Integral Operator of Analytic Functions with Positive Real Part

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ABSTRACT. In this paper, we introduce the integral operator \( I_\beta(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n)(z) \) of analytic functions with positive real part. The radius of convexity of this integral operator when \( \beta = 1 \) is determined. In particular, we get the radius of starlikeness and convexity of the analytic functions with \( \text{Re} \{f(z)/z\} > 0 \) and \( \text{Re} \{f'(z)\} > 0 \). Furthermore, we derive sufficient condition for the integral operator \( I_\beta(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n)(z) \) to be analytic and univalent in the open unit disc, which leads to univalency of the operators \( \int_0^1 (f(t)/t)\alpha \, dt \) and \( \int_0^1 (f'(t))\alpha \, dt \).

1. Introduction and definitions

Let \( A \) denote the class of functions of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disc \( U = \{z : |z| < 1\} \). Further, by \( S \) we shall denote the class of all functions in \( A \) which are univalent in \( U \). A function \( f(z) \) belonging to \( S \) is said to be starlike if it satisfies

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in U)
\]

(1.1)

We denote by \( S^* \) the subclass of \( A \) consisting of functions which are starlike in \( U \). Also, a function \( f(z) \) belonging to \( S \) is said to be convex if it satisfies

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in U)
\]

(1.2)

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We denote by $\mathcal{X}$ the subclass of $\mathcal{A}$ consisting of functions which are convex in $U$.

Let $\alpha_i \in \mathbb{C}$ for all $i = 1, \ldots, n$, $n \in \mathbb{N}$, $\beta \in \mathbb{C}$ with $\text{Re}(\beta) > 0$. We let $I_\beta : \mathcal{A}^n \to \mathcal{A}$ be the integral operator defined by

$$I_\beta(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n)(z) = \left\{ \int_0^z \beta t^{\beta-1} (p_1(t))^{\alpha_1} \cdots (p_n(t))^{\alpha_n} \, dt \right\}^{\frac{1}{\beta}}, \tag{1.3}$$

where $p_i(z)$ are analytic in $U$ and satisfy $p_i(0) = 1$ for all $i = 1, \ldots, n$. Here and throughout in the sequel every many-valued function is taken with the principal branch.

**Remark 1.1.** Note that the integral operator $I_\beta(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n)(z)$ generalizes many operators introduced and studied by several authors, for example:

1. For $p_i(t) = D^\alpha_{1-i} f_i(t)$, $1 \leq i \leq n$, we obtain the following integral operator introduced and studied by Bulut [7]

$$I^m_{\alpha}(f_1, \ldots, f_n)(z) = \left\{ \int_0^z \beta t^{\beta-1} (D^m_{1-i} f_i(t))^{\alpha_i} \cdots (D^m_{1-n} f_n(t))^{\alpha_n} \, dt \right\}^{\frac{1}{\beta}}, \tag{1.4}$$

where $D^m_{1-i} f(z) = z + \sum_{n=2}^\infty \frac{\Gamma(n+1) \Gamma(2-n)}{\Gamma(n+1-\gamma)} (1 + (n-1)) \lambda^m a_n z^n$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the generalized Al-Oboudi operator [2].

2. For $p_i(t) = \frac{f_i(t)}{t}$, $1 \leq i \leq n$, we obtain the integral operator introduced and studied by Frasin [9].

$$I_\beta(f_1, \ldots, f_n)(z) = \left\{ \int_0^z \beta t^{\beta-1} \left( \frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\alpha_n} \, dt \right\}^{\frac{1}{\beta}} \tag{1.5}$$

introduced and studied by Breaz and Breaz [3].

3. For $p_i(t) = \frac{f_i(t)}{t}$, $1 \leq i \leq n$, we obtain the integral operator introduced and studied by Breaz and Breaz [3].

$$I_\beta(f_1, \ldots, f_n; g_1, \ldots, g_n)(z) = \left\{ \int_0^z \left( \frac{f_1 * g_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{f_n * g_n(t)}{t} \right)^{\alpha_n} \, dt \right\}^{\frac{1}{\beta}} \tag{1.6}$$

introduced and studied by Frasin [9].

4. For $p_i(t) = \frac{f_i(t)}{t}$, $1 \leq i \leq n$, we obtain the integral operator introduced and studied by Breaz and Breaz [3].

$$F_n(z) = \left\{ \int_0^z \left( \frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\alpha_n} \, dt \right\}^{\frac{1}{\beta}} \tag{1.7}$$
(5) For $\beta = 1$ and $p_i(t) = f_i'(t); 1 \leq i \leq n$, we obtain the integral operator

$$F_{\alpha_1, \ldots, \alpha_n}(z) = \int_0^z (f_1'(t))^\alpha_1 \cdots (f_n'(t))^\alpha_n \, dt$$

introduced and studied by Breaz et al. [5].

(6) For $\beta = 1$ and $p_i(t) = \frac{R^k f_i(t)}{t}; 1 \leq i \leq n$, we obtain the integral operator introduced in [11]

$$I(f_1, \ldots, f_n)(z) = \int_0^z \left( \frac{R^k f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{R^k f_n(t)}{t} \right)^{\alpha_n} \, dt,$$

where $R^k f(z) = z + \sum_{n=2}^{\infty} C_{k+n-1}^k a_n z^n$, $k \in \mathbb{N}_0$ is Ruscheweyh differential operator [17].

(7) For $\beta = 1$ and $p_i(t) = \frac{D^k f_i(t)}{t}; 1 \leq i \leq n$, we obtain the integral operator introduced and studied by Breaz et al. [4]

$$D^k F(z) = \int_0^z \left( \frac{D^k f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{D^k f_n(t)}{t} \right)^{\alpha_n} \, dt,$$

where $D^k f(z) = z + \sum_{n=2}^{\infty} \frac{n^k a_n z^n}{n!}$, $k \in \mathbb{N}_0$ is Sălăgean differential operator [18].

(8) For $\beta = 1$ and $p_i(t) = \frac{D^k f_i(t)}{t}; 1 \leq i \leq n$, we obtain the following integral operator introduced and studied by Bulut [6]

$$I_n(f_1, \ldots, f_n)(z) = \int_0^z \left( \frac{D^k f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{D^k f_n(t)}{t} \right)^{\alpha_n} \, dt,$$

where $D^k f(z) = z + \sum_{n=2}^{\infty} \frac{(1 + (n-1)\lambda)^k a_n z^n}{n!}$, $0 \leq \lambda \leq 1$, is Al-Oboudi differential operator [2].

(9) For $\beta = 1$ and $p_i(t) = \frac{L(a,c)f_i(t)}{t}; 1 \leq i \leq n$, we obtain the integral operator introduced and studied by Selvaraj and Karthikeyan [19]

$$F_{\alpha}(a, c, z) = \int_0^z \left( \frac{L(a,c) f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{L(a,c) f_n(t)}{t} \right)^{\alpha_n} \, dt,$$

where $L(a, c) f(z) := z + \sum_{n=2}^{\infty} \frac{(a+n-1)^{c-1} a_n z^n}{n!}$ is the Carlson-Shaffer linear operator [8].
(5) For $\beta = 1$, $n = 1$, $\alpha_1 = \alpha$ and $p_1(t) = \frac{f(t)}{t}$, we obtain the integral operator

\[
F_\alpha(z) = \int_0^z \left( \frac{f(t)}{t} \right)^\alpha dt
\]

studied in [13]. In particular, for $\alpha = 1$, we obtain Alexander integral operator introduced in [1]

\[
I(z) = \int_0^z f(t) dt
\]

(6) For $\beta = 1$, $n = 1$, $\alpha_1 = \alpha$ and $p_1(t) = f'(t)$, we obtain the integral operator

\[
G_\alpha(z) = \int_0^z (f'(t))^\alpha dt
\]

studied in [15] (see also [16]).

In the present paper, the radius of convexity of the integral operator defined by (1.3) when $\beta = 1$ are determined. In particular, we get the radius of starlikeness and convexity of the analytic functions with $\text{Re} \{ f(z)/z \} > 0$ and $\text{Re} \{ f'(z) \} > 0$ obtained by MacGregor [12]. Furthermore, we derive sufficient condition for the integral operator $I_\beta(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z)$ to be analytic and univalent in $U$, which leads to univalency of the operators $\int_0^z (f(t)/t)^\alpha dt$ and $\int_0^z (f'(t))^\alpha dt$ obtained by Kim and Merkes [10] and Pfaltzgraff [16].

In the proofs of our main results we need the following lemmas

\textbf{Lemma 1.2([12])}. Let $p(z) = 1 + c_1z + c_2z^2 + \cdots$ be analytic in $U$ and satisfy $p(0) = 1$ with $\text{Re}\{p(z)\} > 0$, then we have

\[
\left| \frac{zp'(z)}{p(z)} \right| < \frac{2|z|}{1 - |z|^2}, \quad (z \in U).
\]

\textbf{Lemma 1.3([14])}. Let $\beta \in \mathbb{C}$ with $\text{Re}(\beta) > 0$. If $f \in A$ satisfies

\[
(1 - |z|^{2\text{Re}(\beta)}) \left| \frac{zf''(z)}{f'(z)} \right| \leq \text{Re}(\beta),
\]

for all $z \in U$, then the integral operator

\[
F_\beta(z) = \left\{ \beta \int_0^z t^{\beta - 1} f'(t) dt \right\}^{\frac{1}{\beta}}
\]
is in the class $S$.

2. Convexity of $I_1(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z)$

In this section, we obtain the radius of convexity of the integral operator $I_\beta(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z)$ defined by (1.3) when $\beta = 1$.

**Theorem 2.1.** Let $p_i(0) = 1$ and $\Re\{p_i(z)\} > 0$ for all $i = 1, \ldots, n$. Then the integral operator defined by

$$I_1(z) = \frac{z}{\prod_{i=1}^n (p_i(t))^{\alpha_i}} dt$$

is convex in $|z| = r < \sqrt{\left(\sum_{i=1}^n \alpha_i\right)^2 + 1 - \sum_{i=1}^n \alpha_i}$, where $\alpha_i > 0$ for all $i = 1, \ldots, n$.

**Proof.** From (2.1), it is easy to see that

$$I_1'(0) = I_1'(0) = 0$$

and

$$I_1''(z) = \prod_{i=1}^n (p_i(z))^{\alpha_i}.$$

Differentiate (2.2) logarithmically with respect $z$, we obtain

$$zI_1''(z) = \sum_{i=1}^n \alpha_i \left(\frac{zp'_i(z)}{p_i(z)}\right)$$

or, equivalently,

$$1 + \frac{zI_1''(z)}{I_1'(z)} = 1 + \sum_{i=1}^n \alpha_i \left(\frac{zp'_i(z)}{p_i(z)}\right).$$

Now, using the estimate (1.16) in Lemma 1.2, from (2.3) it follows that

$$\Re \left\{1 + \frac{zI_1''(z)}{I_1'(z)}\right\} = 1 + \Re \sum_{i=1}^n \alpha_i \left(\frac{zp'_i(z)}{p_i(z)}\right)$$

$$\geq 1 - \sum_{i=1}^n \alpha_i \left|\frac{zp'_i(z)}{p_i(z)}\right|$$

$$\geq 1 - \left\|\frac{2|z|}{1 - |z|^2}\right\| \sum_{i=1}^n \alpha_i, \quad (|z| < 1).$$

If $|z| < \sqrt{\left(\sum_{i=1}^n \alpha_i\right)^2 + 1 - \sum_{i=1}^n \alpha_i}$, then $\Re \left\{1 + \frac{zI_1''(z)}{I_1'(z)}\right\} > 0$. Therefore, $I_1(z)$ is convex in $|z| < \sqrt{\left(\sum_{i=1}^n \alpha_i\right)^2 + 1 - \sum_{i=1}^n \alpha_i}$. This proves the theorem. \qed
Letting $n = 1$, $\alpha_1 = 1$ and $p_1 = p$ in Theorem 2.1, we have

**Corollary 2.2.** Let $p(0) = 1$ and $\Re \{p(z)\} > 0$. Then the integral operator $$\int_0^z p(t)dt$$ is convex in $|z| < \sqrt{2} - 1$.

Letting $p(z) = f(z)/z$ and $p(z) = f'(z)$ in Corollary 2.2, we get the following interesting results due to MacGregor [12].

**Corollary 2.3.** Let $f(z) \in A$. If $\Re \{f(z)/z\} > 0$ for $|z| < 1$, then $f(z)$ is starlike in $|z| < \sqrt{2} - 1$. The result is sharp for the extremal function $f(z) = (z + z^2)/(1 - z^2)$.

**Corollary 2.4.** Let $f(z) \in A$. If $\Re \{f'(z)\} > 0$ for $|z| < 1$, then $f(z)$ is convex in $|z| < \sqrt{2} - 1$. The result is sharp for the extremal function $f(z) = \int_0^z (1 + t)/(1 - t)dt$.

**Remark 2.5.** Taking different choices of $p_i(z)$ as stated in Section 1, Theorems 2.1 leads to new radius of convexity for the integral operators defined in Section 1 when $\beta = 1$.

### 3. Univalency of $I_\beta(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n)(z)$

Next, we obtain the following sufficient condition for the integral operator $I_\beta(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n)(z)$ to be analytic and univalent in $U$.

**Theorem 3.1.** Let $\alpha_i \in \mathbb{C}$ for all $i = 1, \ldots, n$ and $\beta \in \mathbb{C}$ with $\Re(\beta) = a$. If

\begin{equation}
\sum_{i=1}^n \alpha_i \leq \begin{cases} \frac{a}{2} & \text{if } 0 < a \leq 1/2 \\ 1/4 & \text{if } a \geq 1/2. \end{cases}
\end{equation}

Then the integral operator $I_\beta(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n)(z)$ defined by (1.3) is analytic and univalent in $U$, where $p_i(z)$ are analytic in $U$ and satisfy $p_i(0) = 1$ with $\Re \{p_i(z)\} > 0$ for all $i = 1, \ldots, n$.

**Proof.** Define

$$h(z) = \int_0^z \prod_{i=1}^n (p_i(t))^\alpha_i dt,$$

so that, obviously,

\begin{equation}
h'(z) = \prod_{i=1}^n (p_i(z))^\alpha_i.
\end{equation}

Making use of the logarithmic differentiation on both sides of (3.2) and multiplying
by \( z \), we have
\[
(3.3) \quad \frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{zp_i'(z)}{p_i(z)} \right).
\]
From (1.16) and (3.3), we obtain
\[
(1 - |z|^{2a}) \left| \frac{zh''(z)}{h'(z)} \right| \leq (1 - |z|^{2a}) \sum_{i=1}^{n} |\alpha_i| \left| \frac{zp_i'(z)}{p_i(z)} \right| \leq (1 - |z|^{2a}) \left( \frac{2|z|}{1 - |z|^2} \right) \sum_{i=1}^{n} |\alpha_i|.
\]
Since \( \frac{2|z|}{1 - |z|^2} \leq \frac{2}{1 - |z|} \) for \( z \in \mathbb{U} \), we have
\[
(1 - |z|^{2a}) \left| \frac{zh''(z)}{h'(z)} \right| \leq \left( \frac{2(1 - |z|^{2a})}{1 - |z|} \right) \sum_{i=1}^{n} |\alpha_i|.
\]
Define the function \( \Psi : (0,1) \to \mathbb{R} \) by
\[
\Psi(x) = \frac{2(1 - x^{2a})}{1 - x}, \quad (a > 0, x = |z|).
\]
It is easy to show that
\[
(3.4) \quad \Psi(x) \leq \begin{cases} 
2 & \text{if } 0 < a < 1/2 \\
4a & \text{if } a \geq 1/2.
\end{cases}
\]
We thus find from (3.4) and the hypothesis (3.1) that
\[
(1 - |z|^{2a}) \left| \frac{zh''(z)}{h'(z)} \right| \leq a, \quad (z \in \mathbb{U}).
\]
Applying Lemma 1.3 for the function \( h(z) \) with \( \text{Re}(\beta) = a \), we prove that \( I_\beta(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n)(z) \in \mathcal{S} \). This evidently completes the proof of Theorem 3.1.

Letting \( n = 1, \alpha_1 = \alpha, p_1 = p \) in Theorem 3.1, we have

**Corollary 3.2.** Let \( \alpha \in \mathbb{C} \) and \( \beta \in \mathbb{C} \) with \( \text{Re}(\beta) = a \). If
\[
|\alpha| \leq \begin{cases} 
a/2 & \text{if } 0 < a \leq 1/2 \\
1/4 & \text{if } a \geq 1/2.
\end{cases}
\]
Then the integral operator \( I_\beta(p; \alpha)(z) = \left\{ \int_0^z \beta t^{\beta-1} (p(t))^{\alpha} \, dt \right\}^{1/\beta} \) is analytic and univalent in \( \mathbb{U} \), where \( p(z) \) is analytic in \( \mathbb{U} \) and satisfy \( p(0) = 1, \text{Re}\{p(z)\} > 0 \).
Letting $\beta = 1$ in Corollary 3.2, we have

**Corollary 3.3.** Let $p(z)$ be analytic in $U$ and satisfy $p(0) = 1$, $\text{Re}\{p(z)\} > 0$. Then the integral operator $\int_0^z (p(t))^{\alpha} \, dt$ is analytic and univalent in $U$, where $|\alpha| \leq 1/4$; $\alpha \in \mathbb{C}$.

Letting $p(z) = f(z)/z$ in Corollary 3.3, we get the following result obtained by Kim and Merkes [10].

**Corollary 3.4.** Let $f \in S$, and $\alpha \in \mathbb{C}; \ |\alpha| \leq 1/4$. Then the integral operator $\int_0^z (f(t)/t)^{\alpha} \, dt$ is analytic and univalent in $U$.

Letting $p(z) = f'(z)$ in Corollary 3.3, we get the following result obtained by Pfaltzgraff [16].

**Corollary 3.5.** Let $f \in S$, and $\alpha \in \mathbb{C}; \ |\alpha| \leq 1/4$. Then the integral operator $\int_0^z (f'(t))^{\alpha} \, dt$ is analytic and univalent in $U$.

**Remark 3.6.** Taking different choices of $p_i(z)$, where $p_i(z)$ as stated in Section 1, Theorem 3.1 leads to new sufficient conditions for the integral operators defined in Section 1 to be analytic and univalent in $U$.

**References**


