The Signless Laplacian Spectral Radius for Bicyclic Graphs
with $k$ Pendant Vertices

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Abstract. In this paper, we study the signless Laplacian spectral radius of bicyclic graphs
with given number of pendant vertices and characterize the extremal graphs.

1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$
and edge set $E$. The order of a graph is the cardinality of its vertex set. The matrix
$Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of graph $G$, where
$D(G) = diag(d_v, v \in V)$ is the diagonal matrix of vertex degrees of $G$ and $A(G)$ is
the adjacency matrix of $G$. It is known that $Q(G)$ is a positive semi-definite matrix,
we call this matrix the $Q$-matrix and its largest eigenvalue is denoted by $\mu(G)$ or $\mu$
for simplicity. For the background on the Laplacian eigenvalues of a graph, the
reader is referred to [20] and the references therein.

It is well known that the matrix $L(G) = D(G) - A(G)$ is called the Laplacian
matrix, and $\lambda(G) \leq \mu(G)$ (see, for example, [14]), the equality holds if and only if
$G$ is bipartite.

A bicyclic graph is a connected graph with vertex number equal to edge number
minus one. A pendant path in a connected graph is a path attached to a connected
graph. For $S \subset V$, $G[S]$ denotes the subgraph induced by $S$. For $u \in V$, $d_u$ is the
degree of $u$, $N(u)$ is the neighbor set of $u$.

Denote by $C_n$ and $P_n$ the cycle and the path on $n$ vertices, respectively. We
will use $B_n(k)$ to denote the set of bicyclic graphs on $n$ vertices with $k$ pendant
vertices. Let $C_p$ and $C_q$ be two vertex disjoint cycles. Suppose that $v_1$ is a vertex
of $C_p$ and $v_t$ is a vertex of $C_q$. Joining $v_1$ and $v_t$ by a path $v_1 v_2 \cdots v_t$ of length
$t-1$, where $t \geq 1$ and $t = 1$ means identifying $v_1$ with $v_t$, the resulting graph, denoted by
$B(p, t, q)$. The set of bicyclic graphs obtained from $B(p, t, q)$ by attaching trees is
denoted by $B^+_n(k)$. Let $P_{t+1}, P_{p+1}$ and $P_{q+1}$ be three vertex-disjoint paths, where
t, $p, q \geq 1$ and at most one of them is 1. Identifying the three initial vertices and

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109
terminal vertices of them, respectively, the resulting graph is denoted by $P(t, p, q)$. The set of bicyclic graphs obtained from $P(t, p, q)$ by attaching trees is denoted by $B_n^{t, p, q}$. Obviously, $B_n^k = B_n^k(k) \cup B_n^{k+}(k)$. For other notations in graph theory, we follow [1].

The Laplacian spectral radius of unicyclic graphs is well studied. In [17], the upper and lower bounds for Laplacian spectral radius of unicyclic graphs were studied. In [13], the author characterized the maximum Laplacian spectral radius of unicyclic graphs with given number of pendant vertices. We also characterize the extremal graphs.

2. Some lemmas

Lemma 2.1([19]). Let $G$ be a connected graph and $u, v$ be two vertices of $G$. Suppose $v_1, v_2, \ldots, v_s \in N(v) \setminus (N(u) \cup \{u\})$ (1 ≤ $s \leq d_v$), and $G^*$ is the graph obtained from $G$ by deleting the edges $uv_i$ and adding the edges $uv_i$ (1 ≤ $i \leq s$). Let $X = (x_1, x_2, \ldots, x_n)^t$ be the principal eigenvector of $Q(G)$, where $x_i$ corresponds to $v_i$ (1 ≤ $i \leq n$). If $x_u = x_v$, then $\mu(G) < \mu(G^*)$.

We generalize Lemma 2.1 next.

Lemma 2.2. Let $G$ be a connected graph of order $n$ and $S, T$ be its two disjoint nonempty vertex subset. Suppose $S = \{v_1, v_2, \ldots, v_s\}$ and the neighbors of $v_i$ in $T$ are $v_{i1}, v_{i2}, \ldots, v_{il_i}$ ($l_i \geq 1$, $i = 1, 2, \ldots, s$). Let $X = (x_{v_1}, x_{v_2}, \ldots, x_{v_s})^t$ be the Perron vector of $Q(G)$, where $x_{v_k}$ corresponds to the vertex $v_k$ (1 ≤ $k \leq n$). Suppose $x_{v_i} = \max\{x_{v_k} : i = 1, 2, \ldots, s\}$. Let $H$ be the graph obtained from $G$ by deleting edges $v_{ij}$ and adding the edges $v_{ij}$ (i = 2, 3, \ldots, s; j = 1, 2, \ldots, $l_i$). Then we have $\mu(G) < \mu(H)$.

Proof. The proof is similar to that in [11], we present it here for completeness. Obviously,

$$X^t(Q(H) - Q(G))X = X^t(D(H) + A(H) - D(G) - A(G))X$$

$$= \sum_{i=2}^s \sum_{j=1}^{l_i} \left( (x_{v_i} + x_{v_{ij}})^2 - (x_{v_i} + x_{v_{ij}})^2 \right)$$
Thus, the sequence of vertices $v$ and an additional vertex. We call the following two types of paths internal paths subdividing the edge $uv$ internal path of $G$. Let Lemma 2.3([8], [2]).

Proof. The proof of the result is similar to Theorem 4.11 in [14] and we omit it. □

Lemma 2.5([2]). Suppose $G$ is a nontrivial simple connected graph. Let $u$ be a
Lemma 2.6. Let $G$ be a connected graph and $P$ be a pendant path in $G$. Suppose $e$ is an edge in $P$ and $G'$ is the graph obtained from $G$ by subdividing $e$, then we have $\mu(G) < \mu(G')$.

Proof. Since $G$ is a proper subgraph of $G'$, we have $\mu(G) < \mu(G')$. \hfill \Box

3. Main results

Suppose the vertices of the graphs $B(4, 1, 4), B(4, 1, 3), B(3, 1, 3)$ are labeled as in Fig. 1.

Let $B_1$ be the graph on $n$ vertices obtained from $B(4, 1, 4)$ by attaching $k$ paths of almost equal lengths at $u_6$; $B_2$ be the graph obtained from $B(4, 1, 4)$ by attaching $k$ paths of almost equal lengths at $u_4$; $B_3$ be the graph obtained from $B(4, 1, 4)$ by attaching $k$ paths of almost equal lengths at $u_5$. Let $C_1$ be the graph on $n$ vertices obtained from $B(4, 1, 3)$ by attaching $k$ paths of almost equal lengths at $u_5$; $C_2$ be the graph obtained from $B(4, 1, 3)$ by attaching $k$ paths of almost equal lengths at $u_2$. Let $C_3$ be the graph obtained from $B(4, 1, 3)$ by attaching $k$ paths of almost equal lengths at $u_1$; $C_4$ be the graph obtained from $B(4, 1, 3)$ by attaching $k$ paths of almost equal lengths at $u_4$.

Let $D_1$ be the graph on $n$ vertices obtained from $B(3, 1, 3)$ by attaching $k$ paths of almost equal lengths at $u_4$; $D_2$ be the graph on $n$ vertices obtained from $B(3, 1, 3)$ by attaching $k$ paths of almost equal lengths at $u_2$.

Theorem 3.1. Let $G$ be a bicyclic graph in $\mathcal{B}_1^+(k)$. Then $\mu(G) \leq \mu(D_1)$. The equality holds if and only if $G \cong D_1$.

Proof. Let $G$ be a bicyclic graph in $\mathcal{B}_1^+(k)$. Comparing the eigencomponents of the vertices on $B(p, l, q)$, by Lemma 2.2, identifying the roots of the trees attached
to \( B(p, l, q) \), the signless Laplacian spectral radius increases. Next, by Lemmas 2.3, 2.4, contracting the internal path and Lemma 2.5 to make all the pendant paths having almost equal lengths, the signless Laplacian spectral radius again increases. At last, subdividing the pendant paths several times if necessary to keep the order of graphs unchanged, by Lemma 2.6, \( \mu(G) \) increases.

So we conclude the following three cases hold.

1. If \( p \geq q \geq 4 \) and \( l \geq 1 \), then \( \mu(G) \leq \max \{ \mu(B_1), \mu(B_2), \mu(B_3) \} \).
2. If \( p = 4, q = 3 \) and \( l \geq 1 \), then \( \mu(G) \leq \max \{ \mu(C_1), \mu(C_2), \mu(C_3), \mu(C_4) \} \).
3. If \( p = q = 3, l \geq 1 \), then \( \mu(G) \leq \max \{ \mu(D_1), \mu(D_2) \} \).

For case (1), we claim that \( \max \{ \mu(B_1), \mu(B_2), \mu(B_3) \} = \mu(B_1) \).

In fact, for \( B_2 \), consider the eigencomponents corresponding to \( u \) and \( u_4 \), say, \( x_u \) and \( x_{u_4} \). If \( x_u \geq x_{u_4} \), by Lemma 2.1, removing the \( k \) pendant paths to \( u \), we have \( \mu(B_2) < \mu(B_1) \). If \( x_u < x_{u_4} \), by Lemma 2.1, deleting edges \( uu_1, uu_2 \) and adding edges \( u_4u_1, u_4u_2 \), we also have \( \mu(B_2) < \mu(B_1) \). Similarly, for \( B_3 \), consider the eigencomponents corresponding to \( u \) and \( u_6 \), we have \( \mu(B_3) < \mu(B_1) \).

For case (2), we claim that \( \max \{ \mu(C_1), \mu(C_2), \mu(C_3), \mu(C_4) \} = \mu(C_1) \).

In fact, for \( C_2 \), consider the eigencomponents corresponding to \( u \) and \( u_2 \), say, \( x_u \) and \( x_{u_2} \). If \( x_u \geq x_{u_2} \), by Lemma 2.1, removing the \( k \) pendant paths to \( u \), we have \( \mu(C_2) < \mu(C_1) \). If \( x_u < x_{u_2} \), by Lemma 2.1, deleting edges \( uu_3, uu_4 \) and adding edges \( u_2u_3, u_2u_4 \), we also have \( \mu(C_2) < \mu(C_1) \). Similarly, for \( C_3 \), consider the eigencomponents corresponding to \( u \) and \( u_6 \), we have \( \mu(C_3) < \mu(C_1) \); for \( C_4 \), consider the eigencomponents corresponding to \( u \) and \( u_4 \), we have \( \mu(C_4) < \mu(C_1) \).

For case (3), we claim that \( \max \{ \mu(D_1), \mu(D_2) \} = \mu(D_1) \).

This is similar to the above two cases.

At last, we claim that \( \max \{ \mu(B_1), \mu(C_1), \mu(D_1) \} = \mu(D_1) \).

In fact, for \( C_1 \), by Lemma 2.3, contracting edge \( u_1u_5 \) and by Lemma 2.6, subdividing the pendant edge one time, by Lemma 2.5, we get the graph \( D_1 \) and \( \mu(C_1) \) < \( \mu(D_1) \).

For \( B_1 \), contracting edge \( u_1u_5, u_3u_6 \) and by Lemma 2.6, subdividing the pendant edge one time, by Lemma 2.5, we get the graph \( D_1 \) and \( \mu(B_1) \) < \( \mu(D_1) \).

Suppose the vertices of the graphs \( P(3, 1, 3), P(3, 1, 2), P(2, 1, 2) \) are labeled as in Fig. 2.

Let \( E_1 \) be the graph obtained from \( P(2, 1, 2) \) by attaching \( k \) paths of almost equal lengths at \( u_3 \); \( E_2 \) be the graph obtained from \( P(2, 1, 2) \) by attaching \( k \) paths of almost equal lengths at \( u_4 \).

Let \( F_1 \) be the graph on \( n \) vertices obtained from \( P(3, 1, 2) \) by attaching \( k \) paths of almost equal lengths at \( u_3 \); \( F_2 \) be the graph on \( n \) vertices obtained from \( P(3, 1, 2) \) by attaching \( k \) paths of almost equal lengths at \( u_4 \); \( F_3 \) be the graph obtained from \( P(3, 1, 2) \) by attaching \( k \) paths of almost equal lengths at \( u_3 \).

Let \( G_1 \) be the graph on \( n \) vertices obtained from \( P(3, 1, 3) \) by attaching \( k \) paths of almost equal lengths at \( u_4 \); \( G_2 \) be the graph on \( n \) vertices obtained from \( P(3, 1, 3) \) by attaching \( k \) paths of almost equal lengths at \( u_3 \).
Theorem 3.2. Let $G$ be a bicyclic graph in $\mathcal{B}_n^+(k)$. Then $\mu(G) \leq \mu(E_1)$. The equality holds if and only if $G \cong E_1$.

Proof. Similar as in Theorem 3.1, we conclude that the following three cases holds.

1. If $p = l = 2$ and $q = 1$, then $\mu(G) \leq \max\{\mu(F_1), \mu(F_2)\}$.
2. If $p = 3$, $l = 2$ and $q = 1$ or 2, then $\mu(G) \leq \max\{\mu(F_1), \mu(F_2), \mu(F_3)\}$.
3. If $p \geq l \geq 3$, $q \geq 1$, then $\mu(G) \leq \max\{\mu(G_1), \mu(G_2)\}$.

For case (1), we claim that $\max\{\mu(E_1), \mu(E_2)\} = \mu(E_1)$.

In fact, in $E_2$, just consider the eigencomponents of $u_3$ and $u_4$, by Lemma 2.1, we can get the claim.

For case (2), we claim that $\max\{\mu(F_1), \mu(F_2), \mu(F_3)\} = \mu(F_2)$.

In fact, in $F_1$, just consider the eigencomponents of $u_3$ and $u_1$, by Lemma 2.1, we get $\mu(F_1) \leq \mu(F_2)$; in $F_3$, consider the eigencomponents of $u_1$ and $u_5$, by Lemma 2.1, we get $\mu(F_3) \leq \mu(F_2)$, as claimed.

For case (3), we claim that $\max\{\mu(G_1), \mu(G_2)\} = \mu(G_1)$.

In fact, in $G_2$, just consider the eigencomponents of $u_3$ and $u_1$, by Lemma 2.1, we can get the claim.

At last, we claim that $\max\{\mu(E_1), \mu(F_2), \mu(G_1)\} = \mu(E_1)$.

In fact, for $F_2$, by Lemma 2.3, contracting edge $u_2u_3$ and by Lemma 2.6, subdividing the pendant edge one time, by Lemma 2.5, we get the graph $E_1$ and $\mu(F_2) < \mu(E_1)$.

For $G_1$, contracting edge $u_2v_3$, $u_3u_6$ and by Lemma 2.6, subdividing the pendant edge one time, by Lemma 2.5, we get the graph $E_1$ and $\mu(G_1) < \mu(E_1)$. $\square$

Lemma 3.3([4]). Let $G$ be a graph on $n$ vertices with at least one edge and the maximum degree of $G$ be $\Delta$. Then $\mu(G) \geq \Delta + 1$. The equality holds if and only if $G$ is a star.

Lemma 3.4([9]). For a connected graph $G$, we have $\mu(G) \leq \max\{d_u + m_u : u \in V(G)\}$, where $m_u$ satisfies $d_u m_u = \sum_{v \in E(G)} d_v$. The equality holds if and only if $G$ is regular or semiregular bipartite.

Theorem 3.5. Let $G$ be a bicyclic graph in $\mathcal{B}_n(k)$. Then $\mu(G) \leq \mu(D_1)$, the equality holds if and only if $G = D_1$.

Proof. By Theorems 3.1, 3.2, we have $\mu(G) \leq \max\{\mu(D_1), \mu(E_1)\}$. For $D_1$, by Lemma 3.3, we have $\mu(D_1) \geq k + 5$. By Lemma 3.4, $\mu(E_1) < k + 5$. This implies
the result.

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References


