On Certain Class of Multivalent Functions Involving the Cho-Kwon-Srivastava Operator

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Abstract. In this paper a new subclass of multivalent functions with negative coefficients defined by Cho-Kwon-Srivastava operator is introduced. Coefficient estimate and inclusion relationships involving the neighborhoods of p-valently analytic functions are investigated for this class. Further subordination result and results on partial sums for this class are also found.

1. Introduction

Let $S_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$

which are analytic and p-valent in the unit disk $U = \{z : |z| < 1\}$. Also denote by $T_p$ the class of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0; \quad p \in \mathbb{N} = \{1, 2, 3, \ldots\}).$$

For functions

$$f_j(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} \quad (a_{p+k,j} \geq 0; \quad j = 1, 2),$$

in the class $T_p$, the modified Hadamard product $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k}.$$
Saitoh [9] introduced a linear operator:

$$L_p(a,c) : S_p \longrightarrow S_p$$

defined by

$$L_p(a,c) f(z) = \phi_p(a,c; z) * f(z) \quad (z \in U),$$

where

$$\phi_p(a,c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k},\quad (k=0), \quad (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = \begin{cases} 1; & (k=0), \\ a(a+1)(a+2)\ldots(a+k-1); & (k \in N). \end{cases}$$

and $(a)_k$ is the Pochhammer symbol defined by

$$\phi_p(a,c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k},$$

In 2004, Cho, Kwon and Srivastava [3] introduced the following linear operator $I^{\lambda}_p(a,c)$ analogous to $L_p(a,c)$:

$$I^{\lambda}_p(a,c) : S_p \longrightarrow S_p$$

defined by

$$I^{\lambda}_p(a,c) f(z) = \phi^\ast_p(a,c; z) * f(z) \quad (z \in U; \ a,c \in R \setminus \mathbb{Z}^-; \ \lambda > -p; \ f \in A_p),$$

where $\phi_p^\ast$ is the function defined in terms of the Hadamard product (or convolution) by the following condition:

$$\phi_p(a,c; z) * \phi_p^\ast(a,c; z) = \frac{z^p}{(1-z)^{\lambda+p}} .$$

We can easily find from (1.5), (1.6) and (1.7) and for the function $f(z) \in T_p$ that

$$I^{\lambda}_p(a,c) f(z) = z^p - \sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{k!(a)_k} z^{p+k} \quad (z \in U; \ \lambda > -p).$$

It is easily verified from (1.8) that

$$z(I^{\lambda}_p(a+1,c) f)'(z) = a I^{\lambda}_p(a,c) f(z) - (a-p) I^{\lambda}_p(a+1,c) f(z)$$

and

$$z(I^{\lambda}_p(a,c) f)'(z) = (\lambda + p) I^{\lambda+p}_p(a,c) f(z) - \lambda I^{\lambda}_p(a,c) f(z).$$

Also by specializing the parameter $\lambda$, $a$ and $c$ we obtain from (1.8)

$$I^{\lambda}_p(p+1,1) f(z) = f(z), \quad I^{\lambda}_p(p,1) f(z) = \frac{zf'(z)}{p} ,$$
and

\[ I^\lambda_p(a, a) f(z) = D^{n+p-1} f(z) \quad (n > -p), \]

where \( D^{n+p-1} \) is the well-known Ruscheweyh derivative of order \( n + p - 1 \).

Now making use of Cho-Kwon-Srivastava operator \( I^\lambda_p(a, c) \) defined by (1.8), we introduced the following subclass \( H^p(a, b, c, \lambda, \beta) \) of \( p \)-valent analytic function.

**Definition 1.** A function \( f(z) \in T^p \) is said to be in the class \( H^p(a, b, c, \lambda, \beta) \) if it satisfies the following inequality:

\[
(1.11) \quad \left| \frac{1}{b} \left( \frac{z(I^\lambda_p(a, c)f(z))'}{I^\lambda_p(a, c)f(z)} - p \right) \right| < \beta, 
\]

\( (z \in U; \ p \in \mathbb{N}; \ \lambda > -p; \ b \in C \setminus \{0\}; \ 0 < \beta \leq 1) \).

It may be noted that for suitable choice of \( a, b, c \) and \( \lambda \) the class \( H^p(a, b, c, \lambda, \beta) \) extends several classes of analytic and \( p \)-valent functions such as

(i) \( H^p(p + 1, b, 1, 1, \beta) = S_p(b, \beta) = \left\{ f(z) \in A^p : \left| \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - p \right) \right| < \beta \right\} \)

\( (z \in U; \ p \in \mathbb{N}; \ 0 < \beta \leq 1). \)

(ii) \( H^p(p, b, 1, 1, \beta) = C_p(b, \beta) = \left\{ f(z) \in A^p : \left| \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} + 1 - p \right) \right| < \beta \right\} \)

\( (z \in U; \ p \in \mathbb{N}; \ 0 < \beta \leq 1). \)

Where the classes \( S_p(b, \beta) \) and \( C_p(b, \beta) \) are the well know classes of starlike and convex \( p \)-valent functions of complex order. The classes \( S_p(1, \beta) = S^*_p(\beta) \) and \( C_p(1, \beta) = K^*_p(\beta) \) are the classes of starlike and convex \( p \)-valent functions introduced by Owa [5] and studied by Patil and Thakare [6].

Now following the earlier investigation by Goodman [4], Ruscheweyh [8], Al-tintas and Owa [1], Raina and Srivastava [7], Aouf [2] and others, we define the \( \delta \)-neighborhood of a function \( f(z) \in T^p \) by (see, for example, \([5, p. 1668]\))

\[
(1.12) \quad N_\delta(f) = \{ g : g \in T^p, g(z) = z^p - \sum_{k=1}^\infty b_{p+k} z^{p+k} \text{ and } \sum_{k=1}^\infty (k+p)|a_{p+k} - b_{p+k}| \leq \delta \} 
\]

In particular, if

\[ h(z) = z^p \quad (p \in \mathbb{N}), \]

we immediately have

\[
(1.13) \quad N_\delta(h) = \{ g : g \in T^p, g(z) = z^p - \sum_{k=1}^\infty b_{p+k} z^{p+k} \text{ and } \sum_{k=1}^\infty (k+p)|b_{p+k}| \leq \delta \}. 
\]
2. Coefficient estimates

**Theorem 1.** Let the function \( f(z) \in T_p \) be defined by (1.2). Then \( f(z) \in H_p(a, b, c, \lambda, \beta) \) if and only if

\[
\sum_{k=1}^{\infty} \frac{(k + \beta|b|)}{1 - (\lambda p)(c)} \frac{(\lambda + p)(a)}{k(a)} a_{p+k} \leq \beta|b|, \tag{2.1}
\]

\((z \in U; \ p \in N; \ a, c \in R \setminus Z; \ \lambda > -p; \ b \in C \setminus \{0\}; \ 0 < \beta \leq 1)\).

The result is sharp.

**Proof.** Let the function \( f(z) \in T_p \) be defined by (1.2) and belongs to \( H_p(a, b, c, \lambda, \beta) \). Then in view of (1.8) and (1.11) we have

\[
\Re \left\{ \frac{z(I_p f(z))'}{I_p f(z)} - p \right\} > -\beta|b| \quad (z \in U), \tag{2.2}
\]

or, equivalently,

\[
\Re \left\{ \frac{-\sum_{k=1}^{\infty} \frac{(\lambda + p)(c)}{1 - (\lambda)(a)} k a_{p+k} z^k}{1 - \sum_{k=1}^{\infty} \frac{(\lambda + p)(c)}{1 - (\lambda)(a)} a_{p+k} z^k} \right\} > -\beta|b| \quad (z \in U). \tag{2.3}
\]

Setting \( z = r \) \((0 \leq r < 1)\) in (2.3), we have that the expression in the denominator of the left-hand side of (2.3) is positive for \( r = 0 \) and also for all \( r(0 < r < 1) \). Thus by letting \( r \to 1^- \) through real values, (2.3) leads us to the desired assertion (2.1) of Theorem 1.

Conversely, by applying the hypothesis (2.1) and letting \( |z| = 1 \), we find from (1.11) that

\[
\left| \frac{z(I_p f(z))'}{I_p f(z)} - p \right| = \frac{\sum_{k=1}^{\infty} \frac{(\lambda + p)(c)}{1 - (\lambda)(a)} k a_{p+k} z^k}{1 - \sum_{k=1}^{\infty} \frac{(\lambda + p)(c)}{1 - (\lambda)(a)} a_{p+k} z^k} \leq \beta|b| \left( \frac{1 - \sum_{k=1}^{\infty} \frac{(\lambda + p)(c)}{1 - (\lambda)(a)} a_{p+k}}{1 - \sum_{k=1}^{\infty} \frac{(\lambda + p)(c)}{1 - (\lambda)(a)} a_{p+k}} \right) = \beta|b|. \]
Hence by maximum modulus principle we have \( f(z) \in H_p(a, b, c, \lambda, \beta) \), which evidently completes the proof of Theorem.

Our first inclusion relation involving \( N_\delta(h) \) is given in the following theorem.

3. Inclusion relationships involving \( \delta \)-neighborhoods for the class \( H_p(a, b, c, \lambda, \beta) \).

**Theorem 2.** Let

\[
\delta = \frac{a(p + 1)\beta|b|}{c(\lambda + p)(1 + \beta|b|)} \quad (p > |b|),
\]

then

\[
H_p(a, b, c, \lambda, \beta) \subset N_\delta(h).
\]

**Proof.** Let \( f(z) \in H_p(a, b, c, \lambda, \beta) \). Then, in view of Theorem 1, we have

\[
\{1 + \beta|b|\} \frac{c(\lambda + p)}{a} \sum_{k=1}^{\infty} a_{p+k} \leq \sum_{k=1}^{\infty} \{k + \beta|b|\} \frac{(\lambda + p)k(c_k)}{(1)_k(a)_k} a_{p+k} \leq \beta|b|,
\]

which readily yields

\[
\sum_{k=1}^{\infty} a_{p+k} \leq \frac{a\beta|b|}{c(\lambda + p)(1 + \beta|b|)}.
\]

Making use of (2.1) again, in conjunction with (3.4), we get

\[
\sum_{k=1}^{\infty} (k + p) \frac{(\lambda + p)k(c_k)}{(1)_k(a)_k} a_{p+k} + \sum_{k=1}^{\infty} (\beta|b| - p) \frac{(\lambda + p)k(c_k)}{(1)_k(a)_k} a_{p+k} \leq \beta|b|,
\]

or

\[
\frac{c(\lambda + p)}{a} \sum_{k=1}^{\infty} (k + p) a_{p+k} \leq \beta|b| + (p - \beta|b|) \frac{c(\lambda + p)}{a} \sum_{k=1}^{\infty} a_{p+k}
\]

\[
\leq \beta|b| + \frac{\beta|b|(p - \beta|b|)}{(1 + \beta|b|)} = \frac{(1 + p)\beta|b|}{(1 + \beta|b|)}.
\]

Hence

\[
\sum_{k=1}^{\infty} (k + p) a_{p+k} \leq \frac{a(p + 1)\beta|b|}{c(\lambda + p)(1 + \beta|b|)} \quad (p > |b|),
\]

which, by means of (1.13), establishes the inclusion (3.1) asserted by Theorem 2.
Putting (i) $\lambda = c = 1$, $a = p + 1$ and (ii) $\lambda = c = 1$, $a = p$ in Theorem 2, we obtain the following results.

**Corollary 1.** Let

\[(3.6) \quad \delta = \frac{(p + 1)\beta |b|}{1 + \beta |b|} \quad (p > |b|),\]

then

\[(3.7) \quad S_p(b, \beta) \subset N_\delta(h).\]

**Corollary 2.** Let

\[(3.8) \quad \delta = \frac{p\beta |b|}{1 + \beta |b|} \quad (p > |b|),\]

then

\[(3.9) \quad C_p(b, \beta) \subset N_\delta(h).\]

**4. $\delta$-neighborhoods for the class $H_p^{(\alpha)}(a, b, c, \lambda, \beta)$.**

In this section, we determine the neighborhood for the class $H_p^{(\alpha)}(a, b, c, \lambda, \beta)$, which define as follows. A function $f(z) \in T_p$ is said to be in the class $H_p^{(\alpha)}(a, b, c, \lambda, \beta)$ if there exists a functional $g(z) \in H_p(a, b, c, \lambda, \beta)$ such that

\[
\left| \frac{f(z)}{g(z)} - 1 \right| < p - \alpha \quad (z \in U; 0 \leq \alpha < p).
\]

**Theorem 3.** Let $g(z) \in H_p(a, b, c, \lambda, \beta)$ and

\[(4.1) \quad \alpha = p - \frac{\delta \epsilon (\lambda + p)(1 + \beta |b|)}{(p + 1)(\epsilon (\lambda + p)(1 + \beta |b|) - a \beta |b|)},\]

then

\[(4.2) \quad N_\delta(g) \subset H_p^{(\alpha)}(a, b, c, \lambda, \beta).\]

**Proof.** Let $f(z) \in N_\delta(g)$. We find from (1.12)

\[(4.3) \quad \sum_{k=1}^{\infty} (p + k)|a_{p+k} - b_{p+k}| \leq \delta,\]
which readily implies that

\[(4.4) \quad \sum_{k=1}^{\infty} |a_{p+k} - b_{p+k}| \leq \frac{\delta}{(p+1)} \quad (p \in \mathbb{N}).\]

Next, since \(g(z) \in H_p(a, b, c, \lambda, \beta)\), we have from Theorem 1

\[(4.5) \quad \sum_{k=1}^{\infty} b_{p+k} \leq \frac{a\beta|b|}{c(\lambda + p)(1 + \beta|b|)},\]

so that

\[(4.6) \quad \left| \frac{f(z)}{g(z)} - 1 \right| \leq \sum_{k=1}^{\infty} \left| a_{p+k} - b_{p+k} \right| \leq \frac{\delta c(\lambda + p)(1 + \beta|b|)}{(p+1)[c(\lambda + p)(1 + \beta|b|) - a\beta|b|]}
= (p - \alpha),\]

provided that \(\alpha\) is given by (4.1). Thus \(f(z) \in H_p(a, b, c, \lambda, \beta)\). This evidently proves Theorem 3.

Putting (i) \(\lambda = c = 1, a = p + 1\) and (ii) \(\lambda = c = 1, a = p\) in Theorem 3, we obtain the following results.

**Corollary 3.** Let \(g(z) \in S_p(b, \beta)\) and

\[(4.7) \quad \alpha = p - \frac{\delta(1 + \beta|b|)}{(p + 1)},\]

then

\[(4.8) \quad N_{\delta}(g) \subset S_{p}^{(\alpha)}(b, \beta).\]

**Corollary 4.** Let \(g(z) \in C_p(b, \beta)\) and

\[(4.9) \quad \alpha = p - \frac{\delta(1 + \beta|b|)}{1 + p + \beta|b|},\]

then

\[(4.10) \quad N_{\delta}(g) \subset C_{p}^{(\alpha)}(b, \beta).\]
5. Subordination results

The function $f(z)$ is said to be subordinate to $g(z)$ in $U$ written $f(z) \prec g(z)$, if there exist a function $w(z)$ analytic in $U$ such that $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$.

**Definition 2.** A sequence $\{b_{p+k}\}_{k=0}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if for any regular and convex function

$$g(z) = \sum_{k=0}^{\infty} c_{p+k} z^{p+k},$$

with $c_p = 1, z \in U$,

$$\sum_{k=0}^{\infty} b_{p+k} c_{p+k} z^{p+k} \prec g(z) \quad (z \in U).$$

In 1961, Wilf [10] gave the following necessary and sufficient conditions for a sequence to be a subordinating factor sequence:

**Lemma 1.** The sequence $\{b_{p+k}\}_{k=0}^{\infty}$ is a subordinating factor sequence if and only if

$$Re \left\{ 1 + 2 \sum_{k=0}^{\infty} b_{p+k} z^{p+k} \right\} > 0 \quad (z \in U).$$

**Theorem 4.** Let $f(z) \in H_p(a, b, c, \lambda, \beta)$ of the form (1.2) and

$$g(z) = \sum_{k=0}^{\infty} c_{p+k} z^{p+k}, \quad c_p = 1$$

be regular and convex function in $U$, then

$$\frac{c(\lambda + p)(1 + \beta |b|)}{2[c(\lambda + p)(1 + \beta |b|) + a \beta |b|]} (f \ast g) \prec g(z),$$

where

$$(z \in U; \quad p \in N; \quad \lambda > -p; \quad b \in C \setminus \{0\}; \quad 0 < \beta \leq 1).$$

Moreover,

$$Re \{f(z)\} > (-1)^p \frac{a \beta |b| + c(\lambda + p)(1 + \beta |b|)}{c(\lambda + p)(1 + \beta |b|)},$$

and the subordinating result (5.3) is sharp for the maximum factor

$$\frac{c(\lambda + p)(1 + \beta |b|)}{2[c(\lambda + p)(1 + \beta |b|) + a \beta |b|]}.$$
Proof. Let \( f(z) \in H_p(a, b, c, \lambda, \beta) \) of the form (1.2) and
\[
g(z) = \sum_{k=0}^{\infty} c_{p+k} z^{p+k}, \quad c_p = 1
\]
be regular and convex function in \( U \). To show subordination result (5.3), we need to show that
\[
\left\{ \frac{c(\lambda + p)(1 + \beta|b|)a_{p+k}}{2|c(\lambda + p)(1 + \beta|b|) + a\beta|b|} \right\}_{k=0}^{\infty}
\]
is a subordinating factor with \( a_p = 1 \) which in view of Lemma 1 is true if
\[
(5.6) \quad \text{Re} \left\{ 1 + \sum_{k=0}^{\infty} \frac{c(\lambda + p)(1 + \beta|b|)a_{p+k}z^{p+k}}{|c(\lambda + p)(1 + \beta|b|) + a\beta|b|} \right\} > 0 \quad (z \in U).
\]
Since
\[
\{k + \beta|b|\} \frac{(\lambda + p)c_k}{(1)_k(a)_k} \geq \{1 + \beta|b|\} \frac{c(\lambda + p)}{a} > 0 \quad (k \geq 1),
\]
on using Theorem 1, we have for \(|z| = r < 1\),
\[
\text{Re} \left\{ 1 + \frac{c(\lambda + p)(1 + \beta|b|)}{|c(\lambda + p)(1 + \beta|b|) + a\beta|b|} \sum_{k=0}^{\infty} a_{p+k}z^{p+k} \right\}
\]
\[
= \text{Re} \left\{ 1 + \frac{c(\lambda + p)(1 + \beta|b|)}{|c(\lambda + p)(1 + \beta|b|) + a\beta|b|} z^p \right. 
\]
\[
+ \frac{1}{|c(\lambda + p)(1 + \beta|b|) + a\beta|b|} \sum_{k=0}^{\infty} \frac{c(\lambda + p)(1 + \beta|b|)a_{p+k}z^{p+k}}{a_{p+k}z^{p+k}} \}
\[
\geq 1 - \frac{c(\lambda + p)(1 + \beta|b|)z^p}{|c(\lambda + p)(1 + \beta|b|) + a\beta|b|} \sum_{k=1}^{\infty} a \{k + \beta|b|\} \frac{(\lambda + p)c_k}{(1)_k(a)_k} a_{p+k}z^{p+k}
\]
\[
\geq 1 - \frac{c(\lambda + p)(1 + \beta|b|)z^p}{|c(\lambda + p)(1 + \beta|b|) + a\beta|b|} \frac{c(\lambda + p)(1 + \beta|b|) + a\beta|b|}{a\beta|b|}^p
\]
\[
\geq 1 - \frac{c(\lambda + p)(1 + \beta|b|)z^p}{|c(\lambda + p)(1 + \beta|b|) + a\beta|b|} \frac{c(\lambda + p)(1 + \beta|b|) + a\beta|b|}{a\beta|b|} = 0.
\]
This evidently proves the inequality (5.6) and hence the subordination result (5.3). taking \( g(z) = \sum_{k=0}^{\infty} z^{p+k} \) in the subordination result (5.3), we easily get the result (5.4), and for the function
\[
f(z) = z^p - \frac{a\beta|b|}{c(\lambda + p)(1 + \beta|b|)^{p+1}} z^{p+1} \in H_p(a, b, c, \lambda, \beta),
\]
it can be verified that \( \frac{c(\lambda + p)(1 + \beta|b|)}{|c(\lambda + p)(1 + \beta|b|) + a|b|} \) is a maximum factor for the subordination result (4.3).

\[ \lambda + p \]

6. Partial sums

In this section, we determine inequalities involving partial sums of \( f(z) \in T_p \) where the partial sums of \( f(z) \in T_p \) of the form (1.2) is defined as follows:

\[(6.1)\quad f_0(z) = z^p \quad \text{and} \quad f_n(z) = z^p - \sum_{k=1}^{n} a_{p+k}z^{p+k} \quad (a_{p+k} \geq 0; \ n \geq 1).\]

\[\psi_{n+1}(p, a, b, c, \lambda, \beta) = \left\{ \begin{array}{ll}
\frac{\lambda + p}{(1 + \lambda + p)u + 1} & \text{if } \lambda > -p, \\
1 + \beta|b| & \text{if } \lambda = -p, \\
\infty & \text{if } \lambda < -p.
\end{array} \right.\]

\[\psi_{n+1}(p, a, b, c, \lambda, \beta) = n + 1 + \beta|b| \]

\[\psi_n(p, a, b, c, \lambda, \beta) > \psi_{n+1}(p, a, b, c, \lambda, \beta) > 1,\]

we get

\[(6.6)\quad \sum_{k=1}^{n} a_{p+k} + \psi_{n+1}(p, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k} < \sum_{k=1}^{\infty} \psi_k(p, a, b, c, \lambda, \beta) a_{p+k} \leq 1.\]

Set

\[(6.7)\quad g_1(z) = \psi_{n+1}(p, a, b, c, \lambda, \beta) \left\{ \frac{f(z)}{f_n(z)} - \left( 1 - \frac{1}{\psi_{n+1}(p, a, b, c, \lambda, \beta)} \right) \right\},\]
which is analytic in $U$ and $g_0(z)$. If (6.5) holds we find that

$$|g_1(z) - 1| = \frac{\psi_{n+1}(p, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k} z^k}{2 + 2 \sum_{k=1}^{n} a_{p+k} z^k + \psi_{n+1}(p, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k} z^k} \leq \psi_{n+1}(p, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k} \leq 1,$$

which readily yields that $\text{Re}(g_1(z)) > 0$, and hence from (6.6) assertion (6.2) of Theorem 5 is obtained.

Similarly, if we set

$$g_2(z) = (1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)) \left\{ \frac{f_n(z)}{f(z)} - \frac{\psi_{n+1}(p, a, b, c, \lambda, \beta)}{1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)} \right\},$$

(6.8)

and making use of (6.5), we find that

$$\frac{|g_2(z) - 1|}{|g_2(z) + 1|} = \frac{(1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)) \sum_{k=n+1}^{\infty} a_{p+k} z^k}{2 + 2 \sum_{k=1}^{n} a_{p+k} z^k - (1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)) \sum_{k=n+1}^{\infty} a_{p+k} z^k} \leq \frac{(1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)) \sum_{k=n+1}^{\infty} a_{p+k}}{2 - 2 \sum_{k=1}^{n} a_{p+k} - (\psi_{n+1}(p, a, b, c, \lambda, \beta) - 1) \sum_{k=n+1}^{\infty} a_{p+k} \leq 1,$$

which proves the assertion (6.3).

\[ \square \]

References


