Extinction and Permanence of a Holling I Type Impulsive Predator-prey Model

HUNKI BAEK* AND CHANGDO JUNG  
Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea.  
e-mail: hgbaek@knu.ac.kr and cdjung@knu.ac.kr

ABSTRACT. We investigate the dynamical properties of a Holling type I predator-prey model, which harvests both prey and predator and stock predator impulsively. By using the Floquet theory and small amplitude perturbation method we prove that there exists a stable prey-extermination solution when the impulsive period is less than some critical value, which implies that the model could be extinct under some conditions. Moreover, we give a sufficient condition for the permanence of the model.

1. Introduction

One important component of the predator-prey relationship is the predator’s rate of feeding on prey, i.e., the so-called predator’s functional response. Functional response refers to the change in the density of prey attached per unit time per predator as the prey density changes. Based on experiments, Holling [6] gave three different kinds of functional response for different kinds of species to model the phenomena of predation. The basic model we considered is based on the following predator-prey model with Holling type I.

\[
\frac{dx}{dt} = ax(t)(1 - \frac{x(t)}{K}) - \phi(x(t))y(t), \\
\frac{dy}{dt} = -Dy(t) + b\phi(x(t))y(t), \\
(x(0^+), y(0^+)) = (x_0, y_0) = x_0,
\]

with

\[
\phi(x(t)) = \begin{cases} 
 cx(t), & x \leq \nu, \\
 \nu, & x > \nu,
\end{cases}
\]

where \(x(t), y(t)\) denote, respectively, the prey and predator densities. Here, \(a, b, D, K, \nu\) are positive constants and \(K\) represents the environmental capacity.
and $a$ a intrinsic birth rate, $D$ denotes the death rate of the predator, $b$ is the rate of conversion of a consumed prey to a predator, $\phi(x(t))$ is the capture rate of prey per predator or functional response of a predator and $\nu$ is a constant characterizing the threshold of prey concentration above which the predation rate is constant and under which the predation rate is similar to the Lotka-Volterra one.

The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects [1, 2, 3, 5, 7]. Thus, with the idea of periodic forcing and impulsive perturbations, we considered the following predator-prey model.

$$
\begin{align*}
&x'(t) = ax(t)\left(1 - \frac{x(t)}{K}\right) - \phi(x(t))y(t), \\
y'(t) = -Dy(t) + b\phi(x(t))y(t), \\
&\Delta x(t) = -p_1x(t), \\
&\Delta y(t) = -p_2y(t) + q,
\end{align*}
$$

where $\Delta x(t) = x(t^+) - x(t)$, $\Delta y(t) = y(t^+) - y(t)$ and $0 \leq p_1, p_2 < 1$. $T$ is the period of the impulsive immigration or stock of the predator, $q$ is the size of immigration or stock of the predator.

2. Preliminaries

Firstly, we give some notations, definitions and Lemmas which will be useful for our main results.

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^2_+ = \{x = (x(t), y(t)) \in \mathbb{R}^2 : x(t), y(t) \geq 0 \}$. Denote $\mathbb{N}$ the set of all of nonnegative integers and $f = (f_1, f_2)^T$ the right hand of (1.3). Let $V : \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+$, then $V$ is said to be in a class $V_0$ if

1. $V$ is continuous on $(nT, (n+1)T] \times \mathbb{R}_+^2$, and

$$
\lim_{(t,y) \to (nT,x)} V(t,y) = V(nT^+, x)
$$

exists.

2. $V$ is locally Lipschitzian in $x$.

**Definition 2.1.** Let $V \in V_0, (t, x) \in (nT, (n+1)T] \times \mathbb{R}_+^2$. The upper right derivatives of $V(t, x)$ with respect to the impulsive differential system (1.3) is defined as

$$
D^+ V(t, x) = \lim_{h \to 0^+} \sup_{t \geq nT} \frac{1}{h} [V(t+h, x + hf(t, x)) - V(t, x)].
$$

**Remark 2.2.** (1) The solution of the system (1.3) is a piecewise continuous function $x : \mathbb{R}_+ \to \mathbb{R}_+^2$, $x(t)$ is continuous on $(nT, (n+1)T]$, $n \in \mathbb{N}$ and $x(nT^+) = \lim_{t \to nT^+} x(t)$ exists.

2. The smoothness properties of $f$ guarantee the global existence and uniqueness of solution of the system (1.3) (see [7] for the details).
Since \( \frac{dx}{dt} = \frac{dy}{dt} = 0 \) whenever \( x(t) = y(t) = 0, t \neq nT \) and \( x(nT^+) = (1 - p_1)x(nT),\ y(nT^+) = (1 - p_2)y(nT) + q(0 \leq p_i < 1, i = 1, 2, q \geq 0) \). We have the following lemma.

**Lemma 2.3.** Let \( x(t) = (x(t), y(t)) \) be a solution of (1.3). Then we have the following assertions.

1. If \( x(0^+) \geq 0 \) then \( x(t) \geq 0 \) for all \( t \geq 0 \).
2. If \( x(0^+) > 0 \) then \( x(t) > 0 \) for all \( t \geq 0 \).

We show that all solutions of (1.3) are uniformly ultimately bounded.

**Lemma 2.4.** There is an \( M > 0 \) such that \( x(t), y(t) \leq M \) for all \( t \) large enough, where \( (x(t), y(t)) \) is a solution of (1.3).

**Proof.** Let \( x(t) = (x(t), y(t)) \) be a solution of (1.3) and let \( V(t, x) = bx(t) + y(t) \). Then \( V \in V_0 \) if \( t \neq nT \)

\[
D^+V + kV = -\frac{ba}{K}x^2(t) + b(a + k)x(t) + c(k - D)y(t).
\]

Clearly, the right hand of (2.1), is bounded by \( M_0 = \frac{b(a + k)^2K^2}{4ak} \) when \( 0 < k < D \). When \( t = nT, V(nT^+) = bx(nT^+) + y(nT^+) = (1 - p_1)bx(nT) + (1 - p_2)y(nT) + q \leq V(nT) + q \). So we can choose \( 0 < k_0 < D \) and \( M_0 > 0 \) such that

\[
\begin{cases}
D^+V \leq -k_0V + M_0, t \neq nT, \\
V(nT^+) \leq V(nT) + q, t = nT.
\end{cases}
\]

From Lemma 2.2 of [4], we can obtain that

\[
V(t) \leq (V(0^+) - \frac{M_0}{k_0}) \exp(-k_0t) + \frac{q(1 - \exp(-(n + 1)k_0T))}{1 - \exp(-k_0T)} \exp(-k_0(t - nT)) + \frac{M_0}{k_0}
\]

for \( t \in (nT, (n + 1)T] \). Therefore, \( V(t) \) is bounded by \( M = \frac{q \exp(k_0T)}{\exp(k_0T) - 1} \) for sufficiently large \( t \). Hence there is an \( M > 0 \) such that \( x(t) \leq M, y(t) \leq M \) for a solution \( (x(t), y(t)) \) with all \( t \) large enough. \( \square \)

Now, we give the basic properties of the following impulsive differential equation.

\[
\begin{cases}
y'(t) = -Dy(t), t \neq nT, \\
y(t^+) = (1 - p_2)y(t) + q, t = nT, \\
y(0^+) = y_0.
\end{cases}
\]

Then we can easily obtain the following results.
Lemma 2.6. (1) $y^*(t) = \frac{q \exp(-D(t-nT))}{1-(1-p_2)\exp(-DT)}$, $t \in (nT,(n+1)T]$, $n \in \mathbb{N}$ and $y^*(0^+) = \frac{q}{1-(1-p_2)\exp(-DT)}$ is a positive periodic solution of (2.4).

(2) $y(t) = (1-p_2)^{n+1} \left( y(0^+) - \frac{q \exp(-DT)}{1-(1-p_2)\exp(-DT)} \right) \exp(-Dt) + y^*(t)$ is the solution of (2.4) with $y_0 \geq 0$, $t \in (nT,(n+1)T]$ and $n \in \mathbb{N}$.

(3) All solutions $y(t)$ of (1.3) with $y_0 \geq 0$ tend to $y^*(t)$. i.e., $|y(t) - y^*(t)| \to 0$ as $t \to \infty$.

3. Extinction and Permanence

Now, we present a condition which guarantees locally asymptotical stability of the prey-free periodic solution $(0, y^*(t))$.

Theorem 3.1. The solution $(0, y^*(t))$ is locally asymptotically stable if

$$aT < \frac{cq(1-\exp(-DT))}{bD(1-(1-p_2)\exp(-DT))} + \ln \frac{1}{1-p_2}.$$ 

Proof. The local stability of the periodic solution $(0, y^*(t))$ of (1.3) may be determined by considering the behavior of small amplitude perturbations of the solution. So in this case we can take $\phi(x(t)) = cx(t)$. Let $(x(t), y(t))$ be any solution of (1.3). Define $x(t) = u(t), y(t) = y^*(t) + v(t)$. Then they may be written as

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, 0 \leq t \leq T,$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi}{dt} = \begin{pmatrix} a-cy^*(t) & 0 \\ bcy^*(t) & -D \end{pmatrix} \Phi(t)$$

and $\Phi(0) = I$, the identity matrix. The linearization of the third and fourth equation of (1.3) becomes

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \end{pmatrix} = \begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}.$$

Note that all eigenvalues of $S = \begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix} \Phi(T)$ are $\mu_1 = \exp(-dT) < 1$ and $\mu_2 = (1-p_2)\exp(\int_0^T a-cy^*(t)dt)$. Since

$$\int_0^T y^*(t)dt = \frac{q(1-\exp(-DT))}{D(1-(1-p_2)\exp(-DT))}.$$
we have
\[ \mu_2 = (1 - p_2) \exp \left( aT - \frac{cq(1 - \exp(-DT))}{D(1 - (1 - p_2) \exp(-DT))} \right). \]

By Floquet Theory ([4]), \((0, y^*(t))\) is locally asymptotically stable if \(|\mu_2| < 1\), i.e.,
\[ aT < \frac{cq(1 - \exp(-DT))}{D(1 - (1 - p_2) \exp(-DT))} + \ln \frac{1}{1 - p_2}. \]

**Definition 3.2.** The system (1.3) is permanent if there exist \(M \geq m > 0\) such that, for any solution \((x(t), y(t))\) of (1.3) with \(x_0 > 0\),
\[ m \leq \lim \inf_{t \to \infty} x(t) \leq \lim \sup_{t \to \infty} x(t) \leq M \quad \text{and} \quad m \leq \lim \inf_{t \to \infty} y(t) \leq \lim \sup_{t \to \infty} y(t) \leq M. \]

**Theorem 3.3.** The system (1.3) is permanent if
\[ aT > \frac{cq(1 - \exp(-DT))}{D(1 - (1 - p_2) \exp(-DT))} + \ln \frac{1}{1 - p_2}. \]

**Proof.** Let \((x(t), y(t))\) be any solution of (1.3) with \(x_0 > 0\). From Lemma 2.4, we may assume that \(x(t) \leq M, y(t) \leq M\), \(t \geq 0\) and \(M > \frac{a}{c}\). Let \(m_2 = \frac{q \exp(-DT)}{1 - (1 - p_2) \exp(-DT)} - \epsilon_2, \epsilon_2 > 0\). From Lemma 2.5, clearly we have \(y(t) \geq m_2\) for all \(t\) large enough. Now we shall find an \(m_1 > 0\) such that \(x(t) \geq m_1\) for all \(t\) large enough. We will do this in the following two steps.

(Step 1) Since
\[ aT > \frac{cq(1 - \exp(-DT))}{D(1 - (1 - p_2) \exp(-DT))} + \ln \frac{1}{1 - p_2}, \]
we can choose \(m_3 > 0, \epsilon_1 > 0\) small enough such that \(0 < m_3 < \min\{\frac{p_2}{2}, \nu\}\) and \(R = (1 - p_2) \exp \left( aT - \frac{aK}{1 - q \exp(-DT)} - \epsilon_1 T \right) > 1\). Suppose that \(x(t) < m_3\) for all \(t\). Then we get \(y(t) \leq y(t)(-D + \delta)\), where \(\delta = bcm_3\).

By Lemma 2.2 of [4], we have \(y(t) \leq u(t)\) and \(u(t) \to u^*(t), t \to \infty\), where \(u(t)\) is the solution of
\[
\begin{align*}
  u'(t) &= (-D + \delta)u(t), \ t \neq nT, \\
  u(t^+) &= (1 - p_2)u(t) + q, \ t = nT, \\
  u(0^+) &= y_0, 
\end{align*}
\]
and \(u^*(t) = \frac{q \exp((-D + \delta)(t - nT))}{1 - (1 - p_2) \exp((-D + \delta)T)}, t \in (nT, (n + 1)T]\). Then there exists \(T_1 > 0\) such that \(y(t) \leq u(t) \leq u^*(t) + \epsilon_1\) and
\[
x'(t) = x(t)(a - \frac{a}{K}x(t)) - cx(t)y(t) \\
\geq x(t)(a - \frac{a}{K}m_3 - c(u^*(t) + \epsilon_1)) \quad \text{for} \ t \geq T_1. 
\]
Let \( N_1 \in \mathbb{N} \) and \( N_1 T \geq T_1 \). We have, for \( n \geq N_1 \)

\[
\begin{align*}
(3.5) \quad & \begin{cases} 
  x'(t) \geq x(t)(a - \frac{a}{K}m_3 - c(u^*(t) + \epsilon_1)), t \neq nT, \\
  x(t^+) = (1 - p)x(t), t = nT.
\end{cases}
\end{align*}
\]

Integrating (3.5) on \((nT, (n+1)T]\) for \( n \geq N_1 \), we obtain

\[
x((n+1)T) \geq x(nT^+) \exp \left( \int_{nT}^{(n+1)T} a - \frac{a}{K}m_3 - c(u^*(t) + \epsilon_1)dt \right) = x(nT)R.
\]

Then \( x((N_1 + k)T) \geq x(N_1 T)R^k \to \infty \) as \( k \to \infty \) which is a contradiction. Hence there exists a \( t_1 > 0 \) such that \( x(t_1) \geq m_3 \).

(Step 2) If \( x(t) \geq m_3 \) for all \( t \geq t_1 \), then we are done. If not, we may let \( t^* = \inf_{t \geq t_1} \{ x(t) < m_3 \} \). Then \( x(t) \geq m_3 \) for \( t \in [t_1, t^*] \) and, by the continuity of \( x(t) \), we have \( x(t^*) = m_3 \). In this step, we have only to consider two possible cases.

Case 1) \( t^* = nT \) for some \( n \in \mathbb{N} \). Then \( (1 - p_1)m_3 \leq x(t^*) = (1 - p_1)x(t^*) < m_3 \). Select \( n_2, n_3 \in \mathbb{N} \) such that \( (n_2 - 1)T > \frac{\ln(\frac{m_3}{\sigma + 2})}{-\delta + d} \) and \( (1 - p_1)^{n_2}R^{n_3}\exp((n_2 + 1)\sigma) > (1 - p_1)^{n_2}R^{n_3}\exp((n_2 + 1)\sigma) > 1 \), where \( \sigma = a - \frac{a}{K}m_3 - cM < 0 \). Let \( T' = n_2T + n_3T \). In this case we will show that there exists \( t_2 \in (t^*, t^* + T'] \) such that \( x(t_2) \geq m_3 \). Otherwise, by (3.4) with \( u(t^*) = y(t^*) \), we have

\[
u(t) - u^*(t) = (1 - p_2)^{n_1 + 1} \left( u(t^+) - \frac{q \exp((-D + \delta)T)}{1 - (1 - p_2) \exp((-D + \delta)T)} \right) \exp((-D + \delta)(t - t^*))
\]

for \((n - 1)T < t \leq nT \) and \( n_1 + 1 \leq n \leq n_1 + 1 + n_2 + n_3 \). So we get \( |u(t) - u^*(t)| \leq (M + q) \exp((-D + \delta)(t - t^*)) < \epsilon_1 \) and \( y(t) \leq u(t) \leq u^*(t) + \epsilon_1 \) for \( t^* + n_2T \leq t \leq t^* + T' \). Also we get to know that

\[
(3.6) \quad \begin{cases} 
  x'(t) \geq x(t)(a - \frac{a}{K}m_3 - c(u^* + \epsilon_1)), t \neq nT, \\
  x(t^+) = (1 - p_1)x(t), t = nT,
\end{cases}
\]

for \( t \in [t^* + n_2T, t^* + T'] \). As in step 1, we have

\[
x(t^* + T') \geq x(t^* + n_2T)R^{n_3}.
\]

Since \( y(t) \leq M \), we have

\[
(3.7) \quad \begin{cases} 
  x'(t) \geq x(t)(a - \frac{a}{K}m_3 - cM) = \sigma x(t), t \neq nT, \\
  x(t^+) = (1 - p_1)x(t), t = nT,
\end{cases}
\]
for $t \in [t^*, t^* + n_2 T]$. Integrating (3.7) on $[t^*, t^* + n_2 T]$ we have

$$x((t^* + n_2 T)) \geq m_3 \exp(\sigma n_2 T) \geq m_3 (1 - p_1)^{n_2} \exp(\sigma n_2 T) > m_3.$$ 

Thus $x(t^* + T') \geq m_3 (1 - p_1)^{n_2} \exp(\sigma n_2 T) R^{n_3} > m_3$ which is a contradiction. Now, let $t = \inf_{t \geq t^*} \{ x(t) \geq m_3 \}$. Then $x(t) \leq m_3$ for $t^* \leq t < \bar{t}$ and $x(\bar{t}) = m_3$. For $t \in [t^*, \bar{t})$, suppose $t \in (t^* + (k - 1)T, t^* + kT)$, $k \in \mathbb{N}$ and $k \leq n_2 + n_3$. So, we have, for $t \in [t^*, \bar{t})$ from (3.7) we obtain $x(t) \geq x(t^+)((1 - p_1)^{k - 1} \exp((k - 1)^{-1} \sigma T) \exp(\sigma (t - (t^* + (k - 1)T))) \geq m_3 (1 - p_1)^{k} \exp(\sigma T) \leq m_3 (1 - p_1)^{n_2 + n_3} \exp(\sigma (n_2 + n_3) T) \equiv m_1'$.

Case (2) $t^* \neq nT, n \in \mathbb{N}$. Then $x(t) \geq m_3$ for $t \in [t_1, t^*)$ and $x(t^*) = m_3$. Suppose that $t^* \in (n_1' T, (n_1' + 1) T)$ for some $n_1' \in \mathbb{N}$. There are two possible cases.

Case (2a) $x(t) < m_3$ for all $t \in (t^*, (n_1' + 1) T]$. In this case we will show that there exists $t_2 \in [(n_1' + 1) T, (n_1' + 1) T + T^*]$ such that $x_2(t_2) \geq m_3$. Suppose not. i.e., $x(t) < m_3$, for all $t \in [(n_1' + 1) T, (n_1' + 1 + n_2 + n_3) T]$. Then $x(t) < m_3$ for all $t \in (t^*, (n_1' + 1 + n_2 + n_3) T)$. By (3.4) with $u((n_1' + 1) T^*) = g((n_1' + 1) T^*)$, we have

$$u(t) - u^*(t) = \left( u((n_1' + 1) T^*) - \frac{q \exp(-D + \delta)}{1 - (1 - p_2) \exp(-D + \delta)} \right) \exp(-D + \delta)(t - (n_1' + 1) T^*)$$

for $t \in (nT, (n + 1) T]$, $n_1' + 1 \leq n \leq n_1' + 2 + n_3$. A similar argument as in (step 1), we have

$$x((n_1' + 1 + n_2 + n_3) T) \geq x_2((n_1' + 1 + n_2 + n_3) T) R^{n_3}.$$ 

It follows from (3.7) that

$$x((n_1' + 1 + n_2 + n_3) T) \geq m_3 (1 - p)^{n_2 + 1} \exp(\sigma (n_2 + 1) T).$$

Thus $x((n_1' + 1 + n_2 + n_3) T) \geq m_3 (1 - p)^{n_2 + 1} \exp(\sigma (n_2 + 1) T) R^{n_3} > m_3$ which is a contradiction. Now, let $t = \inf_{t \geq t^*} \{ x(t) \geq m_3 \}$. Then $x(t) \leq m_3$ for $t^* \leq t < \bar{t}$ and $x(\bar{t}) = m_3$. For $t \in [t^*, \bar{t})$, suppose $t \in (n_1' T + (k' - 1) T, n_1' T + k' T)$, $k' \in \mathbb{N}$, $k' \leq 1 + n_2 + n_2$, we have $x(t) \geq m_3 (1 - p)^{1 + n_2 + n_3} \exp(\sigma (1 + n_2 + n_3) T) \equiv m_1$. Since $m_1 < m_1'$, so $x(t) \geq m_1$ for $t \in (t^*, \bar{t})$.

Case (2b) There is a $t' \in (t^*, (n_1' + 1) T)$ such that $x_2(t') \geq m_3$. Let $\tilde{t} = \inf_{t \geq t^*} \{ x(t) \geq m_3 \}$. Then $x(t) \leq m_3$ for $t \in [t^*, \tilde{t})$ and $x(\tilde{t}) = m_3$. Also, (3.7) holds for $t \in [t^*, \tilde{t})$. Integrating the equation on $[t^*, t^* + T')$, we can get that $x(t) \geq x(t^*) \exp(\sigma (t - t^*)) \geq m_3 \exp(\sigma T) \geq m_1$. Thus in both case the similar argument can be continued since $x(t) \geq m_1$ for some $t > t_1$. This completes the proof.
References


