An Application of Furuta Inequality to Linear Operator Equations

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Abstract. A class of Hermitian operators \( B \) admitting a positive semidefinite solution of the linear operator equation \( \sum_{j=1}^{n} A^{n-j} X A^{j-1} = B \) for a fixed positive definite operator \( A \) is given via the Furuta inequality.

1. Introduction

The main concern of this paper is to study the linear operator equation (a general form of the Lyapunov equation \( AX +XA = B \))

\[
\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B
\]

where \( A > 0 \) (positive definite) and \( B \) is a Hermitian operator on a Hilbert space \( \mathcal{H} \), and positive semidefinite solution \( X \) is sought.

In [1], Bhatia and Uchiyama obtained an explicit form of the unique solution of (1.1) (when no point of the spectrum of \( A \) is on the negative real axis) by

\[
X = \frac{\sin \pi/n}{\pi} \int_{0}^{\infty} (t + A^n)^{-1} B (t + A^n)^{-1} t^{1/n} dt
\]

This implies in particular that if \( A \) and \( B \) are positive semidefinite, then so is \( X \), an analogue of one of the important facts for the Lyapunov equation. However for positive definite \( A \) but for general Hermitian operator \( B \), it is non-trivial to determine the positive semidefiniteness of the solution \( X \). The main purpose of this paper is to find a class of Hermitian operators \( B \) that assure the positive semidefiniteness of the solution for a fixed positive definite operator \( A \).

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In [2], Chan and Kwong have established the existence of positive semidefinite solution of \( A^2X + XA^2 = A(AB + BA)A \), equivalently,

(1.2) \[ AX + XA = A^{1/2}(A^{1/2}B + BA^{1/2})A^{1/2} \]

for positive definite matrix \( A \) and positive semidefinite matrix \( B \) via the inequality

(1.3) \[ (BA^2B)^{1/2} \geq B^2, \quad A \geq B \geq 0 \] (positive semidefinite),

a special type of the Furuta inequality:

\[ (B^2A^rB^2)^{1/2} \geq B^{r+q}, \quad A \geq B \geq 0 \]

for any \((r, p) \in D_q := \{(r, \alpha) \in [0, \infty) \times [0, \infty) : (1 + r)q \geq r + \alpha \}, \quad q \geq 1 \). Also, see [5] for the matrix equation

(1.4) \[ AX + XA = f(A)B + Bf(A) \]

where \( f \) is a matrix monotone function. These results are quite non-trivial since the right-hand side of (1.2) or (1.4) fails to be non-negative, in general.

Recently T. Furuta obtained the existence of semidefinite solution of (1.1) for a special \( B \) as follows:

**Theorem A [4].** Let \( A \) be a positive definite operator and \( B \) be a positive semidefinite operator. Let \( m \) and \( n \) be natural numbers. There exists positive semidefinite operator solution \( X \) of the following operator equation:

\[
\sum_{j=1}^{n} A^{n-j}XA^{j-1} = A^{\frac{n-r}{2+m}} \left( \sum_{j=1}^{m} A^{\frac{n-r}{2+m-j}}BA^{\frac{n-r}{2+m-j}} \right) A^{\frac{n-r}{2+m}}
\]

for \( r \) such that

\[
\begin{cases} 
  r \geq 0 & \text{if } n \geq m \quad \text{(i)} \\
  r \geq \frac{m-n}{n-1} & \text{if } m \geq n \geq 2 \quad \text{(ii)}
\end{cases}
\]

This includes the case \( r = n = m = 2 \), yielding the equation (1.2).

In this paper we generalize the result of Furuta by finding a general class of Hermitian operators \( B \) that assure the existence of positive semidefinite solution of (1.1). For \( A > 0 \) and \( \alpha \geq 0 \), we consider the set \( \mathcal{F}_A^{(\alpha)} \) of all (Hermitian operator) derivatives at \( t = 0 \)

\[ \frac{d}{dt} \bigg|_{t=0} f(t)^\alpha \]

where \( f : (-\epsilon, \epsilon) \to \text{Herm}({\mathcal{H}}) \) varies over differentiable functions defined near \( t = 0 \) satisfying

(1.5) \[ f(t) \geq f(0) = A, \quad \text{for all } t \geq 0. \]

One directly sees that the map defined by \( f(t) = A + tB \) with \( A > 0 \) and \( B \geq 0 \) satisfies (1.5) and eventually yields \( \sum_{j=1}^{m} A^{m-j}BA^{j-1} \in \mathcal{F}_A^{(m)} \).
The main result of this paper is the following Theorem 1.1 which is further extension of Theorem A.

**Theorem 1.1.** Let $A > 0$. Then the operator equation

\[
\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{r_n}{r+\alpha}} BA^{\frac{r_n}{r+\alpha}}
\]

has a (unique) positive semidefinite solution for any $B \in F^{(\alpha)}_A$ and $(r, \alpha) \in D_n$.

2. **Properties of $F^{(\alpha)}_A$**

Let $\mathcal{H}$ be a Hilbert space and let $\text{Herm}(\mathcal{H})$ be the Banach space of all Hermitian operators on $\mathcal{H}$. Let $P(\mathcal{H})$ be the open convex cone of all positive definite operators on $\mathcal{H}$.

**Definition 2.1.** For $\epsilon > 0$, we denote $F_\epsilon$ by the set of differentiable functions $f : (-\epsilon, \epsilon) \to \text{Herm}(\mathcal{H})$ satisfying

\[ f(t) \geq f(0) > 0 \quad \forall t \geq 0. \]

We also define

\[ F = \bigcup_{\epsilon > 0} F_\epsilon, \quad F_A = \{ f \in F : f(0) = A \} \]

**Remark 2.2.** Let $f \in F_A$ defined on $(-\epsilon, \epsilon)$. From $A > 0$ and continuity of $f$ we can find a small $\epsilon' < \epsilon$ so that $f(t) > 0$ for all $t \in (-\epsilon', \epsilon')$.

**Example 2.3.** (1) Let $\mu, \nu : (-\epsilon, \epsilon) \to \mathbb{R}$ be increasing and differentiable functions such that $\mu(0) = 1$ and $\nu(0) = 0$. Then for $A > 0$ and $B \geq 0$, $f(t) = \mu(t)A + \nu(t)B$ belongs to $F_A$.

(2) Let $B \geq A > 0$, and let $0 < \epsilon < 1$. Let $\mu, \nu : (-\epsilon, \epsilon) \to (0, \infty)$ be increasing and differentiable functions with $\mu(0) = 1$. Then $f(t) = \mu(t)A\#\nu(t)B$ belongs to $F_A$, where $X\#Y = X^{1/2}(X^{-1/2}YX^{-1/2})^tX^{1/2}$ denotes the weighted geometric mean of $X$ and $Y$. Indeed for $t \geq 0$,

\[ f(t) = \mu(t)A\#\nu(t)B \geq \mu(t)A\#\nu(t)A = \mu(t)^{1-t}\nu(t)^tA \geq \mu(0)A(= f(0)) = A \]

where the first inequality follows from the L"owner-Heinz inequality:

\[ A\#B \leq A'\#B', \quad t \in [0, 1], \quad 0 \leq A \leq A', 0 \leq B \leq B'. \]

(3) The set $F_A$ is closed under the geometric and arithmetic mean operations; if $f_1, f_2 \in F_A$ defined on $(-\epsilon_1, \epsilon_1)$ and $(-\epsilon_2, \epsilon_2)$ respectively, then

\[ f_1 \# f_2, \quad \frac{f_1 + f_2}{2} \in F_A \]
where the functions are on a small interval $(-\epsilon', \epsilon'), \epsilon' < \min\{\epsilon_1, \epsilon_2\}$. We may assume that $f_1$ and $f_2$ have positive definite values on $(-\epsilon', \epsilon')$. It then follows from the Löwner-Heinz inequality that

$$(f_1 \# f_2)(t) = f_1(t) \# f_2(t) \geq f_1(0) \# f_2(0) = (f_1 \# f_2)(0)) = A \# A = A, t \geq 0$$

and similarly for the arithmetic mean $\frac{f_1 + f_2}{2}$.

(4) For $\alpha \geq 0$, we consider the power map $\hat{\alpha}(X) = X^\alpha$ on the convex cone of positive definite operators. For $f \in \mathcal{F}_A$, one can find $\epsilon > 0$ such that $f(t) > 0$ for all $t \in (-\epsilon, \epsilon)$ by Remark 2.2. Define $\hat{\alpha}(f) := \hat{\alpha} \circ f$ on $(-\epsilon, \epsilon)$. Then $\hat{\alpha}(\mathcal{F}_A) \subset \mathcal{F}_{A^\alpha}$ for any $\alpha \in [0, 1]$ from the Löwner-Heinz inequality; if $f \in \mathcal{F}_A$, then $f(t) \geq f(0) = A$ and hence

$$(\hat{\alpha} \circ f)(t) = f(t)^\alpha \geq f(0)^\alpha = (\hat{\alpha} \circ f)(0), \ t \geq 0.$$ 

Proposition 2.4 (Invariancy under congruence transformations). Let $A > 0$ and let $M$ be an invertible operator on $\mathcal{H}$. Then

$$\Gamma_M(\mathcal{F}_A) = \mathcal{F}_{\Gamma_M(A)},$$

where $\Gamma_M(X) = MXM^*$. 

Proof. Let $f : (-\epsilon, \epsilon) \to \text{Herm}(\mathcal{H})$ be a member of $\mathcal{F}_A$. Then $(\Gamma_M \circ f)(t) = Mf(t)M^*$ for all $t \in (-\epsilon, \epsilon)$ and

$$(\Gamma_M \circ f)(t) = Mf(t)M^* \geq Mf(0)M^* = (\Gamma_M \circ f)(0) > 0$$

and hence $\Gamma_M \circ f \in \mathcal{F}_{\Gamma_M(A)}$. This implies that $\Gamma_M(\mathcal{F}_A) \subset \mathcal{F}_{\Gamma_M(A)}$. Since $M$ is invertible and $\Gamma_M^{-1} = \Gamma_{M^{-1}}$, the equality $\Gamma_M(\mathcal{F}_A) = \mathcal{F}_{\Gamma_M(A)}$ holds. \qed

For $f \in \mathcal{F}_A$, the map $t \mapsto f(t)^\alpha$ composed by the power map $\hat{\alpha}(X) = X^\alpha$ is differentiable on an appropriate interval $(-\epsilon', \epsilon')$, in particular at $t = 0$ (Remark 2.2).

Definition 2.5. For $A > 0$ and $\alpha \geq 0$, we define

$$\mathcal{F}_A^{(\alpha)} = \left\{ \frac{d}{dt} \bigg|_{t=0} f(t)^\alpha : f \in \mathcal{F}_A \right\}.$$

Note that $\mathcal{F}_A^{(\alpha)}$ consists of Hermitian operators and $\mathcal{F}_A^{(0)} = \{0\}$.

Example 2.6. (1) Let $A > 0, B \geq 0$ and $m \in \mathbb{N}$. For differentiable and increasing functions $\mu, \nu : (-\epsilon, \epsilon) \to \mathbb{R}$ with $\mu(0) = 1$ and $\nu(0) = 0$ (see Examples 2.3 (1)),

$$m\mu'(0)A^m + \nu'(0) \sum_{j=1}^m A^{m-j}BA^{j-1} \in \mathcal{F}_A^{(m)}.$$

This follows by taking $f(t) = \mu(t)A + \nu(t)B$. In particular, $\sum_{j=1}^m A^{m-j}BA^{j-1} \in \mathcal{F}_A^{(m)}$ with $\mu(t) = 1$ and $\nu(t) = t$. Furthermore, $\mathcal{F}_A^{(1)}$ contains all positive semidefinite operators.
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(2) For $A, B > 0$ with $AB = BA$ and $B \geq A$,

$$A(\log B - \log A) \in \mathcal{F}_A^{(1)}$$

by taking $f(t) = A^{\#} B = A^{1-t} B^t$.

(3) The case $A > 0$ and $0 < \alpha < 1$: We consider the power map $\hat{\alpha}(X) = X^\alpha$ defined on the convex cone of positive definite operators. Then the derivative of $\hat{\alpha}$ at $A$ is a linear map whose action is given by ([1])

$$D\hat{\alpha}(A)(X) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \left[ \int_0^t e^{-sA} X e^{-(t-s)A} ds \right] t^{-(r+1)} dt.$$

In this case,

$$\mathcal{F}_A^{(\alpha)} = \left\{ \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \left[ \int_0^t e^{-sA} f'(0) e^{-(t-s)A} ds \right] t^{-(r+1)} dt : f \in \mathcal{F}_A \right\}.$$

**Lemma 2.7 (Unitary invariance).** For any unitary operator $U$,

$$\Gamma_U(\mathcal{F}_A^{(\alpha)}) = \mathcal{F}_{UAU^*}^{(\alpha)}.$$

**Proof.** This follows from that

$$\Gamma_U(\mathcal{F}_A^{(\alpha)}) = \left\{ \left. \frac{d}{dt} \bigg|_{t=0} f(t)^\alpha \right| U^* : f \in \mathcal{F}_A \right\}$$

$$= \left\{ \left. \frac{d}{dt} \bigg|_{t=0} (U f(t) U^*)^\alpha \right| : f \in \mathcal{F}_A \right\}$$

$$= \mathcal{F}_{UAU^*}^{(\alpha)}. \quad \Box$$

3. Proof of the main result

**Definition 3.1.** For $q \geq 1$,

$$\mathcal{D}_q := \{(r, p) \in [0, \infty) \times [0, \infty) : (1 + r)q \geq r + p\}.$$

We note that the domain $\mathcal{D}_q$ coincides with that of parameters satisfying the Furuta inequality:

**Theorem 3.2 (Furuta, [3]).** If $A \geq B \geq 0$ then

$$(B^\frac{r}{2} A^p B^\frac{r}{2})^{\frac{1}{r}} \geq B^{\frac{r+2p}{r}}$$

for any $(r, p) \in \mathcal{D}_q$. 
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We consider the operator equation

\[ \sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{r_n}{n+\alpha}} B A^{\frac{r_n}{n+\alpha}}, \quad A > 0, \quad B \in \text{Herm}(H). \]  

Here we assume \( r_n^\alpha = 0 \) for \( r = \alpha = 0 \).

**Proof of Theorem 1.1.** If \( (r, \alpha) = (0, 0) \), then the right hand side of (1.6) is the zero matrix from \( F(0) A = \{0\} \) and hence the equation has the trivial solution \( X = 0 \).

Suppose that \( (r, \alpha) \neq (0, 0) \). Let \( B \in F(\alpha) \). Let \( f \in F_A \) and let \( \epsilon > 0 \) such that \( f(t) > 0 \) for all \( t \in (-\epsilon, \epsilon) \) and \( B = \frac{d}{dt}|_{t=0} f(t)^{\alpha} \). We consider the curve

\[ Y(t) := (A^\frac{r_n}{n} f(t)^{\alpha} A^\frac{r_n}{n})^\frac{1}{n}, \quad t \in (-\epsilon, \epsilon). \]

Then \( Y(\cdot) \) is differentiable and \( Y(0) = (A^\frac{r_n}{n} A^\frac{r_n}{n})^\frac{1}{n} = A^\frac{r_n}{n^{\frac{1}{n}}}. \) By definition and the Furuta inequality, \( f(t) \geq f(0) = A \) and

\[ Y(t) = (A^\frac{r_n}{n} f(t)^{\alpha} A^\frac{r_n}{n})^\frac{1}{n} \geq A^\frac{r_n}{n^{\frac{1}{n}}} = Y(0) \]

for all \( t \in [0, \epsilon) \). This implies that

\[ X := Y'(0) = \lim_{t \downarrow 0} \frac{Y(t) - Y(0)}{t} \geq 0. \]

By differentiating \( Y^n(t) = A^\frac{r_n}{n} f(t)^{\alpha} A^\frac{r_n}{n} \) at \( t = 0 \), we have

\[ \sum_{j=1}^{n} A^{\frac{r_n}{n}(n-j)} X A^{\frac{r_n}{n}(j-1)} = \sum_{j=1}^{n} Y(0)^{n-j} X Y(0)^{j-1} \]

\[ = \frac{d}{dt}|_{t=0} [A^\frac{r_n}{n} f(t)^{\alpha} A^\frac{r_n}{n}] \]

\[ = A^\frac{r_n}{n} \frac{d}{dt}|_{t=0} f(t)^{\alpha} A^\frac{r_n}{n} = A^\frac{r_n}{n} B A^\frac{r_n}{n}. \]

Replacing \( A \) to \( A^\frac{r_n}{n} \) yields \( B \in F(\alpha) \) and

\[ \sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{r_n}{n+\alpha}} B A^{\frac{r_n}{n+\alpha}}. \]

This completes the proof of Theorem 1.1.

**Remark 3.3.** (1) It is natural to see whether Theorem 1.1 holds for positive semidefinite operator \( A \) or not. We don’t have an answer yet.
(2) The (right) differentiability of \( Y(t) \) at \( t = 0 \) is enough in the proof of Theorem 1.1 and so more general \( f \) can be thought: \( f : [0, \epsilon) \to \text{Herm}(\mathbb{H}) \) is (right) differentiable at \( t = 0 \) and \( f(t) \geq f(0) > 0 \) for all \( t \in [0, \epsilon) \).

**Remark 3.4.** If \( A = \text{diag}(a_1, a_2, \ldots, a_l) > 0 \), then the (unique) solution of

\[
\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B
\]

is given by \( X = \left( \frac{a_i^m - a_j^m}{a_i^{m} - a_j^{m}} b_{ij} \right)_{l \times l} \), where \( B = \left( b_{ij} \right)_{l \times l} \). When \( a_i = a_j \) the quotient \( \frac{a_i^m - a_j^m}{a_i^{m} - a_j^{m}} \) is interpreted to mean \( \frac{1}{n a_i^{m+j}} \) (See [1]). Our result shows that if \( B \in A^{r \times n} F^{(a)}_{A^{r}} A^{r \times n} \) with \( (r, \alpha) \in \mathbb{D}_n \) then the matrix \( X \) with entries \( x_{ij} = \frac{a_i - a_j}{a_i^{n} - a_j^{n}} b_{ij} \) is positive semidefinite. This provides a construction of positive semidefinite matrices depending on the set \( F^{(a)}_{A^{r}} \) of Hermitian matrices and \( (r, \alpha) \in \mathbb{D}_n \).

For instance, if \( (r, m) \in \mathbb{D}_n \) with \( m \in \mathbb{N} \), then \( \sum_{j=1}^{m} A^{m-j} B A^{j-1} \in F^{(m)}_{A^{r}} \) for any \( B \geq 0 \) by Example 2.6(1) and hence the unique semidefinite solution of

\[
\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{r \times n} \left( \sum_{j=1}^{m} A^{n(m-j)} B A^{n(j-1)} \right) A^{r \times n}
\]

is given by

\[
X = \left( (a_i a_j) \frac{a_i^{n} - a_j^{n}}{a_i^{n} - a_j^{n}} b_{ij} \right)_{l \times l}.
\]

Conversely, the \( l \times l \) matrix \( X \) in (3.2) is positive semidefinite for any \( B = (b_{ij})_{l \times l} \geq 0 \) and \( (r, m) \in \mathbb{D}_n \).

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