Weakly Hyponormal Composition Operators and Embry Condition

Mi Ryeong Lee
Faculty of Liberal Education, Kyungpook National University, Daegu 702-701, Korea
e-mail: leemr@knu.ac.kr

Jung Woi Park
Department of Mathematics, College of Natural Sciences, Kyungpook National University, Daegu 702-701, Korea
e-mail: ttangc@nate.com

Abstract. We investigate the gaps among classes of weakly hyponormal composition operators induced by Embry characterization for the subnormality. The relationship between subnormality and weak hyponormality will be discussed in a version of composition operator induced by a non-singular measurable transformation.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $L(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $A$ in $L(\mathcal{H})$ is normal if $A^*A = AA^*$. An operator $A$ is subnormal if $A$ is the restriction of a normal operator to an invariant subspace. In [5], the Bram-Halmos criterion states that an operator $A$ is subnormal if and only if $\sum_{i,j=0}^n \langle A^i f_j, A^j f_i \rangle \geq 0$ for all $\{f_i\}_{i=0}^n$ in $\mathcal{H}$ and any $n \in \mathbb{N}$. Another well-known condition for the subnormality is Embry criterion which states that an operator $A$ is subnormal if and only if $\sum_{i,j=0}^n \langle A^i f_j + A^j f_i \rangle \geq 0$ for all $\{f_i\}_{i=0}^n$ in $\mathcal{H}$ and any $n \in \mathbb{N}$ ([6]). Recall that $A$ is $n$-hyponormal if $\sum_{i,j=0}^n \langle A^i f_j, A^j f_i \rangle \geq 0$ for all $\{f_i\}_{i=0}^n$ in $\mathcal{H}$ and any $n \in \mathbb{N}$ ([5],[9],[10]). Recall that an operator $A$ is $E(n)$-hyponormal if $\sum_{i,j=0}^n \langle A^{i+j} f_i, A^{i+j} f_j \rangle \geq 0$ for any $f_0, f_1, \cdots, f_n$ in $\mathcal{H}$([7]). Note that $E(n)$-hyponormality is weaker than $n$-hyponormality. In [7], $E(n)$-hyponormality was discussed as a bridge between subnormality and weak hyponormalities in $L(\mathcal{H})$.

In this note, we discuss $E(n)$-hyponormality for composition operators induced by a non-singular measurable transformation which is applied to being distinct the classes of $E(n)$-hyponormality. In Section 2, we show that the subnormality and

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2. Relationship between subnormality and $E(n)$-hyponormality

We now introduce definitions and well-known facts in reference [5] and [3] which provide good materials for our work.

**Basic Properties (BP)**

(i) Put an $2 \times 2$-operator matrix of $\widetilde{A} := \begin{pmatrix} A & b \\ b^* & c \end{pmatrix}$, where $A \in M_n(\mathbb{C})$, $b \in \mathbb{C}^n$ and $c \in \mathbb{C}$. If $A \geq 0$ and rank $\widetilde{A} = \text{rank} \ A$, then $\widetilde{A} \geq 0$. If $A \geq 0$ and rank $\widetilde{A} = \text{rank} \ A$, then $\widetilde{A} \geq 0$.

(ii) Let $A = (a_{ij})_{i,j=0}^\infty$ be an infinite Hermitian matrix and let $A_k$ be the truncation of $A$ to the first $(k+1)$ rows and columns. Assume that $A \geq 0$ and $\det(A_k) = 0$ for some $k$. Then $\det(A_l) = 0$ for all $l \geq k$.

(iii) For $\widetilde{A} \in M_{n+1}(\mathbb{C})$ and $1 \leq k \leq n$, let $\widetilde{A}_k \in M_k(\mathbb{C})$ be the truncation of $\widetilde{A}$. If $\det(\widetilde{A}_k) > 0$ for $1 \leq k \leq n$ and $\det(\widetilde{A}) \geq 0$, then $\widetilde{A} \geq 0$. (This is called the Nested Determinants Test.)

(iv) Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and let $T$ be a non-singular measurable transformation $T : X \to X$ (i.e., $\mu \circ T^{-1} \ll \mu, T^{-1} \mathcal{F} \subset \mathcal{F}$). Then there exist the (first) Radon-Nikodym derivative $h = \frac{d\mu \circ T^{-1}}{d\mu}$ and the $n$-th Radon-Nikodym derivative, $h_n = \frac{d\mu \circ T^{-n}}{d\mu}$ $(n \geq 1)$. And it holds that $\int_{T^{-1}A} f \circ T \ d\mu = \int_A h \cdot f \ d\mu$.

(v) The composition operator $C_T : L^2(X, \mathcal{F}, \mu) \to L^2(X, \mathcal{F}, \mu)$ is defined by $C_T f = f \circ T$ for all $f \in L^2(X, \mathcal{F}, \mu)$. We assume that $C_T$ is continuous (i.e., $\|C_T\| = \|h\|_{1/2} < \infty$).

Let $\mathcal{F}$ be the $\sigma$-algebra by all subsets of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $l \in \mathbb{N}$, we consider a point mass measure $\mu_l$ on $\mathbb{N}_0$ determined as follows:

$$\begin{cases} 1,1,\cdots,1, c_1, c_2, \cdots, c_l, (c_1)^2, (c_2)^2, \cdots, (c_l)^2, (c_1)^3, \cdots, (c_l)^3, (c_1)^4, \cdots \\ \end{cases}$$

with $c_i > 0$ ($i = 1, \cdots, l$). Let $(\mathbb{N}_0, \mathcal{F}, \mu_l)$ be the $\sigma$-finite measure space as above. Define a measurable non-singular transformation $T_l$ on $\mathbb{N}_0$ by $T_l(k) = 0$ for $k = 0, 1, 2, \cdots, l$ and $T_l(k) = k - l$ for $k \geq l + 1$.

**Proposition 2.1.** For a fixed number $l \in \mathbb{N}$, let transformation $T_l$ and measure $\mu_l$ be defined as above. Then the $n$-th Radon-Nikodym derivatives $h_n(k)$ with $h_0(k) \equiv 1$, $h_1(k) = k - l$ for $k \geq l + 1$. If $h_n(k) = 0$ for $0 \leq k \leq l - 1$, then

$$h_n(k) = \begin{cases} k - l, & \text{if } k \geq l + 1 \\ 0, & \text{if } 0 \leq k \leq l - 1 \end{cases}$$
\( n \geq 1, k \in \mathbb{N}_0 \) are expressed by the followings;

\[
h_n(0) = 1 + \sum_{1 \leq j \leq l} \frac{c_j^n - 1}{c_j - 1}, \quad h_n(k) = (c_r)^n \text{ for } k = l \cdot q + r, \quad q \geq 0 \text{ and } r = 1, \ldots, l.
\]

**Proof.** For each \( n \geq 1 \), we show that the \( \sigma \)-algebra \( T^{-n}F \) is generated by the sets \( \{0, 1, 2, \ldots, nl\} \), \( \{nl + 1\} \), \( \{nl + 2\}, \ldots \). It follows from the definition of \( n \)-th Radon-Nikodym derivatives \( h_n(k) \) that

\[
h_n(0) = \frac{\mu \circ T^{-n}(0)}{\mu(0)} = \mu(\{0, 1, 2, \ldots, nl\}) = 1 + \sum_{1 \leq j \leq l} \frac{c_j^n - 1}{c_j - 1}.
\]

On the other hand, for \( k \neq 0 \), we write \( k = lq + r \) for \( q \geq 0 \) and \( r = 1, 2, \ldots, l \). So \( T^{-n}(k) = nl + k \) and \( \mu \circ T^{-n}(k) = \mu(\{nl + k\}) = c_r^{n+q} \). Hence

\[
h_n(k) = \frac{\mu \circ T^{-n}(k)}{\mu(k)} = c_r^{n+q} c_r^{q} = c_r^n
\]

for all \( n, k \geq 1 \). Hence the proof is complete. \( \Box \)

For positive integers \( m \) and \( n \), we set

\[
J_n^{(m)} = \{(j_1, \ldots, j_n) : 1 \leq j_1 < j_2 < \cdots < j_n \leq m, \quad j_i \in \mathbb{N}\}
\]

with \( J_n^{(m)} = \emptyset \) for \( n > m \). We denote for \((j_1, \ldots, j_n) \in J_n^{(m)} \) and \( n \geq 1 \),

\[
c_{j_1, \ldots, j_n} \equiv \prod_{i=1}^n c_{j_i}.
\]

**Lemma 2.2.** For \( l \in \mathbb{N} \), let \( d_n = 1 + \sum_{1 \leq j \leq l} \frac{c_j^n - 1}{c_j - 1} \quad (n \geq 1) \) with \( d_0 = 1 \).

Then \( \{v_i, v_{i+1}, \ldots, v_{i+l+1}\} \) is linearly dependent for all \( i \in \mathbb{N}_0 \) where \( v_i = (d_i, d_{i+1}, \ldots, d_{i+l+1}) \in \mathbb{C}^{l+2} \) for all \( i \in \mathbb{N}_0 \). In particular, the infinite matrix with row vectors \( v_i, v_{i+1}, \ldots, v_{i+l+1} \) \((i \geq 0)\) has rank \( l + 1 \).

**Proof.** For simple notations, we write \( J_i := J_i^{(l)} \) for all \( i = 2, 3, \ldots, l-1 \). Put

\[
a_0 = (-1)^l \prod_{1 \leq j \leq l} c_j, \quad a_1 = (-1)^{l-1} \left( \prod_{1 \leq j \leq l} c_j + \sum_{(j_1, \ldots, j_{l-1}) \in J_{l-1}} c_{j_1, \ldots, j_{l-1}} \right),
\]

\[
a_2 = (-1)^{l-2} \left( \sum_{(j_1, \ldots, j_{l-1}) \in J_{l-1}} c_{j_1, \ldots, j_{l-1}} + \sum_{(j_1, \ldots, j_{l-2}) \in J_{l-2}} c_{j_1, \ldots, j_{l-2}} \right), \ldots,
\]

\[
a_{l-1} = (-1)^l \left( \sum_{(j_1, j_2) \in J_2} c_{j_1, j_2} + \sum_{1 \leq j \leq l} c_j \right), \quad a_l = \sum_{1 \leq j \leq l} c_j + 1.
\]

For simple calculations, we can obtain that \( \sum_{0 \leq j \leq l} a_j d_{j+l+1} = d_{l+l+1} \) for all \( i \in \mathbb{N}_0 \).

Hence the set \( \{v_i, v_{i+1}, \ldots, v_{i+l+1}\} \) is linearly dependent for all \( i \in \mathbb{N}_0 \). \( \Box \)
For a σ-finite measure space \((X, \mathcal{F}, \mu)\), it follows from [7] that the composition operator \(C_T\) on the space \(L^2(X, \mathcal{F}, \mu)\) is \(E(n)\)-hyponormal for a positive integer \(n\) if and only if the \((n+1) \times (n+1)\) matrix \((h_{i+j}(x))_{i,j=0}^{n} \geq 0\) for all \(x \in X\) with respect to \(\mu\), where \(h_{n}(x)\) is the \(n\)-th Radon-Nikodym derivative with \(h_0(x) \equiv 1\). Then we obtain the following theorem.

**Theorem 2.3.** For \(l \in \mathbb{N}\), let \(C_{T_l}\) be a composition operator on the space \(L^2(\mathbb{N}_0, \mathcal{F}, \mu_1)\). Then \(C_{T_l}\) is \(E(l)\)-hyponormal if and only if \(C_{T_l}\) is subnormal.

*Proof.* Let \(l \in \mathbb{N}\). According to the remark above this theorem, we obtain that the composition operator \(C_{T_l}\) is \(E(l)\)-hyponormal if and only if the \((l+1) \times (l+1)\) matrix \((h_{i+j}(k))_{i,j=0}^{l} \geq 0\) for almost all \(k\), where \(h_{n}(k)\) is \(n\)-th Radon-Nikodym derivatives. For the case \(k \neq 0\), using the Proposition 2.1, we see that each column vectors of the infinite matrix \((h_{i+j}(k))_{i,j=0}^{\infty}\) is linearly dependent and its rank is 1. So from BP(i), we have that the infinite matrix \((h_{i+j}(k))_{i,j=0}^{\infty} \geq 0\). Hence \(C_{T_l}\) is subnormal.

Finally we only show the result for the case \(k = 0\). For brevity, we write \(h_{n} := h_{n}(0)\) for all \(n \geq 1\) and \(h_0 = 1\). By Proposition 2.1 and Lemma 2.2, we see that the \((l+1) \times (l+1)\) matrix \((h_{i+j})_{i,j=0}^{l} \geq 0\) has rank \(l + 1\). And by BP(i), rank \((h_{i+j})_{i,j=0}^{l} = l + 1 = \text{rank} \,(h_{i+j})_{i,j=0}^{n} = 0\) for all \(n \geq l + 1\). Also, from the condition \((h_{i+j})_{i,j=0}^{l} \geq 0\), we can obtain that \((h_{i+j})_{i,j=0}^{n} \geq 0\) for all \(n \geq 1\). Hence the composition operator \(C_{T_l}\) is subnormal. The converse implication is obvious. \(\square\)

**Corollary 2.4.** For \(l \in \mathbb{N}\), let \(C_{T_l}\) be a composition operator on the space \(L^2(\mathbb{N}_0, \mathcal{F}, \mu_1)\). Then \(C_{T_l}\) is \(E(l)\)-hyponormal if and only if \(C_{T_l}\) is \(l\)-hyponormal.

*Proof.* We note that \(n\)-hyponormality implies \(E(n)\)-hyponormality for each \(n \in \mathbb{N}\). From Theorem 2.3, we can have the assertion. \(\square\)

In addition, we show formulae of determinants for the matrix \((h_{i+j})_{i,j=0}^{n} \geq 0\) in the following proposition.

**Proposition 2.5.** For \(l \in \mathbb{N}\), we have that

\[
\det (h_{i+j}(0))_{i,j=0}^{n} = \begin{cases} 
\prod_{(j_1,j_2) \in J(l)} (c_{j_1} - c_{j_2})^2 \cdot D_l & \text{for } n = l, \\
0 & \text{for } n \geq l + 1,
\end{cases}
\]

where

\[
D_l = \sum_{r=0}^{l} (-1)^{l-r} (l+1-r) \sum_{(i_1,\cdots,i_r) \in J(r)} c_{i_1,\cdots,i_r}.
\]

In particular, \(\det (h_{i+j}(k))_{i,j=0}^{n} = 0\) for all \(k \neq 0\) and \(n \geq 1\).

*Proof.* From the Proposition 2.1 and Lemma 2.2, we can obtain the result. \(\square\)

**Remark 2.6.** From Theorem 2.3 and Proposition 2.5, we can see that the matrix \((h_{i+j}(k))_{i,j=0}^{n} \geq 0\) for all \(k \in \mathbb{N}\) and \(n \geq 1\). i.e., the composition operator \(C_{T_l}\) is always subnormal.
3. Distinctions of \(E(n)\)-hyponormalities

In our constructed model, we want to show the distinctions of \(E(n)\)-hyponormalities for each \(n \in \mathbb{N}\). Owing to Theorem 2.3, we can see that disjointness of \(E(n)\)-hyponormal operators comes from only cases \(n = 1, 2, \ldots, l\) for the given positive integer number \(l\). So we show that the gaps between \(E(n)\)-hyponormal operators step by step for given number \(n\).

3.1. \(E(1)\)-hyponormal but not \(E(2)\)-hyponormal. For \(k \in \mathbb{N}_0\) and \(n = 1, 2\), we set

\[
RE(2, n) = \{ (c_1, c_2) : C_{T_2} \text{ is } E(n)\text{-hyponormal} \}
\]

and

\[
RD(2, n) = \{ (c_1, c_2) : \det \Delta_i > 0 \ (i = 1, \ldots, n - 1) \text{ and } \det \Delta_n \geq 0 \},
\]

where \(\Delta_i = (h_{i+j}(0))_{t,j=0}^l\) for \(l = 1, 2, \ldots\). Then we can obtain that \(RE(2, n) = RD(2, n), n = 1, 2\). In fact, from BP(iii), we have that \(RD(2, n) \subset RE(2, n)\). To show the reverse implication, let \((c_1, c_2) \in RE(2, n)\), i.e., \(\Delta_n \geq 0\) for all \(k \in \mathbb{N}_0\) and \(n = 1, 2, \ldots\). Suppose that there exists \((\alpha_1, \alpha_2)\) such that \(\det \Delta_1 = c_1 + c_2 - 6 = 0\) for \(c_1 > 0\) and \(c_2 > 0\). Since \(\det \Delta_2 = (c_1 - c_2)^2(3 - 2c_1 - 2c_2 + c_1c_2)\), if we put \(f(c_1, c_2) := 3 - 2c_1 - 2c_2 + c_1c_2\), then we can have that \(f(\alpha_1, \alpha_2) < 0\), which is contradicts to \(\Delta_2 \geq 0\). Hence we have the following assertions;

\(C_{T_2}\) is \(E(1)\)-hyponormal \iff \(c_1 + c_2 - 6 \geq 0\) for \(c_1 > 0, c_2 > 0\)

and

\(C_{T_2}\) is \(E(2)\)-hyponormal \iff \(3 - 2c_1 - 2c_2 + c_1c_2 \geq 0\) for \(c_1 > 0, c_2 > 0\).

Remark 3.1. More specially, to see the gaps between \(E(n)\)-hyponormalities for \(n = 1, 2\), in \(\mathbb{R}^1\), we restrict \(d = 2c\) with the positive number \(c\). Put

\[I_i = \{ c > 0 : C_{T_2} = E(i)\text{-hyponormal} \}
\]

for \(i = 1, 2\). Then we have two intervals, \(I_2 = [\alpha, \infty) \subsetneq I_1 = [2, \infty)\), where \(\alpha = \frac{3 + \sqrt{3}}{2}\).

3.2. \(E(2)\)-hyponormal but not \(E(3)\)-hyponormal. From now on, because of conveniences of calculations, we will look for the gaps in \(\mathbb{R}^l\) about the classes of \((E(n))\)-hyponormal composition operators for each positive integer \(n\). Put each point mass \(c_j = j \cdot c\) for \(j = 1, 2, \ldots, l\) for a positive number \(c\). For \(k \in \mathbb{N}_0\) and \(n = 1, 2, 3\), we set \(RE(3, n) = \{ c > 0 : \Delta_n \geq 0 \}\) and

\[
RD(3, n) = \{ c > 0 : \det \Delta_i > 0 \ (i = 1, \ldots, n - 1) \text{ and } \det \Delta_n \geq 0 \},
\]

where \(\Delta_n = (h_{i+j}(0))_{t,j=0}^n\). Then we can obtain that \(RE(3, n) = RD(3, n)\) for \(n = 1, 2, 3\). Indeed, from simple calculations, \(\det \Delta_1 = 6(c - 2)\) and \(\det \Delta_2 = 4c^2(5c^2 - \)
15c + 6) = 0 for c > 0. Suppose that there exists \( \alpha_0 \geq 2 \) such that \( 5c^2 - 15c + 6 = 0 \). Since \( \det \Delta_3 = 8c^6(-2 + 9c - 11c^2 + 3c^3) \), if we put \( f(c) := -2 + 9c - 11c^2 + 3c^3 \), then we can have that \( f(\alpha_0) = -\frac{30\alpha_0}{5} + \frac{2}{5} < 0 \) (because \( \alpha_0 \geq 2 \)), which contradicts to \( \Delta_3 \geq 0 \). If we denote an interval \( I_n = \{ c > 0 : \sigma_{T_n} \text{ is } E(n)\text{-hyponormal} \} \) for \( n = 1, 2, 3 \), then we have the following relationships for \( E(n)\text{-hyponormalities} \),

\[
I_3 = [\alpha_3, \infty) \subset I_2 = [\alpha_2, \infty) \subset I_1 = [2, \infty),
\]

where \( \alpha_2 \approx 2.525, \alpha_3 \approx 2.618 \).

3.3. Algorithm. Throughout previous examples, we provide the following algorithm giving the distinctions of \( E \)-hyponormalities for a fixed integer \( l \geq 3 \) and a constant \( c > 0 \).

I. Set a matrix \( \Omega = (h_{i+j})_{i,j=0}^\infty \), where each \( h_m := h_m(0) \) is the same as in Proposition 2.1.

II. Compute the determinants of matrices \( \Omega_k \) for \( k = 1, 2, \cdots, l \). Put \( d_k(c) = \det \Omega_k \) for \( k = 1, 2, \cdots, l \). Then \( d_1(c) = \frac{l(l+1)}{2}(c-2) \). So we take \( \alpha_1(= c) > 2 \).

III. Find polynomial remainder \( R_k(c) \) of \( d_k(c) \),

\[
d_k(c) = \left( \sum_{1 \leq j \leq l} j^{2k-1}c^{2k-1}d_{k-1}(c) + R_k(c) \right), \quad 2 \leq k \leq l.
\]

IV. For each \( \alpha_{k-1} > 2, 2 \leq k \leq l \), check \( R_k(\alpha_{k-1}) < 0 \), where \( \alpha_{k-1} \) is the greatest root of \( d_{k-1}(c) = 0 \).

V. Find \( E(l, n) = \{ c > 0 : d_k > 0, d_n \geq 0, 1 \leq k \leq n-1 \} \) for \( n = 1, 2, \cdots, l \).

3.4. Some estimations. Using Algorithm, we can obtain mutually disjoint values \( \alpha_1, \alpha_2, \cdots, \alpha_l \) satisfying \( \alpha_n \in E(l, n) \) (\( n = 1, \cdots, l \)) for some low numbers permitted by computer estimations, which means that the classes of \( E(n)\text{-hyponormal operators are distinct} \), i.e., \( E(l, n-1) \setminus E(l, n) = \{ \alpha_{n-1}, \alpha_n \} \) for such low numbers.

For examples, we give the numerical values \( \alpha_{l-1} \) and \( \alpha_l \) in the Table 3.1 which show the distinct classes of \( E(n)\text{-hyponormal operators for } 1 \leq n \leq l, 2 \leq l \leq 10 \), where the values of \( \alpha_1 \) are approximated ones.

<table>
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<th>( l = 2 )</th>
<th>( l = 3 )</th>
<th>( l = 4 )</th>
<th>( l = 5 )</th>
<th>( l = 6 )</th>
<th>( l = 7 )</th>
<th>( l = 8 )</th>
<th>( l = 9 )</th>
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Table 3.1
References


