} \frac{a+b}{(b-a)^2} \left[ \frac{1}{b-a} \ln \left( \frac{b}{a} \right) - \frac{2}{a+b} \right] \).

Hence we obtain \( \tilde{\omega}(x) = K \). In the same way, we obtain \( \omega(y) = K \). \( \square \)

Xie proved the following lemma in [8, Lemma 2.2].

**Lemma 2.2.** If \( a, b > 0 \), \( a \neq b \) and \( \alpha > 0 \) for \( 0 < \varepsilon < p \), we have

\[ \int_0^\infty \frac{u^{1-\frac{p}{\varepsilon}}}{(1+au)^2 (1+bu)^2} \, du = K + o(1), \quad \varepsilon \to 0^+. \]

**Lemma 2.3.** If \( p > 1 \) (or \( 0 < p < 1 \)), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( a, b > 0 \), \( a \neq b \), \( \alpha > 0 \) and \( 0 < \varepsilon < p \), setting

\[ I := \int_1^\infty \left[ \int_0^\infty \frac{y^{\alpha(2-\frac{1}{\varepsilon})-1}}{(x^\alpha + ay^\alpha)^2 (x^\alpha + by^\alpha)^2} \, dy \right] x^{\alpha(2-\frac{1}{\varepsilon})-1} \, dx \]
then we have

\[ \frac{1}{\alpha^2} (K + o(1)) = O(1) \leq I \leq \frac{1}{\alpha^2} (K + o(1)), \quad \varepsilon \to 0^+. \]

Proof. For fixed \( x \), \( (1+au)^2 (1+bu)^2 > (a+b)u \), setting \( y^\alpha = x^\alpha u \), then we obtain the following inequality by (2.4).

\[ I = \int_1^\infty x^{-\alpha \varepsilon - 1} \left[ \int_0^\infty \frac{u^{1-\frac{1}{\varepsilon}} \, du}{(1+au)^2 (1+bu)^2} \right] \, dx \]

\[ = \int_1^\infty x^{-\alpha \varepsilon - 1} \left[ \int_0^\infty \frac{u^{1-\frac{1}{\varepsilon}} \, du}{(1+au)^2 (1+bu)^2} \right] \, dx \]

\[ - \int_1^\infty x^{-\alpha \varepsilon - 1} \left[ \int_0^\infty \frac{u^{1-\frac{1}{\varepsilon}} \, du}{(1+au)^2 (1+bu)^2} \right] \, dx \]

\[ \geq \frac{1}{\alpha^2} (K + o(1)) - \frac{1}{\alpha^2} \left( \int_1^\infty x^{-1} \left( \int_0^\infty u^{-\frac{1}{\varepsilon}} \, du \right) \, dx \right) \]

\[ = \frac{1}{\alpha^2} (K + o(1)) - \frac{1}{\alpha^2} \left( \frac{1}{\alpha + \varepsilon} \right) \]

\[ = \frac{1}{\alpha^2} (K + o(1)) - O(1). \]
Theorem 2.1. If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, a, b > 0, a \neq b, \alpha > 0 \) and \( f(x), g(x) \geq 0, \) such that \( 0 < \int_0^\infty \frac{1}{x^{p(\alpha-1)+1}} f^p(x) dx < \infty \) and \( 0 < \int_0^\infty \frac{1}{x^{q(\alpha-1)+1}} g^q(x) dx < \infty, \) then
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + ay^\alpha)^2} dxdy < K \left( \int_0^\infty \frac{1}{x^{p(\alpha-1)+1}} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_0^\infty \frac{1}{x^{q(\alpha-1)+1}} g^q(x) dx \right)^{\frac{1}{q}},
\]
where the constant factor \( K \) is the best possible and \( K \) is defined by (2.3).

Proof. By Hölder’s inequality, with weight (see [7]) and (2.1)-(2.3), we have
\[
J := \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + ay^\alpha)^2} dxdy
\leq \int_0^\infty \int_0^\infty \frac{1}{(x^\alpha + ay^\alpha)^2} \left[ \frac{2^{\alpha-1}}{x^\alpha} f(x) \right] \left[ \frac{2^{\alpha-1}}{y^\alpha} g(y) \right] dxdy
\leq \left\{ \int_0^\infty \frac{1}{x^{p(\alpha-1)+1}} f^p(x) dx \right\} \left\{ \int_0^\infty \frac{1}{y^{q(\alpha-1)+1}} g^q(y) dy \right\}.
\]
(2.7)

If (2.7) takes the form of equality, then the exists constants \( M \) and \( N, \) such that they are not all zero, and (see [7])
\[
M \left( \frac{y}{y^{(\alpha-1)+1}} \right) f^p(x) = N \left( \frac{x}{x^{(\alpha-1)+1}} \right) g^q(y)
\]
a.e. in \((0, \infty) \times (0, \infty).\) Hence, there exists a constant \( C, \) such that
\[
M x^{-p(\alpha-1)} f^p(x) = N y^{-q(\alpha-1)} g^q(y) = C
\]
a.e. in \((0, \infty).\) We claim that \( M = 0. \) In fact, if \( M \neq 0, \) then \( x^{-p(\alpha-1)-1} f^p(x) = \frac{C}{Mx} \) a.e. in \((0, \infty), \) which contradicts the fact that \( 0 < \int_0^\infty x^{-p(\alpha-1)-1} f^p(x) dx < \infty. \)
In the same way, we claim that \( N = 0 \). This is a contradiction. Hence by (2.7), we have (2.6).

If the constant factor \( K \) in (2.6) is not the best possible, then there exists a positive constant \( H \) (with \( H < K \)), such that (2.6) is still valid if we replace \( K \) by \( H \). For \( 0 < \varepsilon < p \) small enough, setting \( f_\varepsilon \) and \( g_\varepsilon \) as: \( f_\varepsilon (x) = g_\varepsilon (x) = 0 \), for \( x \in (0, 1) \); \( f_\varepsilon (x) = x^{\alpha (2 - \frac{p}{q}) - 1} \), \( g_\varepsilon (x) = x^{\alpha (2 - \frac{p}{q}) - 1} \), for \( x \in [1, \infty) \), then we have

\[
H \left\{ \int_0^1 \frac{1}{x^{\alpha (2 - \frac{p}{q}) + 1}} f_\varepsilon^p (x) \, dx \right\} \frac{1}{q} \left\{ \int_0^1 \frac{1}{x^{\alpha (2 - \frac{p}{q}) + 1}} g_\varepsilon^q (x) \, dx \right\} \frac{q}{p} = H \frac{1}{\alpha}.
\]

By (2.5), we have

\[
\int_0^1 \int_0^1 \frac{f_\varepsilon (x) g_\varepsilon (y)}{(x^\alpha + ay^\alpha)^2 (x^\beta + by^\beta)^2} \, dx \, dy \geq \frac{1}{\alpha \varepsilon} (K + o(1)) - O(1).
\]

Hence, we find

\[
\frac{1}{\alpha \varepsilon} (K + o(1)) - O(1) < \frac{H}{\alpha} \quad \text{or} \quad (K + o(1)) - \alpha \varepsilon O(1) < H.
\]

For \( \varepsilon \to 0^+ \), it follows that \( K \leq H \). This contradicts the fact that \( H < K \). Hence the constant factor \( K \) in (2.6) is the best possible.

\[\square\]

**Theorem 2.2.** If \( 0 < p < 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( a, b > 0 \), \( a \neq b \), \( \alpha > 0 \) and \( f(x) \), \( g(x) \geq 0 \), such that \( 0 < \int_0^1 \frac{1}{x^{\alpha (2 - \frac{p}{q}) + 1}} f^p (x) \, dx < \infty \) and \( 0 < \int_0^1 \frac{1}{x^{\alpha (2 - \frac{p}{q}) + 1}} g^q (x) \, dx < \infty \), then

\[
(2.8) \quad K \left( \int_0^1 \frac{1}{x^{\alpha (2 - \frac{p}{q}) + 1}} f^p (x) \, dx \right)^{\frac{1}{p}} \left( \int_0^1 \frac{1}{x^{\alpha (2 - \frac{p}{q}) + 1}} g^q (x) \, dx \right)^{\frac{1}{q}} > K \int_0^1 \frac{f(x) g(y)}{(x^\alpha + ay^\alpha)^2 (x^\beta + by^\beta)^2} \, dx \, dy,
\]

where the constant factor \( K \) is the best possible and \( K \) is defined by (2.3).

**Proof.** By the reverse Hölder’s inequality with weight (see [7]) and the same way of giving (2.7), we obtain (2.8).

If the constant factor \( K \) in (2.8) is not the best possible, then there exists a positive constant \( H \) (with \( H > K \)), such that (2.8) is still valid if we replace \( K \) by \( H \). For \( 0 < \varepsilon < p \) small enough, setting \( f_\varepsilon \) and \( g_\varepsilon \) as: \( f_\varepsilon (x) = g_\varepsilon (x) = 0 \), for
\( x \in (0, 1); \) \( f_\varepsilon(x) = x^{\alpha(2 - \frac{\varepsilon}{\xi}) - 1}; \) \( g_\varepsilon(x) = x^{\alpha(2 - \frac{\xi}{\varepsilon}) - 1}, \) for \( x \in [1, \infty), \) then we have

\[
H \left\{ \int_0^\infty \frac{1}{x^{p(2\alpha - \frac{\varepsilon}{\xi}) - 1 + 1}} f_\varepsilon^p(x) \, dx \right\}^{\frac{1}{p'}} \left\{ \int_0^\infty \frac{1}{x^{p(2\alpha - \frac{\xi}{\varepsilon}) - 1 + 1}} g_\varepsilon^p(x) \, dx \right\}^{\frac{1}{p'}}
= H \left\{ \int_1 x^{-\alpha \varepsilon - 1} \, dx \right\}^{\frac{1}{p'}} \left\{ \int_1 x^{-\alpha \xi - 1} \, dx \right\}^{\frac{1}{p'}} = H \frac{1}{\alpha \varepsilon}.
\]

By (2.5), we have

\[
\int_0^\infty \int_0^\infty \frac{f_\varepsilon(x) g_\varepsilon(y)}{(x^{\alpha} + ay^\alpha)^2 (x^{\alpha} + by^\alpha)^2} \, dx \, dy = \int_1 \int_0^{\frac{1}{\alpha \varepsilon}} \frac{y^{\alpha(2 - \frac{\varepsilon}{\xi}) - 1} dy}{(x^{\alpha} + ay^\alpha)^2 (x^{\alpha} + by^\alpha)^2} x^{\alpha(2 - \frac{\xi}{\varepsilon}) - 1} \, dx \leq \frac{1}{\alpha \varepsilon} (K + o(1)).
\]

Hence, we find

\[
\frac{1}{\alpha \varepsilon} (K + o(1)) > \frac{H}{\alpha \varepsilon} \quad \text{or} \quad (K + o(1)) > H.
\]

For \( \varepsilon \to 0^+ \), it follows that \( K \geq H. \) This contradicts the fact that \( H > K. \) Hence the constant factor \( K \) in (2.8) is the best possible.

**Theorem 2.3.** Under the same assumption of Theorem 2.1 we have

\[
\int_0^\infty y^{2\alpha p - 1} \left( \int_0^\infty \frac{f(x)}{(x^{\alpha} + ay^\alpha)^2 (x^{\alpha} + by^\alpha)^2} \, dx \right)^p \, dy < K^p \int_0^\infty \frac{f^p(x)}{x^{2\alpha(2 - \frac{\varepsilon}{\xi}) - 1}} \, dx,
\]

where the constant factor \( K^p \) is the best possible. Inequalities (2.9) and (2.6) are equivalent.

**Proof.** Setting \( g(y) = y^{2\alpha p - 1} \left( \int_0^\infty \frac{f(x)}{(x^{\alpha} + ay^\alpha)^2 (x^{\alpha} + by^\alpha)^2} \, dx \right)^{p-1} \), by (2.6), we have

\[
\int_0^\infty y^{-q(2\alpha - 1) - 1} g^q(y) \, dy = \int_0^\infty y^{2\alpha p - 1} \left( \int_0^\infty \frac{f(x)}{(x^{\alpha} + ay^\alpha)^2 (x^{\alpha} + by^\alpha)^2} \, dx \right)^p \, dy = \int_0^\infty \int_0^\infty \frac{f(x) g(y)}{(x^{\alpha} + ay^\alpha)^2 (x^{\alpha} + by^\alpha)^2} \, dx \, dy \leq K \left( \int_0^\infty \frac{f^p(x)}{x^{2\alpha(2 - \frac{\varepsilon}{\xi}) - 1}} \, dx \right)^{\frac{1}{p}} \left( \int_0^\infty \frac{g^q(y)}{y^{q(2\alpha - 1) - 1}} \, dy \right)^{\frac{1}{q}}.
\]

\[
0 < \int_0^\infty y^{-q(2\alpha - 1) - 1} g^q(y) \, dy \leq K^p \int_0^\infty \frac{f^p(x)}{x^{p(2\alpha - 1) + 1}} \, dx < \infty.
\]
Hence by (2.6), (2.10) and (2.11) preserve the form of strict inequalities, and we have (2.9). By Hölder’s inequality, we have
\[
\left( \int_0^{\infty} y^{2\alpha p - 1} \left( \int_0^{\infty} \frac{f(x)}{(x^\alpha + a y^\gamma)^2} dx \right)^p dy \right)^\frac{1}{p} = \left( \int_0^{\infty} y^{2\alpha p - 1 + \frac{2}{p}} \left( \int_0^{\infty} \frac{f(x)}{(x^\alpha + a y^\gamma)^2} dx \right)^p dy \right)^\frac{1}{p} \left( \int_0^{\infty} y^{q(2\alpha - 1) - 1} g(y) dy \right)^\frac{1}{q}.
\]
(2.12)

Then by (2.9), we have (2.6). Hence inequalities (2.6) and (2.9) are equivalent.

If the constant factor in (2.9) is not the best possible, then by (2.12), we can get a contradiction that the constant factor in (2.6) is not the best possible. □

**Theorem 2.4.** Under the same assumption of Theorem 2.2 we have
\[
\left( \int_0^{\infty} y^{2\alpha p - 1} \left( \int_0^{\infty} \frac{f(x)}{(x^\alpha + a y^\gamma)^2} dx \right)^p dy \right)^\frac{1}{p} > K^p \int_0^{\infty} \frac{f^p(x)}{x^{p(2\alpha - 1) + 1}} dx,
\]
where the constant factor $K^p$ is the best possible. Inequalities (2.13) and (2.8) are equivalent.

*Proof.* The proof of Theorem 2.3 is the similar. □

**References**


