Sandwich Results for Certain Subclasses of Multivalent Analytic Functions Defined by Srivastava-Attiya Operator

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Abstract. In this paper, we obtain some applications of first order differential subordination and superordination results involving the operator $J_{s,b}^{\lambda}$ for certain normalized $p$-valent analytic functions associated with that operator.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$ and let $H[a,p]$ be the subclass of $H(U)$ consisting of functions of the form:

(1.1) $f(z) = a + a_pz^p + a_{p+1}z^{p+1} + \ldots$ $(a \in \mathbb{C}; p \in \mathbb{N} = \{1,2,\ldots\}).$

Also, let $A(p)$ denote the class of functions of the form:

(1.2) $f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p}z^{k+p}$ $(p \in \mathbb{N}),$

and let $A_1 = A(1)$.

If $f, g \in A(p)$, we say that $f$ is subordinate to $g$, written $f \prec g$ if there exists a Schwarz function $w$, which (by definition) is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)), z \in U$. Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence (cf., e.g., [5], [9] and [10]):

$f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

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Let \( p, h \in H(U) \) and let \( \varphi(r, s, t; z) : C^3 \times U \to C \). If \( p \) and \( \varphi(p(z), zp'(z), z^2p''(z); z) \) are univalent functions in \( U \) and if \( p \) satisfies the second-order superordination

\[
(1.3) \quad h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z),
\]

then \( p \) is a solution of the differential superordination (1.3). Note that if \( f \) is subordinate to \( g \), then \( g \) is superordinate to \( f \). An analytic function \( q \) is called a subordinant of (1.3), if \( q(z) \prec p(z) \) for all functions \( p \) satisfying (1.3). An univalent subordinant \( q \) that satisfies \( q(z) \prec q(z) \) for all subordinants of (1.3) is called the best subordinant. Recently, Miller and Mocanu [11] obtained sufficient conditions on the functions \( h, q \) and \( \varphi \) for which the following implication holds:

\[
(1.4) \quad h(z) \prec \varphi \left( p(z), zp'(z), z^2p''(z); z \right) \Rightarrow q(z) \prec p(z).
\]

Using the results of Miller and Mocanu [11], Bulboaca [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [3]. Ali et al. [1], have used the results of Bulboaca [4] to obtain sufficient conditions for normalized analytic functions to satisfy:

\[
q_1(z) \prec \frac{zf''(z)}{f'(z)} \prec q_2(z),
\]

where \( q_1 \) and \( q_2 \) are univalent functions in \( U \) with \( q_1(0) = q_2(0) = 1 \). Also, Tuneski [20] obtained a sufficient condition for starlikeness of \( f \) in terms of the quantity \( \frac{f'''(z)f(z)}{(f'(z))^2} \). Recently, Shanmugam et al. [17] obtained sufficient conditions for the normalized analytic functions \( f \) to satisfy

\[
q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)
\]

and

\[
q_1(z) - \frac{z^2f'(z)}{(f(z))^2} \prec q_2(z).
\]

They [17] also obtained results for functions defined by using Carlson-Shaffer operator.

For functions \( f \) given by (1.1) and \( g \in A(p) \) given by \( g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \), the Hadamard product (or convolution) of \( f \) and \( g \) is defined by

\[
(f \ast g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g \ast f)(z).
\]
We begin our investigation by recalling that a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (see [19])

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s},$$

$$a \in \mathbb{C}\setminus\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; \mathbb{Z}_0^- = \mathbb{Z}\setminus\mathbb{N}, \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}; s \in \mathbb{C}$$

when $|z| < 1; R\{s\} > 1$ when $|z| = 1$.

Recently, Srivastava and Attiya [18] (see also [8], [13] and [14]) introduced and investigated the linear operator $J_{s;b} : A_1 \rightarrow A_1$, defined in terms of the Hadamard product by

$$J_{s;b}f(z) = G_{s;b}(z) * f(z) \quad (z \in U; b \in \mathbb{C}\setminus\mathbb{Z}_0^-; s \in \mathbb{C}),$$

where for convenience,

$$G_{s;b} = (1 + b)^s[\Phi(z, s, b) - b^{-s}] \quad (z \in U).$$

In [21], Wang et al. defined the operator $J_{s,b}^{\lambda,p} : A(p) \rightarrow A(p)$ by

$$J_{s,b}^{\lambda,p}f(z) = f_{s,b}^{\lambda,p}(z) * f(z)$$

$$(z \in U; b \in \mathbb{C}\setminus\mathbb{Z}_0^-; s \in \mathbb{C}; \lambda > -p; p \in \mathbb{N}; f \in A(p)),$$

where

$$f_{s,b}^{\lambda,p}(z) * f_{s,b}^{\lambda,p}(z) = \frac{z^p}{(1 - z)^{\lambda+p}}$$

and

$$f_{s,b}^{\lambda,p}(z) = z^p + \sum_{k=1}^{\infty} \frac{(\lambda + p)^k}{k!} \left(\frac{p + b}{k + p + b}\right)^s a_{k+p}z^{k+p} \quad (z \in U; p \in \mathbb{N}).$$

It is easy to obtain from (1.6), (1.7) and (1.8) that

$$J_{s,b}^{\lambda,p}f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\lambda + p)^k}{k!} \left(\frac{p + b}{k + p + b}\right)^s a_{k+p}z^{k+p},$$

where $(\gamma)_k$, is the Pochhammer symbol defined in terms of the Gamma function $\Gamma$, by

$$(\gamma)_k = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \begin{cases} 1 & (k = 0) \\ \gamma(\gamma + 1)\ldots(\gamma + k - 1) & (k \in \mathbb{N}). \end{cases}$$

We note that

$$J_{0,b}^{1-p,p}f(z) = f(z) \quad (f \in A(p)).$$
Using (1.9), it is easy to verify that (see [21])

\[ (1.10) \quad z \left( J_{s+b}^{\lambda,p} f \right)'(z) = (p+b) J_{s+b}^{\lambda,p}(f)(z) - b J_{s+b+1}^{\lambda,p}(f)(z) \]

and

\[ (1.11) \quad z \left( J_{s+b}^{\lambda,p} f \right)'(z) = (p+\lambda) J_{s+b+1}^{\lambda+1,p}(f)(z) - \lambda J_{s+b}^{\lambda,p}(f)(z). \]

It should be remarked that the linear operator \( J_{s+b}^{\lambda,p} \) is generalization of many other linear operators considered earlier. We have:

1. \( J_{0}^{\lambda,p} f(z) = D^{\lambda+p-1} f(z) \) (\( \lambda > -p, p \in \mathbb{N} \)), where \( D^{\lambda+p-1} \) is the \((\lambda+p-1)\)-th order Ruscheweyh derivative of a function \( f(z) \in A(p) \) (see [7]);
2. \( J_{1}^{\lambda,p} f(z) = J_{v}^{\lambda,p} f(z) \) (\( v > -p \)), where the generalized Bernardi-Libera-Livingston operator \( J_{v}^{\lambda,p} \) was studied by Choi et al. [6];
3. \( J_{m,0}^{\lambda,p} f(z) = I_{m}^{\lambda,p} f(z) \) (\( m \in \mathbb{N}_{0} = \mathbb{N} \cup \{0\} \)), where for \( p = 1 \) the integral operator \( I_{m}^{1} \) was introduced and studied by Salagean [15];
4. \( J_{1}^{\lambda,p} f(z) = J_{1}^{\lambda,p} f(z) \) (\( \sigma > 0 \)), where the integral operator \( I_{1}^{\sigma} \) was studied by Shams et al. [16] and Aouf et al. [2];
5. \( J_{\gamma,0}^{\lambda,p} f(z) = P_{\tau}^{\gamma} f(z) \) (\( \gamma \geq 0, \tau > 1 \)), where the integral operator \( P_{\tau}^{\gamma} \) was introduced and studied by Patel and Sahoo [12].

In this paper, we obtain sufficient conditions for the normalized analytic function \( f \) defined by using the operator \( J_{s+b}^{\lambda,p} \) to satisfy:

\[ q_{1}(z) \prec \left( \frac{z^{p}}{\lambda^{\lambda,p} f(z)} \right)^{\mu} \prec q_{2}(z) \]

and \( q_{1} \) and \( q_{2} \) are given univalent functions in \( U \).

2. Definitions and preliminaries

In order to prove our results, we shall need the following definition and lemmas.

**Definition 1 ([11]).** Let \( Q \) be the set of all functions \( f \) that are analytic and injective on \( U \setminus E(f) \), where

\[ E(f) = \{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \}, \]

and are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(f) \).

**Lemma 1 ([9]).** Let \( q \) be univalent in the unit disc \( U \), and let \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \), with \( \varphi(w) \neq 0 \) when \( w \in q(U) \). Set

\[ (2.1) \quad Q(z) = z q'(z) \varphi(q(z)) \] and \( h(z) = \theta(q(z)) + Q(z) \)
suppose that
(i) \( Q \) is a starlike function in \( U \),
(ii) \( \text{Re}\left\{ \frac{zh'(z)}{Q(z)} \right\} > 0, z \in U. \)

If \( p \) is analytic in \( U \) with \( p(0) = q(0) \), \( p(U) \subseteq D \) and

\[
\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)),
\]
then \( p(z) \prec q(z) \), and \( q \) is the best dominant of \( (2.2) \).

**Lemma 2 ([4]).** Let \( q \) be a convex univalent function in \( U \) and \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \). Suppose that
(i) \( \text{Re}\left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} > 0 \) for \( z \in U \),
(ii) \( Q(z) = zq'(z)\varphi(q(z)) \) is starlike univalent in \( U \).

If \( p \in H[q(0), 1] \cap Q \), \( p(U) \subseteq D \), \( \theta(p(z)) + zp'(z)\varphi(p(z)) \) is univalent in \( U \), and

\[
\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)),
\]
then \( q(z) \prec p(z) \), and \( q \) is the best subordinant of \( (2.3) \).

3. Applications to the operator \( J_{s,b}^{p} \) and sandwich theorems

Unless otherwise mentioned, we shall assume in the reminder of this paper that \( b \in \mathbb{C}\setminus \mathbb{Z}^{-} \), \( s \in \mathbb{C}, p \in \mathbb{N}, \lambda > -p, \gamma, \tau, \zeta \in \mathbb{C}, \Omega, \mu \in \mathbb{C}^{*} = \mathbb{C} \setminus \{0\}, z \in U \) and the powers are understood as principle values.

**Theorem 1.** Let \( q(z) \) be analytic and univalent in \( U \) with \( q(z) \neq 0 \). Suppose that

\[
\frac{zq'(z)}{q(z)} \text{ is starlike univalent in } U.
\]

Let

\[
\text{Re}\{1 + \frac{\gamma}{\Omega} q(z) + \frac{2\zeta}{\Omega} (q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} \} > 0,
\]

and

\[
\chi(f, s, b, \lambda, p, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left( \frac{z^p}{J_{s,b}^{\lambda,p} f(z)} \right)^{\mu} + \zeta \left( \frac{z^p}{J_{s,b}^{\lambda,p} f(z)} \right)^{2\mu} + \Omega \mu \left( p - \frac{z}{J_{s,b}^{\lambda,p} f(z)} \right).
\]

If \( q \) satisfies the following subordination:

\[
\chi(f, s, b, \lambda, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq'(z)}{q(z)}.
\]
Then

\[
\left( \frac{z^p}{J_{s,b}^p f(z)} \right)^\mu \prec q(z)
\]

and \( q \) is the best dominant.

**Proof.** Define a function \( p(z) \) by

\[
p(z) = \left( \frac{z^p}{J_{s,b}^p f(z)} \right)^\mu (z \in U).
\]

Then the function \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \).

Therefore, differentiating (3.5) logarithmically with respect to \( z \) and using the identity (1.10) in the resulting equation, we have

\[
\tau + \gamma \left( \frac{z^p}{J_{s,b}^p f(z)} \right)^\mu + \zeta \left( \frac{z^p}{J_{s,b}^p f(z)} \right)^{2\mu} + \Omega \left( \frac{z}{J_{s,b}^p f(z)} \right) p = \frac{z}{p(z)}. \tag{3.6}
\]

Using (3.3) and (3.6), we have

\[
\tau + \gamma p(z) + \zeta (p(z))^2 + \Omega \frac{z^p(z)}{p(z)}. \tag{3.7}
\]

Setting

\[
\theta(w) = \tau + \gamma w + \zeta w^2 \text{ and } \varphi(w) = \frac{\Omega}{w}
\]

it can be easily observed that \( \theta \) is analytic in \( \mathbb{C} \), \( \varphi \) is analytic in \( \mathbb{C}^* \) and \( \varphi(w) \neq 0 \ (w \in \mathbb{C}^*) \). Hence, the result now follows by using Lemma 1.

Taking \( q(z) = \frac{1 + Az}{1 + Bz} \ (-1 \leq B < A \leq 1) \) in Theorem 1, the condition (3.1) reduces to

\[
\text{Re}\{1 + \frac{2}{\Omega} \left( \frac{1 + Az}{1 + Bz} \right) + \frac{2\zeta}{\Omega} \left( \frac{1 + Az}{1 + Bz} \right)^2 - \frac{(A - B)z}{(1 + Az)(1 + Bz)} - \frac{2Bz}{1 + Bz} \} > 0. \tag{3.9}
\]

hence, we obtain the following corollary.

**Corollary 1.** Let \( f(z) \in A(p) \), assume that (3.9) holds true, \(-1 \leq B < A \leq 1\) and (3.10)

\[
\chi(f, s, b, \lambda, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma \left( \frac{1 + Az}{1 + Bz} \right) + \zeta \left( \frac{1 + Az}{1 + Bz} \right)^2 + \Omega \frac{(A - B)z}{(1 + Az)(1 + Bz)}.
\]
where \( \chi(f; s, b, \lambda, p, \gamma, \tau, \zeta, \Omega, \mu) \) is given by (3.2), then

\[
\left( \frac{z^p}{f'(z)} \right)^{\mu} < \frac{1 + Az}{1 + Bz},
\]

and \( \frac{1 + Az}{1 + Bz} \) is the best dominant of (3.10).

Taking \( q(z) = \left( \frac{1 + z}{1 - z} \right)^v \) \( (0 < v \leq 1) \) in Theorem 1, the condition (3.1) reduces to

(3.11) \[ \Re\{1 + \frac{\gamma}{\Omega} \left( \frac{1 + z}{1 - z} \right)^v + \frac{2\zeta}{\Omega} \left( \frac{1 + z}{1 - z} \right)^{2v} - \frac{2z^2}{1 - z^2} \} > 0, \]

hence, we obtain the following corollary.

**Corollary 2.** Let \( f(z) \in A(p) \), assume that (3.11) holds true, \( 0 < v \leq 1 \) and

(3.12) \[ \chi(f; s, b, \lambda, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma \left( \frac{1 + z}{1 - z} \right)^v + \zeta \left( \frac{1 + z}{1 - z} \right)^{2v} + \Omega \frac{2 vz}{(1 - z)^2}, \]

where \( \chi(f; s, b, \lambda, p, \gamma, \tau, \zeta, \Omega, \mu) \) is given by (3.2), then

\[
\left( \frac{z^p}{f'(z)} \right)^{\mu} < \left( \frac{1 + z}{1 - z} \right)^v,
\]

and \( \left( \frac{1 + z}{1 - z} \right)^v \) is the best dominant of (3.12).

Putting \( s = 0 \) and \( \lambda = 1 - p \) \( (p \in \mathbb{N}) \) in Theorem 1, we obtain the following corollary.

**Corollary 3.** Let \( q(z) \) be analytic and univalent in \( U \) with \( q(z) \neq 0 \). Suppose that \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( U \). If \( f(z) \in A(p) \), assume that (3.1) holds true and

(3.13) \[ G(f; p, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left( \frac{z^p}{f(z)} \right)^{\mu} + \zeta \left( \frac{z^p}{f(z)} \right)^{2\mu} + \Omega \mu \left( p - \frac{zf'(z)}{f(z)} \right). \]

If \( q \) satisfies the following subordination:

(3.14) \[ G(f; p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq'(z)}{q(z)}. \]

Then

\[
\left( \frac{z^p}{f(z)} \right)^{\mu} \prec q(z)
\]
and \( q \) is the best dominant of (3.14).

Putting \( p = 1 \) in Corollary 3, we obtain the following corollary.

**Corollary 4.** Let \( q(z) \) be analytic and univalent in \( U \) with \( q(z) \neq 0 \). Suppose that \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( U \). If \( f(z) \in A \), assume that (3.1) holds true and

\[
K(f, p, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left( \frac{z}{f(z)} \right)^2 + \Omega \mu \left( 1 - \frac{z'f(z)}{f(z)} \right).
\]

If \( q \) satisfies the following subordination:

\[
K(f, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q(z) + \zeta q(z)^2 + \Omega \frac{zq'(z)}{q(z)}.
\]

Then

\[
\left( \frac{z}{f(z)} \right)^\mu \prec q(z)
\]

and \( q \) is the best dominant of (3.16).

Putting \( s = 0 \) in Theorem 1, we obtain the following corollary.

**Corollary 5.** Let \( q(z) \) be analytic and univalent in \( U \) with \( q(z) \neq 0 \). Suppose that \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( U \). If \( f(z) \in A(p) \), assume that (3.1) holds true and

\[
D(f, p, \lambda, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left( \frac{z^p}{D^{\lambda+p-1}f(z)} \right)^2 + \Omega \mu \left( p - \frac{z(D^{\lambda+p-1}f(z)'^p}{D^{\lambda+p-1}f(z)} \right).
\]

If \( q \) satisfies the following subordination:

\[
D(f, p, \lambda, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q(z) + \zeta q(z)^2 + \Omega \frac{zq'(z)}{q(z)}.
\]

then

\[
\left( \frac{z^p}{D^{\lambda+p-1}f(z)} \right)^\mu \prec q(z)
\]

and \( q \) is the best dominant of (3.18).

Putting \( s = 1, b = v(v > -p) \) and \( \lambda = 1 - p(p \in \mathbb{N}) \) in Theorem 1, we obtain the following corollary.

**Corollary 6.** Let \( q(z) \) be analytic and univalent in \( U \) with \( q(z) \neq 0 \). Suppose that \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( U \). If \( f(z) \in A(p) \), assume that (3.1) holds true and

\[
(f, v, p, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left( \frac{z^p}{I_{v,p}f(z)} \right)^2 + \Omega \mu \left( p - \frac{z(I_{v,p}f(z)'^p}{I_{v,p}f(z)} \right).
\]
If $q$ satisfies the following subordination:

\[(3.20) \quad (f, v, p, \beta, \alpha, \eta, \mu) \prec \tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq'(z)}{q(z)},\]

then

\[\left( \frac{z^p}{J_{v,p} f(z)} \right)^\mu \prec q(z)\]

and $q$ is the best dominant of (3.20).

Putting $s = m$ ($m \in \mathbb{N}_0$), $b = 0$ and $\lambda = 1 - p$ ($p \in \mathbb{N}$) in Theorem 1, we obtain the following corollary.

**Corollary 7.** Let $q(z)$ be analytic and univalent in $U$ with $q(z) \neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. If $f(z) \in A(p)$, assume that (3.1) holds true and

\[(3.21) \quad S(f, m, p, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left( \frac{z^p}{I^m_{p} f(z)} \right)^\mu + \zeta \left( \frac{z^p}{I^m_{p} f(z)} \right)^{2\mu} + \Omega \mu \left( p - \frac{z^m_{p} f(z)}{I^m_{p} f(z)} \right).\]

If $q$ satisfies the following subordination:

\[(3.22) \quad S(f, m, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq'(z)}{q(z)},\]

then

\[\left( \frac{z^p}{I^m_{p} f(z)} \right)^\mu \prec q(z)\]

and $q$ is the best dominant of (3.22).

Putting $s = \sigma$ ($\sigma > 0$), $b = 1$ and $\lambda = 1 - p$ ($p \in \mathbb{N}$) in Theorem 1, we obtain the following corollary.

**Corollary 8.** Let $q(z)$ be analytic and univalent in $U$ with $q(z) \neq 0$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. If $f(z) \in A(p)$, assume that (3.1) holds true and

\[(3.23) \quad \varphi(f, \sigma, p, \gamma, \tau, \zeta, \Omega, \mu) = \tau + \gamma \left( \frac{z^p}{I^\sigma_{p} f(z)} \right)^\mu + \zeta \left( \frac{z^p}{I^\sigma_{p} f(z)} \right)^{2\mu} + \Omega \mu \left( p - \frac{z^\sigma_{p} f(z)}{I^\sigma_{p} f(z)} \right).\]

If $q$ satisfies the following subordination:

\[(3.24) \quad \varphi(f, \sigma, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq'(z)}{q(z)},\]

then

\[\left( \frac{z^p}{I^\sigma_{p} f(z)} \right)^\mu \prec q(z)\]
and \( q \) is the best dominant of (3.24).

**Theorem 2.** Let \( q \) be a convex univalent function in \( U \), \( q(z) \neq 0 \) and \( \frac{z q'(z)}{q(z)} \) be starlike univalent in \( U \). Assume that

\[
\text{(3.25)} \quad \operatorname{Re} \left\{ \frac{2 \zeta}{\Omega} (q(z))^2 + \frac{\gamma}{\Omega} q(z) \right\} > 0.
\]

If \( f \in A(p), 0 \neq \left( \frac{z^p}{J_{s,b}^p f(z)} \right)^\mu \in H[q(0), 1] \cap Q, \chi(f, s, b, p, \gamma, \tau, \zeta, \Omega, \mu) \) univalent in \( U \), and

\[
\text{(3.26)} \quad \tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{z q'(z)}{q(z)} \prec \chi(f, s, b, p, \gamma, \tau, \zeta, \Omega, \mu),
\]

where \( \chi(f, s, b, p, \gamma, \tau, \zeta, \Omega, \mu) \) is given by (3.2), then

\[
q(z) \prec \left( \frac{z^p}{J_{s,b}^p f(z)} \right)^\mu,
\]

and \( q \) is the best subordinant of (3.26).

**Proof.** Taking

\[
\theta(w) = \tau + \gamma w + \zeta w^2 \quad \text{and} \quad \varphi(w) = \frac{\Omega}{w},
\]

it is easily observed that \( \theta \) is analytic in \( \mathbb{C} \), \( \varphi \) is analytic in \( \mathbb{C}^* \) and \( \varphi(w) \neq 0 \) \((w \in \mathbb{C}^*)\). Since \( q \) is a convex (univalent) function it follows that

\[
\operatorname{Re} \left\{ \frac{\theta'(q(z))}{\varphi(q(z))} \right\} = \operatorname{Re} \left\{ \frac{2 \zeta}{\Omega} (q(z))^2 + \frac{\gamma}{\Omega} q(z) \right\} q'(z) > 0.
\]

Thus the assertion (3.26) of Theorem 2 follows by an application of Lemma 2. \( \square \)

Putting \( s = 0 \) and \( \lambda = 1 - p \) \((p \in \mathbb{N})\) in Theorem 2, it is easy to check that the assumption (3.25) holds, we obtain the following corollary.

**Corollary 9.** Let \( q \) be a convex univalent function in \( U \), \( q(z) \neq 0 \) and \( \frac{z q'(z)}{q(z)} \) be starlike univalent in \( U \). If \( f \in A(p), 0 \neq \left( \frac{z^p}{J_{s,b}^p f(z)} \right)^\mu \in H[q(0), 1] \cap Q \) and \( G(f, p, \gamma, \tau, \zeta, \Omega, \mu) \) univalent in \( U \), where \( G(f, p, \gamma, \tau, \zeta, \Omega, \mu) \) is given by (3.13), then

\[
\text{(3.27)} \quad \tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{z q'(z)}{q(z)} \prec G(f, p, \gamma, \tau, \zeta, \Omega, \mu),
\]

implies

\[
q(z) \prec \left( \frac{z^p}{f(z)} \right)^\mu.
\]
and \( q \) is the best dominant of (3.27).

Putting \( s = 0 \) in Theorem 2, it is easy to check that the assumption (3.25) holds, we obtain the following corollary.

**Corollary 10.** Let \( q \) be a convex univalent function in \( U \), \( q(z) \neq 0 \) and \( \frac{zq'(z)}{q(z)} \) be starlike univalent in \( U \). If \( f \in A(p), 0 \neq \left( \frac{z^p}{f(z)} \right)^\mu \in H[q(0), 1] \cap Q \) and \( D(f, p, \lambda, \gamma, \tau, \zeta, \Omega, \mu) \) univalent in \( U \), where \( D(f, p, \lambda, \gamma, \tau, \zeta, \Omega, \mu) \) is given by (3.17), then

\[
\tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq'(z)}{q(z)} \prec D(f, p, \lambda, \gamma, \tau, \zeta, \Omega, \mu),
\]

implies

\[
q(z) \prec \left( \frac{z^p}{D\lambda+p-1f(z)} \right)^\mu
\]

and \( q \) is the best dominant of (3.28).

Putting \( s = 1, b = v (v > p) \) and \( \lambda = 1 - p (p \in \mathbb{N}) \) in Theorem 2, it is easy to check that the assumption (3.25) holds, we obtain the following corollary.

**Corollary 11.** Let \( q \) be a convex univalent function in \( U \), \( q(z) \neq 0 \) and \( \frac{zq'(z)}{q(z)} \) be starlike univalent in \( U \). If \( f \in A(p), 0 \neq \left( \frac{z^p}{f(z)} \right)^\mu \in H[q(0), 1] \cap Q \) and \( f, v, p, \gamma, \tau, \zeta, \Omega, \mu \) univalent in \( U \), where \( (f, v, p, \gamma, \tau, \zeta, \Omega, \mu) \) is given by (3.19), then

\[
\tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq'(z)}{q(z)} \prec (f, v, p, \gamma, \tau, \zeta, \Omega, \mu),
\]

implies

\[
q(z) \prec \left( \frac{z^p}{Jv,pf(z)} \right)^\mu
\]

and \( q \) is the best dominant of (3.29).

Putting \( s = m (m \in \mathbb{N}_0), b = 0 \) and \( \lambda = 1 - p (p \in \mathbb{N}) \) in Theorem 2, it is easy to check that the assumption (3.25) holds, we obtain the following corollary.

**Corollary 12.** Let \( q \) be a convex univalent function in \( U \), \( q(z) \neq 0 \) and \( \frac{zq'(z)}{q(z)} \) be starlike univalent in \( U \). If \( f \in A(p), 0 \neq \left( \frac{z^p}{f(z)} \right)^\mu \in H[q(0), 1] \cap Q \) and \( S(f, m, p, \gamma, \tau, \zeta, \Omega, \mu) \) univalent in \( U \), where \( S(f, m, p, \gamma, \tau, \zeta, \Omega, \mu) \) is given by (3.21), then

\[
\tau + \gamma q(z) + \zeta (q(z))^2 + \Omega \frac{zq'(z)}{q(z)} \prec S(f, m, p, \gamma, \tau, \zeta, \Omega, \mu),
\]
implies

\[ q(z) \prec \left( \frac{z^p}{F^m f(z)} \right) ^\mu \]

and \( q \) is the best dominant of (3.30).

Putting \( s = \sigma (\sigma > 0), b = 1 \) and \( \lambda = 1 - p (p \in \mathbb{N}) \) in Theorem 2, it is easy to check that the assumption (3.25) holds, we obtain the following corollary.

**Corollary 13.** Let \( q \) be a convex univalent function in \( U \), \( q(z) \neq 0 \) and \( zq_1'(z) \) be starlike univalent in \( U \). If \( f \in A(p), 0 \neq \left( \frac{z^p}{J^p f(z)} \right) ^\mu \in H[q(0), 1] \cap Q \) and \( \varphi(f, \sigma, p, \gamma, \tau, \zeta, \Omega, \mu) \) univalent in \( U \), where \( \varphi(f, \sigma, p, \gamma, \tau, \zeta, \Omega, \mu) \) is given by (3.23), then

\[ (3.31) \quad \tau + \gamma q_1(z) + \zeta (q_1(z))^2 + \Omega \frac{zq_1'(z)}{q_1(z)} \prec \varphi(f, \sigma, p, \gamma, \tau, \zeta, \Omega, \mu), \]

implies

\[ q(z) \prec \left( \frac{z^p}{J^p f(z)} \right) ^\mu \]

and \( q \) is the best dominant of (3.31).

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem.

**Theorem 3.** Let \( q_1 \) be convex univalent in \( U \) and \( q_2 \) be univalent in \( U \). Suppose that \( q_1 \) and \( q_2 \) satisfies (3.1) and (3.25), respectively. If \( f \in A(p), \left( \frac{z^p}{J^p f(z)} \right) ^\mu \in H[q(0), 1] \cap Q \) and \( \chi(f, s, b, \lambda, p, \gamma, \tau, \zeta, \Omega, \mu) \) is univalent in \( U \), where \( \chi(f, s, b, \lambda, p, \gamma, \tau, \zeta, \Omega, \mu) \) is defined in (3.2), then

\[ (3.32) \quad \tau + \gamma q_1(z) + \zeta (q_1(z))^2 + \Omega \frac{zq_1'(z)}{q_1(z)} \prec \chi(f, s, b, \lambda, p, \gamma, \tau, \zeta, \Omega, \mu) \prec \tau + \gamma q_2(z) + \zeta (q_2(z))^2 + \Omega \frac{zq_2'(z)}{q_2(z)}, \]

implies

\[ q_1(z) \prec \left( \frac{z^p}{J^p f(z)} \right) ^\mu \prec q_2(z) \]

and \( q_1, q_2 \) are respectively the best subordinant and dominant of (3.32).

Putting \( s = 0 \) and \( \lambda = 1 - p (p \in \mathbb{N}) \) in Theorem 3, we obtain the following corollary.

**Corollary 14.** Let \( q_1 \) be convex univalent in \( U \) and \( q_2 \) univalent in \( U \), \( q_1 \neq 0 \) and
\( q_2 \neq 0 \) in \( U \). Suppose that \( q_1 \) and \( q_2 \) satisfies (3.1) and (3.25), respectively. If \( f \in A(p), \left( \frac{z^p f'(z)}{f(z)} \right)^\mu \in H[q(0), 1] \cap Q \) and \( G(f, p, \gamma, \tau, \zeta, \Omega, \mu) \) is univalent in \( U \), where \( G(f, p, \gamma, \tau, \zeta, \Omega, \mu) \) is defined in (3.13), then

\[
\tau + \gamma q_1(z) + \zeta (q_1(z))^2 + \Omega \frac{z q_1'(z)}{q_1(z)} < G(f, p, \gamma, \tau, \zeta, \Omega, \mu) < \tau + \gamma q_2(z) + \zeta (q_2(z))^2 + \Omega \frac{z q_2'(z)}{q_2(z)},
\]

implies

\[
q_1(z) < \left( \frac{z^p}{f(z)} \right)^\mu < q_2(z)
\]

and \( q_1, q_2 \) are respectively the best subordinant and dominant of (3.33).

**Remark.** Combining: (1) Corollary 5 and Corollary 10; (2) Corollary 6 and Corollary 11; (3) Corollary 7 and Corollary 12; (4) Corollary 8 and Corollary 13, we obtain similar sandwich theorems for the corresponding operators.

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**References**


