On the Polynomial of the Dunwoody (1, 1)-knots

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Abstract. There is a special connection between the Alexander polynomial of (1, 1)-knot and the certain polynomial associated to the Dunwoody 3-manifold ([3], [10] and [13]). We study the polynomial(called the Dunwoody polynomial) for the (1,1)-knot obtained by the certain cyclically presented group of the Dunwoody 3-manifold. We prove that the Dunwoody polynomial of (1, 1)-knot in $S^3$ is to be the Alexander polynomial under the certain condition. Then we find an invariant for the certain class of torus knots and all 2-bridge knots by means of the Dunwoody polynomial.

1. Introduction

We begin with the fact that every closed 3-manifold has a spine called the Heegaard diagram, from which one can obtain the presentation for the group; however, not all group presentations arise from the spines of 3-manifolds. Therefore, determining which cyclic presentations of groups correspond to spines of closed 3-manifolds is an open problem.

In 1968, L. Neuwirth introduced an algorithm for the construction of a connected closed orientable 3-manifold from 2-complex, which corresponds to a group presentation([17]). In 1994, M. J. Dunwoody introduced the 6-tuples yielding a family of genus $n$ Heegaard diagrams of closed orientable 3-manifolds called the Dunwoody 3-manifold ([10]). In 2000, author in [19] first proposed that the branched set in the quotient space of the Dunwoody 3-manifold is a (1,1)-knot in $S^3$. In other words, at least one cyclic symmetry on the Dunwoody 3-manifold induces a (1,1)-knot. In 2004, it was shown that some classes of knots represented by the Dunwoody 3-manifolds contain all (1,1)-knots in $S^3$ ([5]).

Conversely, for a given (1,1)-knot $K$, it is an interesting problem to determine a type of the Dunwoody 3-manifold representing $K$ even if it is not unique. Until

* Corresponding Author.
Received May 8, 2010; accepted February 19, 2012.
2010 Mathematics Subject Classification: Primary 57M12, 57M25, 57M27, 57M50; Secondary 57M05, 57M10, 57M15, 57M60.
Key words and phrases: Torus knot, (1, 1)-knot, (1, 1)-decomposition, Dunwoody 3-manifold, Alexander polynomial, Heegaard splitting, Heegaard diagram.
This work was supported by Dong-eui University Grant(2010AA088).
now these problems for all 2-bridge knots and some torus knots were solved in [11], [14] and [19]. For example, the explicit type for the torus knot $T(p, q)$ satisfying $q \equiv \pm 1 \mod p$ has been obtained in [1] and [5], and for the torus knot $T(p, q)$ satisfying $q \equiv \pm 2 \mod p$, the type has been obtained in [13]. Furthermore, in [6], it has been obtained for all torus knots with bridge number at most three. However to determining types of Dunwoody 3-manifolds for all torus knots is still unknown.

The Dunwoody 3-manifold plays an important role in determining which cyclically presented group corresponds to a 3-manifold. Indeed, in order to study a 3-manifold with some particular group as the fundamental group, the Dunwoody 3-manifold has a Heegaard diagram from which one can obtain a presentation for the group. Thus, to find the Dunwoody 3-manifold, one must seek a cyclically presented group associated with a 3-manifold. Furthermore, as in [12], the branched covering space of the spatial $\Theta$-curve containing $(1, 1)$-knots as the constituent knots is related to the Dunwoody 3-manifold. Therefore the concept of the Dunwoody 3-manifold is important in knot, branched covering and graph theories.

In section 2, we introduce a set of 4-tuples representing all $(1, 1)$-knots, which is determined by two permutations, and so 3-manifolds related to a set of 4-tuples are containing the Dunwoody 3-manifolds. In particular, we prove that the strongly-cyclic branched covering space of the Dunwoody $(1, 1)$-knot represented by the certain 4-tuples is homeomorphic to the Dunwoody 3-manifold. Moreover we show some conditions of the Dunwoody $(1, 1)$-knot representing a torus knot; and we also discuss about the type of the Dunwoody 3-manifold representing the torus knot.

In Section 3, we show that the fundamental group of the Dunwoody 3-manifold admits a cyclic presentation, which is independent of results in [4] and [16]; and we define the Dunwoody polynomial from the cyclic presentation. As the main result, we show that the Dunwoody polynomial for the Dunwoody $(1, 1)$-knot in $S^3$ is the Alexander polynomial under some condition. For the results in [3] and [4], they are shown the connection between the Dunwoody polynomial and the projection of Alexander polynomial into $\mathbb{Z}[t]/(t^n - 1)$ for some $n > 1$. Moreover we show that a certain numerical number from the Dunwoody polynomial is an invariant for some torus knots and all 2-bridge knots of $(1, 1)$-knots. This result gives an answer to a question in [10]. In this note, all lens spaces will be assumed to include $S^3$ but not $S^1 \times S^2$. The basic facts about lens spaces are covered in [18].

2. On the Dunwoody $(1, 1)$-knots

Let $(V_1, V_2)$ be a Heegaard splitting with genus $n$ of a closed orientable 3-manifold $M$. A properly embedded disc $D$ in the handlebody $V_2$ is called a meridian disc of $V_2$ if cutting $V_2$ along $D$ yields a handlebody of genus $n - 1$. A collection of $n$ mutually disjoint meridian discs $\{D_i\}$ of $V_2$ is called a complete system of meridian discs of $V_2$ if cutting $V_2$ along $\cup_i D_i$ gives a 3-ball. Let $c_i$ denote the 1-sphere $\partial D_i$ which lies in the closed orientable surface $\partial V_1 = \partial V_2$ of genus $n$. Then the system is
said to be a Heegaard diagram of \( M \) denoted by \((V_1; c_1, c_2, \cdots, c_n)\). Let \( M \) be a lens space and \( K \) be a knot in \( M \). Then the pair \((M, K)\) admits a \((1, 1)\)-decomposition if there exists a Heegaard splitting of genus one \((V_1, K_1) \cup_\varphi (V_2, K_2)\) of \((M, K)\) such that \((V_1; c_1)\) is a Heegaard diagram of \( M \), and \( K_1 \subset V_1 \) and \( K_2 \subset V_2 \) are properly embedded trivial arcs, where \( \varphi \) is an attaching homeomorphism.

We now introduce the Dunwoody \((1, 1)\)-decomposition of \((M, K)\) determined by two permutations and 4-tuples \((a, b, c, r)\), where \( a > 0, b \geq 0, c \geq 0, r \in \mathbb{Z}_d \), and \( d = 2a + b + c \). Let \( m^+ \) and \( m^- \) be two circles with each other different orientations, and let \( X^+ = \{1, 2, \cdots, d\} \) and \( X^- = \{-1, -2, \cdots, -d\} \) be sets of \( d \) vertices in \( m^+ \) and \( m^- \), respectively. We now define two permutations \( \alpha \) and \( \beta \) as below, where all numbers are under \( \mod \ d \).

\[
\alpha(j) = \begin{cases} 
  d - j + 1 & \text{if } 1 \leq j \leq a \\
  -j - c & \text{if } a + 1 \leq j \leq a + b \\
  -j + b & \text{if } a + b + 1 \leq j \leq a + b + c \\
  -d - j - 1 & \text{if } -a \leq j \leq -1 
\end{cases}
\]

and

\[
\beta(j) = \begin{cases} 
  -j + r & \text{if } r < j \\
  -j + r - d & \text{otherwise}
\end{cases}
\]

The cycle expressions of \( \alpha \) and \( \beta \) are the following.

\[
\alpha = (1, d)(2, d - 1)(3, d - 2) \cdots (a, d - a + 1) \\
(a + 1, -(a + c + 1)) \cdots (a + b, -(a + c + b)) \\
(a + b + 1, -(a + 1)) \cdots (a + b + c, -(a + c)) \\
(-1, -d)(-2, -(d - 1)) \cdots (-a, -(d - a + 1)) \text{ (**)}
\]

and

\[
\beta = (1, -(1 - r))(2, -(2 - r)) \cdots (j, -(j - r)) \\
\cdots (d, -(d - r)). \text{ (***)}
\]

We note that each 2-cycle in \( \alpha \) consists of the end points of a curve connecting \( m^+ \) and \( m^- \) or themselves as the rule of Figure 1; and each 2-cycles in \( \beta \) generates a meridian disk \( m \), by gluing the corresponding points in \( m^+ \) and \( m^- \) via \( \beta \). For example, \((r, -(r - r))\) means that the number \( r \) of \( m^+ \) is identified with the number \(- (r - r) = -0 = -d \) in \( m^- \). Thus \( \beta \alpha \) determines the genus one solid torus \( V_1 \) and the disjoint simple closed curves on \( \partial V_1 \).

**Theorem 2.1.** If \( L \) is the number of the disjoint simple closed curves on \( \partial V_1 \), then there are two permutations \( \alpha \) and \( \beta \) such that \(|\alpha| - |\beta| + 2L = |\beta\alpha|\), where \(|\cdot|\) means the number of disjoint cycles in a permutation.

**Proof.** Let \( d = 2a + b + c \). Let \( X^+ = \{1, 2, \cdots, d\} \) and \( X^- = \{-d, -d + 1, \cdots, -1\} \) be sets of \( d \) points in \( m^+ \) and \( m^- \), respectively. Then the permutation \( \alpha \) is consisting of
d 2-cycles by two end points of line segments connecting $m^+$ and $m^-$ or themselves on $\partial V_1$, and the permutation $\beta$ is consisting of $d$ 2-cycles connecting $m^+$ and $m^-$ on $\partial V_1$. We now define an equivalence relation on $X = X^+ \cup X^-$ by

$$x \sim y \text{ if } y = (\beta \alpha)^i(x) \text{ or } y = \alpha(\beta \alpha)^i(x) \text{ for some } i.$$ 

Then we call the equivalence classes of $X$ under the relation the orbits of $\beta \alpha$. Let $l$ be a simple closed curve on $\partial V_1$ and $x$ be a point on $m^+$ meeting $l$. Then $l$ is determined by the repeated applications of $\alpha$ and $\beta$ as follows;

$$x, \alpha(x), \beta \alpha(x), \alpha \beta \alpha(x), \ldots, \alpha \beta \cdots \alpha(x),$$

which forms an orbit of $\beta \alpha$. Each orbit of $\beta \alpha$ determines a simple closed curve in $\partial V_1$. Let $Y_1, \ldots, Y_L$ be orbits of $\beta \alpha$. If $x \in Y_i$ and $d$ is the smallest positive integer such that $(\alpha \beta)^d(x) = x$, then on $Y_i$, $\beta \alpha$ is expressed as a product $\beta_i \alpha_i$ of two disjoint permutations $\alpha_i$ and $\beta_i$ of the same length:

$$\alpha_i = (x, \beta \alpha(x), (\beta \alpha)^2(x), \ldots, (\beta \alpha)^{d-1}(x))$$

and

$$\beta_i = (\alpha(x), \alpha \beta \alpha(x), \ldots, (\alpha \beta)^{d-1} \alpha(x)).$$

Furthermore the $\beta_i \alpha_i$ are pairwise disjoint and

$$\beta \alpha = (\beta \alpha_1 \cdots \beta \alpha_L)(\beta \alpha_{L+1} \cdots \beta \alpha_2)(\beta \alpha_1).$$

Moreover

$$|\beta \alpha| = |\beta \alpha_L| + \cdots + |\beta \alpha_1| = 2L.$$

Since two consecutive cycles in $\beta \alpha$ determine a simple closed curve (which is isotopic to $c_1 = \partial D_1$) on $\partial V_1$, we assume that $l$ is the simple closed curve determined by $\alpha$ and $\beta$ on $\partial V_1$ whenever $L = 1$. Let $K_1$ be a trivial arc in $V_1$ such that $K_1 \cap \partial V_1 = \partial K_1$, which is situated inside the bigons bounded by 2-cycles $(1,d)$ and $(-1,-d)$ as shown in Figure 1. Then a set of 4-tuples of integers

$$D = \{(a, b, c, r) | a > 0, b \geq 0, c \geq 0, \}$$

Figure 1: The Dunwoody $(1,1)$-decomposition $D(a, b, c, r)$
admits a $(1,1)$-decomposition of $(M,K)$ called the Dunwoody $(1,1)$-decomposition.

For each $(a,b,c,r)$ in $\mathcal{D}$, we denote the Dunwoody $(1,1)$-decomposition of $(M,K)$ by $D(a,b,c,r)$. (See Figure 1.) Moreover, we denote a $(1,1)$-knot $K$ represented by $D(a,b,c,r)$ by $K(a,b,c,r)$, and call it the Dunwoody $(1,1)$-knot. We note that every $(1,1)$-knot can be represented by the Dunwoody $(1,1)$-knot and vice versa[5]. The representation of a $(1,1)$-knot by Dunwoody $(1,1)$-decomposition is not unique. For example, both $K(1,3,4,7)$ and $K(2,1,4,4)$ represent the pretzel knot $P(-2,3,7)$ which is a $(1,1)$-knot as was mentioned in [19]. The subset of $\mathcal{D}$ representing all torus knots was determined by [11], [14] and [19]. However, the subset of $\mathcal{D}$ representing all torus knots is not yet determined completely. In [1], [5], [13] and [6] we have Dunwoody $(1,1)$-decompositions representing the certain class of torus knots.

We now construct a family of 3-manifolds which are the $n$-fold strongly-cyclic coverings branched over Dunwoody $(1,1)$-knots. Let $M$ be a lens space and $K$ be a Dunwoody $(1,1)$-knot in $M$. Then the $n$-fold cyclic covering of $M$ branched over $K$ is completely defined by an epimorphism $C : H_1(M-K) \to \mathbb{Z}_n$, where $\mathbb{Z}_n$ is the cyclic group of order $n$. Let $r_1$ be a generator of $\partial V_1$, which is the boundary of the meridian disk meeting with $K_1$ at one point and let $r_2$ be a generator of $\partial V_1$, which is the longitude curve meeting with $r_1$ at one point. Then every curve of $\partial V_1$ determined by two permutations $\alpha$ and $\beta$ is generated by $r_1$ and $r_2$. In other words, the orbit $l$ of $\beta \alpha$ is generated by $r_1$ and $r_2$. We define $l_i$ $(1 \leq i \leq 6)$ from the oriented curve $l$ on $D(a,b,c,r)$ as follows.

- $l_1$ is the number of left directed arrows from $m^+$ or $m^-$ to $m^+$ or $m^-$ in $a$ edges respectively.
- $l_2$ is the number of right directed arrows from $m^+$ or $m^-$ to $m^+$ or $m^-$ in $a$ edges respectively.
- $l_3$ is the number of arrows directed from $m^+$ to $m^-$ in $b$ edges.
- $l_4$ is the number of arrows directed from $m^-$ to $m^+$ in $b$ edges.
- $l_5$ is the number of arrows directed from $m^+$ to $m^-$ in $c$ edges.
- $l_6$ is the number of arrows directed from $m^-$ to $m^+$ in $c$ edges.

From now on for $D(a,b,c,r)$ we let $p = (l_1 + l_5) - (l_4 + l_6)$, $q = (l_1 + l_3) - (l_2 + l_4)$ and $d = 2a + b + c$. If $p = \pm 1$ or $p = 0$, then $M$ is $S^3$ or $S^1 \times S^2$([11] and [12]), respectively. Thus $p \neq 0$ if $M$ is not $S^1 \times S^2$. We have $\pi(M) = \langle x \rangle \times \mathbb{Z}$ and $H_1(M-K) = \langle r_1, r_2 | pr_2 + qr_1 \rangle = \mathbb{Z} \oplus \mathbb{Z}_{\gcd(p,q)}$. By definition, the $n$-fold cyclic covering $f$ of $M$ branched over $K$ is called strongly-cyclic if the branching index of $K$ is $n$. That is, the fiber $f^{-1}(x)$ of each point $x \in K$ contains a single point. Therefore the homology class of a meridean loop $r_1$ around $K$ is mapped by $C$ in a generator of $\mathbb{Z}_n$, say $C(r_1) = 1$, and so there exists an $n$-fold strongly-cyclic
covering space $\overline{M}$ of $M$ branched over $K$ if and only if there is $s = C(r_2) \in \mathbb{Z}_n$ such that $ps + q \equiv 0 \mod n$. We call the diagram in Figure 2 a Heegaard diagram of $\overline{M}$ and denote it by $D_n(a, b, c, r, s)$. If the Dunwoody $(1, 1)$-knot $K$ is in $S^3$, the strongly-cyclic branched covering is the same as cyclic branched covering. Indeed the $n$-fold cyclic branched covering of $K$ in $S^3$ always exists and is unique up to equivalence for $n > 1$ because $H_1(S^3 - K) = \mathbb{Z}$, the homology class $r_1$ is mapped by $C$ in a generator of $\mathbb{Z}_n$ and $s = C(r_2) = -q$.

We have proved the following.

**Theorem 2.2.** Let $D(a, b, c, r)$ be the Dunwoody $(1, 1)$-decomposition of $(M, K)$ and $n > 1$. Then $\overline{M}$ is homeomorphic to a 3-manifold if and only if there is an integer $s$ such that $ps + q \equiv 0 \mod n$.

We notice that $D_n(a, b, c, r, s)$ satisfies the conditions for the Heegaard diagram of the Dunwoody 3-manifold considered in [10]. Thus we have the following.

**Corollary 2.3.** Let $M$ be a lens space and $K$ be a Dunwoody $(1, 1)$-knot in $M$. Then the $n$-fold strongly-cyclic covering space $\overline{M}$ of $M$ branched over $K$ is homeomorphic to the Dunwoody 3-manifold.

**Corollary 2.4.** [12] $D(a, b, c, r)$ is the $(1, 1)$-decomposition of $(S^3, K)$ if and only if $|p| = 1$.

From the result of Corollary , if $d = 2a + b + c$ is even, then $D(a, b, c, r)$ cannot be a $(1, 1)$-decomposition of $(S^3, K)$ because $d$ has the same parity of $p$. (See [15] for detail.)

Generally, for the following set

$$S = \{(a, b, c, r) | a > 0, b \geq 0, c \geq 0, \quad d = 2a + b + c, r \in \mathbb{Z}_d, |\alpha\beta| = 2L\},$$
we suppose that $L \geq 2$ is the number of simple closed curves determined by $\alpha$ and $\beta$ on $\partial V_1$. Given an $(a, b, c, r) \in S$, it is possible to represent a link in lens spaces containing $S^3$ and $S^1 \times S^2$. Thus the orientable 3-manifold $M$ in existing is a generalization of the Dunwoody 3-manifold introduced in [10], called the generalized Dunwoody 3-manifold. (See [15], [13] or [7] for some examples.)

We let $L = 1$. That is, for each $(a, b, c, r) \in S$, $K(a, b, c, r)$ is the Dunwoody $(1, 1)$-knot in a lens space or $S^3$. We now consider the Dunwoody $(1, 1)$-knot representing the torus knot. The torus knot is a knot embedded in the standard torus $T$ in $S^3$. Regarding $T$ as the boundary of tubular neighborhood of trivial knot in $S^3$, we take a meridian-longitude system $(m_j)$ where

\[(2.1)\]

\[
K(a, b, c, r) = \begin{cases} \frac{k_j - 1}{2} \\ b = 1 \\ c = \frac{(k_j + 1)(k_j - 1)(k_j - 2)}{2k_j} \\ r = \frac{-k_j + (k_j - 1)^2k_j^2 - k_j^2 + 2 - (k_j)^3}{2k_j}, \end{cases}
\]

and (ii) $T(k_1, k_2)$ with $k_2 \equiv -2 \mod k_1$ is represented by $K(a, b, c, r)$ where

\[(2.2)\]

\[
\begin{cases} a = \frac{k_j - 1}{2} \\ b = 1 \\ c = \frac{(k_j)^2k_j - 2(k_j)^2 - k_j^2 - 2}{2k_j} \\ r = \frac{1}{2}(k_j)^2 - \frac{3}{2}. \end{cases}
\]

In the following theorem the conditions for $K(a, b, c, r)$ to represent a torus knot will be given where $|X_1 \cap X_2|$ means the number of intersecting points between two sets $X_1$ and $X_2$.

**Theorem 2.5.** Let $D(a, b, c, r)$ be the Dunwoody $(1, 1)$-decomposition of $(S^3, K)$. Suppose that $m$ is the meridian disk determined by $\beta$ and $l$ is the simple closed curve defined by $\beta \alpha$ such that $|K_1 \cap K_2| = 2$, $|K_1 \cap l| = k_1$, and $|K_2 \cap m| = k_2$, for some coprime integers $k_1$ and $k_2$. Then $K(a, b, c, r)$ is $T(k_1, k_2)$, where $k_1 = 2a + b$ and $k_2 \leq c + 2$.

**Proof.** There exists a Heegaard splitting of genus one $(V_1, K_1) \cup \phi(V_2, K_2)$ of $(S^3, K)$,
where $V_1$ and $V_2$ are solid tori, $K_1 \subset V_1$ and $K_2 \subset V_2$ are properly embedded trivial arcs, and $\phi : (\partial V_2, \partial K_2) \rightarrow (\partial V_1, \partial K_1)$ is an attaching homeomorphism. Since $|K_1 \cap K_2| = 2$, $K_1$ and $K_2$ do not meet each other except the bigons determined by the 2-cycles $(1, d)$ and $(-1, -d)$. Thus the meridian-longitude system $(m, l)$ satisfies $|K_1 \cap l| = 2a + b$ and $|K_2 \cap m| \leq c + 2$. Let $k_1 = 2a + b$ and $k_2 = |K_2 \cap m|$ be integers satisfying $\gcd(k_1, k_2) = 1$. Then $K = K_1 \cup_\phi K_2$ is homologous to $k_1 m + k_2 l$ in $V_1$. Therefore $K(a, b, c, r)$ is $T(k_1, k_2)$.

The inequality $k_2 \leq c + 2$ in Theorem 2.5 is the generalization of Theorem 4.2(iii) in [8]. That is, $K(1, 0, 2k - 1, 2)$ is equivalent to $T(2k + 1, 2)$. We also note that the Dunwoody $(1, 1)$-decomposition $D(a, b, c, r)$ representing $T(k_1, k_2)$ with $k_2 \equiv \pm 2 \mod k_1$ satisfies the conditions in Theorem 2.5.

**Example 1.** Let $D(1, 2, 3, 3)$ be a $(1, 1)$-decomposition of $(S^3, K) = (V_1, K_1) \cup_\phi (V_2, K_2)$ and $|[(l_3 + l_4) - (l_3 + l_4)]| = 1$. (See Figure 3.) Then $|K_1 \cap l| = 4$, $|K_2 \cap m| = 5$, and $|K_1 \cap K_2| = 2$, which are marked by circled numbers, numbers and dots respectively in Figure 3. Thus $D(1, 2, 3, 3)$ satisfies the conditions of Theorem 2.5 and so $K(1, 2, 3, 3)$ is $T(4, 5)$.

![Figure 3: A $(1, 1)$-decomposition $D(1, 2, 3, 3)$ of $T(4, 5)$](image)

3. The Alexander polynomial vs the Dunwoody polynomial

In this section, we show that (i) the certain polynomial of the Dunwoody $(1, 1)$-knot in $S^3$ is the Alexander polynomial, and that (ii) if $K(a, b, c, r)$ is the Dunwoody $(1, 1)$-knot representing 2-bridge knot or some torus knots, then the number $d = 2a + b + c$ is an invariant for $K(a, b, c, r)$. For the Dunwoody polynomial, (i) gives an answer to a question in [10].

**Theorem 3.1.** The $n$-fold strongly-cyclic branched covering of the Dunwoody $(1, 1)$-knot in a lens space admits a cyclically presented fundamental group.

**Proof.** When we consider the Dunwoody $(1, 1)$-knot $K(a, b, c, r)$ in a lens space, $\beta \alpha$ has two cycles of length $d$ such that $(\beta \alpha)^d(x) = x$ for each $x$ on $D(a, b, c, r)$ by Theorem 2.1. Thus the $n$-fold strongly-cyclic branched covering of $K(a, b, c, r)$ is homeomorphic to the Dunwoody 3-manifold $D_n(a, b, c, r, s)$. Since $ps + q \equiv 0 \mod n$, $n$,
the path corresponding to this cycle connects the endpoint labelled 1 in the hole labelled 0 to the endpoint labelled 1 in the hole labelled \( ps + q \) under mod \( n \). That is, the condition \( ps + q \equiv 0 \) mod \( n \) ensures that the path corresponding to the cycles is a simple closed curve with an orientation. Since \( D_n(a, b, c, r, s) \) has \( n \) simple closed curves, each path starting at the endpoint labelled 1 in the hole labelled \( i \) corresponding to the cycles of \( \beta \alpha \) will be connected to the endpoint labelled 1 in the hole labelled 7 under mod \( n \). With notations in [13], \( w(C_i) \) (resp. \( w(C_i') \)) is a cyclic presentation obtained by reading off simple closed curves around the hole labelled \( i \) (resp. \( i' \)). Thus the identification of \( C_i \) and \( C_i' \) by \( r \) on \( D_n(a, b, c, r, s) \) induces \( w(C_i) \approx_r w(C_i') \). If \( i = 0 \), then

\[
\eta^s(c)\eta^{s-1}(b)\eta_1^{-1}(u^{-1}) \approx_r abc^{-1}(a^{-1}),
\]

from which we have a cyclic presentation for the fundamental group. \( \square \)

For the specific example, let the Dunwoody \((1, 1)\)-knot \( K(a, b, c, r) \) represent \( T(p, q) \) such that \( p \) is odd and \( q \equiv \pm 2 \) mod \( p \). Then the \( n \)-fold cyclic covering of \( S^3 \) branched over \( K(a, b, c, r) \) satisfies \( \eta^s(c) \eta^{s-1}(b) \eta_1^{-1}(u^{-1}) \approx_r abc^{-1}(a^{-1}) \) (for reference see [13]), where the parameter \( s \) is equal to \(-s \) in [13].

From [3], [9] and [20], we recall the definition of the Alexander polynomial of a knot in compact connected 3-manifold. We also note that every finitely generated abelian group \( G \) is a direct sum of a torsion-free part \( F \) and a torsion-part \( T(G) \). For the group \( G \), we denote its integral group ring by \( \mathbb{Z}[G] \). In particular, the first homology group \( H_1(N) \) of a compact connected 3-manifold \( N \) has a decomposition

\[
H_1(N) \cong F(H_1(N)) \oplus T(H_1(N)).
\]

The projection \( J : H_1(N) \rightarrow H_1(N)/T(H_1(N)) \) induces the ring homomorphism \( J' : \mathbb{Z}[H_1(N)] \rightarrow \mathbb{Z}[H_1(N)/T(H_1(N))] \). If \( k \) is the first Betti number of \( N \) and \( t_1, \cdots, t_k \) are generators of \( H_1(N)/T(H_1(N)) \), then we have

\[
\mathbb{Z}[H_1(N)/T(H_1(N))] \cong \mathbb{Z}[t_1, t_1^{-1}, \cdots, t_k, t_k^{-1}].
\]

Let \( h : \pi_1(N, *) \rightarrow H_1(N) \) be the Hurewitz homomorphism, where \( * \) is a fixed point in \( N \). Denote \( E_1(N) \subset \mathbb{Z}[H_1(N)] \) and \( E'_1(N) \) with the first elementary ideal of \( \pi_1(N, *) \) and the smallest principal ideal of \( \mathbb{Z}[H_1(N)/T(H_1(N))] \) containing \( J'(E_1(N)) \), respectively. The generator \( \triangle_N \) of \( E'_1(N) \) is well-defined up to multiplication by units of \( \mathbb{Z}[t_1, t_1^{-1}, \cdots, t_k, t_k^{-1}] \) and is said to be the Alexander polynomial of \( N \). Let \( K \) be a knot in a compact connected 3-manifold \( M \). Then the Alexander polynomial \( \triangle_N \) of \( N = M - K \) is the Alexander polynomial of \( K \) and it will be denoted by \( \triangle_K \) instead of \( \triangle_N \) for a knot \( K \).

Let \( R \) be a unital commutative ring and let

\[
G_\alpha \cong \langle x_0, x_1, \cdots, x_{n-1} | r_0, \cdots, r_{n-1} \rangle
\]
be a finitely-presented $R$-module, where each relation $r_i$ is a linear combination of the generators $x_j$: $r_i = a_{ij}x_0 + \cdots + a_{i(n-1)}x_{n-1}$. In other words, $G_n$ is generated as an $R$ module by the elements $x_0, \cdots, x_{n-1}$, and $r_0 = 0, \cdots, r_{n-1} = 0$ are relations among the $x_j$’s. Then we can define a presentation matrix to be an $n \times n$ matrix with entries $a_{ij}$ for $0 \leq i \leq n - 1, 0 \leq j \leq n - 1$. An Alexander matrix is a presentation matrix for $H_1(X)$ as a $\mathbb{Z}[t, t^{-1}]$ module, where $X$ is the infinite cyclic cover of the knot complement $N$. The ideal generated by the Alexander matrix is the Alexander ideal of the knot, so the Alexander ideal is principal([18], P.207). Any generator of this principal ideal is the Alexander polynomial $\Delta_K$ for a knot $K$. In fact, it was discovered by Alexander [2] in the 1920s, early in the history of topology, using the homology of the infinite cyclic cover of a knot complement.

In this section, let $M$ be a lens space and denoted by $L(p, q')$, where $p$ and $q'$ are relatively prime. Considering the Dunwoody $(1,1)$-knot $K$ in $M$, we have $F(H_1(N)) \cong \mathbb{Z}$ and $T(H_1(N)) \cong \mathbb{Z}_{gcd(p, q')}$. The Dunwoody polynomial $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ is the Alexander polynomial of $K$. In particular, for $K$ in $\mathbb{S}^3$, $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ has the properties (i) $\Delta_K(t) = \Delta_K(t^{-1})$ and (ii) $\Delta_K(1) = \pm 1$. Vice verse, every polynomial in $\mathbb{Z}[t, t^{-1}]$ satisfying (i) and (ii) is the Alexander polynomial of a knot in $\mathbb{S}^3$. However, by Theorem B in [20], for the Alexander polynomial of $K$ in $M$, it is true for the condition (i), but the condition (ii) is no longer true. See also [21].

Now we introduce the Dunwoody polynomial of the Dunwoody $(1,1)$-knot $K = K(a, b, c, r)$, and study the connections between the Dunwoody polynomial and the Alexander polynomial for $K$. Let $n > 1$. Then the $n$-fold strongly-cyclic branched covering $D_n(a, b, c, r, s)$ of $K$ in $M$ admits a cyclically presented fundamental group by Corollary 2.3 and Theorem 3.1. Due to the cyclic symmetry of $D_n(a, b, c, r, s)$, the fundamental group has the cyclic presentation induced by a single word $w(x_0, x_1, \cdots, x_{n-1})$ as following:

$$G_n(w(x_0, x_1, \cdots, x_{n-1})) = \langle x_0, x_1, \cdots, x_{n-1} \rangle$$

$$\theta^j(w(x_0, x_1, \cdots, x_{n-1})), 0 \leq j \leq n - 1$$

where $\theta$ is the automorphism on the free group $F_n = \langle x_0, \cdots, x_{n-1} \rangle$ of rank $n$ defined by $\theta(x_i) = x_{i+1}$ and all indices are taken under mod $n$. Since $\theta$ is an automorphism of order $n$, the relations

$$\{\theta^j(w(x_0, x_1, \cdots, x_{n-1})), 0 \leq j \leq n - 1\}$$

are independent of $j$ with $0 \leq j \leq n - 1$, that is, for any $0 \leq j \leq n - 1$,

$$G_n(w(x_0, x_1, \cdots, x_{n-1})) \cong G_n(\theta^j(w(x_0, x_1, \cdots, x_{n-1})))$$

The relations $\theta^j(w(x_0, x_1, \cdots, x_{n-1})), 0 \leq j \leq n - 1$, are determined by $n$ disjoint simple closed curves on $D_n(a, b, c, r, s)$. For a relation $w(x_0, x_1, \cdots, x_{n-1}) \in$
{θj(w(x0, x1, · · · , xn−1))|0 ≤ j ≤ n − 1}, the relation w(x0, x1, · · · , xn−1) is said to be principal if all indices in w(x0, x1, · · · , xn−1) are independent of n. For the cyclic presentation $G_n(w(x_0, x_1, · · · , x_{n−1}))$ by $w(x_0, x_1, · · · , x_{n−1})$, we obtain the abelianized word $\sum_{i=0}^{k} a_i x_i$, $a_i \in \mathbb{Z}$, of w and a polynomial $f_n(t) = \sum_{i=0}^{k} a_i t^i \in \mathbb{Z}[t, t^{-1}]$ obtained by substituting $t^i$ into $x_i$ is called the Dunwoody polynomial determined by $D_n(a, b, c, r, s)$. Moreover, by the multiplications of $± t^j$ ($j \in \mathbb{Z}$), we can normalize $f_n(t) \in \mathbb{Z}[t, t^{-1}]$ in order to have the polynomial with a positive constant term and positive exponents in $\mathbb{Z}[t]$. Let $f_n(t) \in \mathbb{Z}[t]$ and $n > 1$. Then $f_n(t)w(w(0, j, \cdots , n−1)) = \sum_{i=0}^{k} a_i t^i$ is the Dunwoody polynomial determined by $D_n(a, b, c, r, s)$).

Thus $D_n(a, b, c, r, s)$ admits the principal cyclic presentation $G_n(w)$ if w is principal.

**Example 2.** For $n > 4$, $π(D_n(3, 1, 2, 2, −1))$ has a cyclic presentation induced by

$$w(x_0, x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4^{-1} x_2 x_2^{-1} x_1 x_0^{-1}.$$ 

Thus $π(D_n(3, 1, 2, 2, −1))$ is presented by

$$\langle x_0, x_1, \cdots , x_{n−1}|θ(w(x_1 x_2 x_3 x_4^{-1} x_2 x_2^{-1} x_1 x_0^{-1}),$$

$$0 \leq j \leq n − 1)\rangle.$$ 

Thus $f_n(t) = 1 − 2t + t^2 − 2t^3 + t^4$ is the Dunwoody polynomial determined by $D_n(3, 1, 2, 2, −1)$. Since

$$w(x_0, x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4^{-1} x_2 x_2^{-1} x_1 x_0^{-1}$$ 

is independent of $n(>4)$, it is a principal relation. Indeed, the Dunwoody (1, 1)-knot $K(3, 1, 2, 2)$ represents the knot 942 in the knot table classified by Rolfsen. It is interesting to note that the Dunwoody polynomial $f_n(t)$ for $n > 4$ is the Alexander polynomial of 942.

For the Dunwoody (1, 1)-knot $K = K(a, b, c, r)$ in $S^3$, the following shows that $f_n(t)$ is to be the Alexander polynomial of $K$.

• If $w_k = w(x_0, x_1, \cdots , x_k)$ is the principal relation of the Dunwoody 3-manifold represented by $D_n(a, b, c, r, s)$ for some $n$, then $f_n(t)$ is the Alexander polynomial of $K$ in $S^3$.

In particular, the Dunwoody (1, 1)-knot $K = K(a, b, c, r)$ defined in (2.1) and (2.2) for $K = T(k, hk ± 2)$ with $h, k > 0$ induces the Dunwoody 3-manifold represented by $D_n(a, b, c, r, s)$ for some $s$. Using the principal relation, say $w$, of $D_n(a, b, c, r, s)$, we can obtain the Dunwoody polynomial $f_n(t)$ such that $f_n(t) \cong \Delta_K(t)$. For example, let $K$ be a torus knot $T(5, 7)$ satisfying $7 \equiv 2 \text{ mod } 5$. On $D_n(1, 1, 12, 23, −5)$ with $n > 29$, we can obtain a relation $w$ as the following

$$w = x_0^{-1} x_1^{-1} x_6^{-1} x_9^{-1} x_{16}^{-1} x_{17}^{-1} x_{12}^{-1} x_7^{-1} x_8^{-1} x_9^{-1} x_{13}^{-1} x_{18}^{-1} x_{23}^{-1} x_{24}^{-1}$$
Then the relation \( w \) is principal in \( D_n(2,1,12,13,-5) \) for condition \( n > 29 \). Thus it was shown in [13] that \( f_w(t) \) is the Alexander polynomial of \( K \).

For each knot \( K \) in \( S^3 \), we denote the projection of \( \triangle_K(t) \) into \( \mathbb{Z}[t]/(t^n-1) \) with \( \triangle_K^n(t) \). The following corollary shows the connections between the projection of Alexander polynomial into \( \mathbb{Z}[t]/(t^n-1) \) and the Dunwoody polynomial for the Dunwoody \((1,1)\)-knot in \( S^3 \).

**Corollary 3.2([4]).** Let \( D(a,b,c,r) \) be the Dunwoody \((1,1)\)-decomposition of \((S^3,K)\) and \( n > 1 \). Then \( f_w^n(t) = \triangle_K^n(t) \) in \( \mathbb{Z}[t]/(t^n-1) \), where \( \cong \) means equal up to units.

The following corollary is required the degree of the Alexander polynomial in order to obtain the Dunwoody polynomial.

**Corollary 3.3([4]).** Let \( K \) be the Dunwoody \((1,1)\)-knot representing \( T(k,hk \pm 1) \) with \( h,k > 0 \) and \( n > 1 \). Suppose that \( f_w^n(t) \) is the Dunwoody polynomial associated to the cyclic presentation obtained by applying Theorem 7 in [4]. Then \( f_w^n(t) = \triangle_K(t) \) in \( \mathbb{Z}[t]/(t^n-1) \) if \( n > \deg(\triangle_K(t)) \), where \( \cong \) means equal up to units.

For example, no more the result of Corollary 3.3 is true for \( T(5,7) \) and \( \deg(\triangle_K(t)) = 24 \). In fact, for \( n = 25 \), the relation \( w \) is not principal because \( x_{23} \) and \( x_{24}^{-1} \) in \( w \) are equal to \( x_{-2} \) and \( x_{-1}^{-1} \) under \( D_{25}(2,1,12,13,-5) \), respectively, that is, \( w \) is not independent of \( 25 \). Thus \( w \) is equal to the relation

\[
x_0^{-1}x_{1}^{-1}x_6^{-1}x_9^{-1}x_1^{-1}x_7^{-1}x_{13}^{-1}x_{18}^{-1}x_{19}^{-1}x_{20}^{-1}x_{14}^{-1}x_{10}^{-1}
\]

on \( D_{25}(2,1,12,13,-5) \). In case of Corollary 3.3, we have to know the degree of \( \triangle_K(t) \) in order to show that \( f_w^n(t) \) is to be the Alexander polynomial of \( K \). However, without the condition for the degree of \( \triangle_K(t) \), we can show that \( f_w^n(t) \) is to be the Alexander polynomial of the torus knot \( K \) (generally, the Dunwoody \((1,1)\)-knot in \( S^3 \)) from properties itself.

As the main result of this section, we show that the Dunwoody polynomial is to be the Alexander polynomial. In other words we give the condition for \( n \) in order that \( D_n(a,b,c,r,s) \) admits always the principal cyclic presentation.

Given \( p \) and \( q \) defined on \( D(a,b,c,r) \) which is the Dunwoody \((1,1)\)-decomposition determined by two permutations \( \alpha \) and \( \beta \) such that \( |\beta \alpha| = 2 \), we recall that \( D_n(a,b,c,r,s) \) satisfies \( ps + q \equiv 0 \) mod \( n \) for some \( n > 1 \) and \( s \in \mathbb{Z} \). First of all, we define a cyclic sequence from \( D_n(a,b,c,r,s) \) as follows. We set

\[
A^+ = \{1,2,\cdots,a\},
\]

\[
B^+ = \{a+1,a+2,\cdots,a+b\},
\]

\[
C^+ = \{a+b+1,a+b+2,\cdots,a+b+c\},
\]
\[ E^+ = \{a + b + c + 1, a + b + c + 2, \ldots, a + b + c + d = a\}, \]
\[ A^- = \{-1, -2, \ldots, -a\}, \]
\[ C^- = \{-a - 1, -a - 2, \ldots, -a - c\}, \]
\[ B^- = \{-a - c - 1, -a - c - 2, \ldots, -a - c - b\}, \quad \text{and} \]
\[ E^- = \{-a - c - b - 1, -a - c - b - 2, \ldots, -a - c - b - a = d\}. \]

Then \( A^+ \cup B^+ \cup C^+ \cup E^+ = X^+ \) and \( A^- \cup C^- \cup B^- \cup E^- = X^- \). For each \( 0 \leq i \leq n - 1 \), let \( C_i \) be the \( i \)-th meridian disk of the Heegaard diagram \( D_n(a, b, c, r, s) \) as in Figure 2, and \( C_j \) the \( i \)-th meridian disk of the Heegaard diagram \( D_n(a, b, c, r, s) \).

For \( 0 \leq i \leq n - 1 \) and \( 1 \leq j \leq d \), a point \((i, j)\) on \( D_n(a, b, c, r, s) \) means the number \( j \) in \( i \)-th meridian disk \( C_i \), and a point \((i, -j)\) means the number \(-j\) in \( i \)-th meridian disk \( C_i \). So if \((i, j) \in D_n(a, b, c, r, s)\) is a starting point at \( C_i \), then \( \theta^{n-1}(i, j) = (i + n - 1, j) \) is a starting point at \( \theta^{n-1}(C_i) \). We define the rules corresponding to \( i \) on \( D_n(a, b, c, r, s) \) and \( a \) on \( D(a, b, c, r) \) by

\[
\begin{align*}
(i, a) & \to (i + 1, a(a)) \quad \text{if} \quad a \in A^+ \\
(i, b) & \to (i + s + 1, a(b)) \quad \text{if} \quad b \in B^+ \\
(i, c) & \to (i + s, a(c)) \quad \text{if} \quad c \in C^+ \\
(i, e) & \to (i - 1, a(e)) \quad \text{if} \quad e \in E^+
\end{align*}
\]

The rules corresponding to \( \tilde{i} \) on \( D_n(a, b, c, r, s) \) and \( \alpha \) on \( D(a, b, c, r) \) are defined by

\[
\begin{align*}
(\tilde{i}, -a) & \to (\tilde{i} + 1, a(-a)) \quad \text{if} \quad -a \in A^- \\
(\tilde{i}, -c) & \to (\tilde{i} - s, a(-c)) \quad \text{if} \quad -c \in C^- \\
(\tilde{i}, -b) & \to (\tilde{i} - (s + 1), a(-b)) \quad \text{if} \quad -b \in B^- \\
(\tilde{i}, -e) & \to (\tilde{i} - 1, a(-e)) \quad \text{if} \quad -e \in E^{-}
\end{align*}
\]

Moreover, the identification between \( i \)-th meridian disk \( C_i \) and \( \tilde{i} \)-th meridian disk \( \tilde{C}_i \) on \( D_n(a, b, c, r, s) \) is defined by

\[
\begin{align*}
(i, x) & \to (\tilde{i}, \beta(x)) \quad \text{if} \quad x \in X^+ \\
(\tilde{i}, -x) & \to (i, \beta(-x)) \quad \text{if} \quad -x \in X^-. 
\end{align*}
\]

By the property of the \( n \)-fold strongly-cyclic branched covering space, we have the following.

**Lemma 3.4.** Let \((0, 1)\) be a starting point on \( D_n(a, b, c, r, s) \) and \( d = 2a + b + c \). Then we have \((\beta \alpha)^d(0, 1) = (ps + q, 1)\).

**Proof.** From \( \alpha \) and \( \beta \) defined in Theorem 2.1, \( \beta \alpha \) with length \( d \) determines the simple closed curve in \( D(a, b, c, r) \) with the starting point \((0, 1)\). The simple closed curve is lifted to \( n \) simple closed curves on \( D_n(a, b, c, r, s) \) which is determined by (3.1), (3.2) and (3.3). For \( 0 \leq i \leq n - 1 \) and \( 1 \leq j \leq d \), if \((i, j) \in D_n(a, b, c, r, s)\) is
a starting point of a curve of the $n$ simple closed curves on $D_n(a, b, c, r, s)$, we have $(\beta \alpha)^2(i, j) = (ps + q + i, j)$ because of $ps + q \equiv 0 \mod n$. In particular, let $(0, 1)$ be a starting point on $D_n(a, b, c, r, s)$. Then the proof follows from the above result. □

For $0 \leq i \leq n - 1$ and $1 \leq j \leq d$, if $(i, j) \in D_n(a, b, c, r, s)$ is a starting point, we have a sequence

$$(i, j) \to (\beta \alpha)(i, j) \to (\beta \alpha)^2(i, j) \cdots \to (\beta \alpha)^{d-1}(i, j) \to (\beta \alpha)^d(i, j) = (ps + q + i, j).$$

The sequence from $(i, j)$ to $(ps + q + i, j)$ determined by (3.1), (3.2) and (3.3) is called a cyclic sequence of $D_n(a, b, c, r, s)$. We note that the cyclic sequences of $D_n(a, b, c, r, s)$ are independent of the choice of the starting points on itself.

We now suppose that $(0, 1)$ is a starting point which is the number 1 in 0-th meridian disk of $D_n(a, b, c, r, s)$. Since $1 \in A^+$ and $a(1) = d$, we have $(0, 1) \to (1, d)$ by (3.1). Since $r < d$, $\beta(d) = -d + r$ and so $(1, d) \to (1, -d + r)$ by (3.3). Thus $(0, 1) \to (1, -d + r)$ under $\beta \alpha$, or $\alpha(0, 1) = (1, -d + r)$. Applying repeated process, we obtain $(\beta \alpha)^2(0, 1) = (ps + q, 1)$ by Lemma 3.4. We define a relation

$$w = x_0\frac{x_0^+}{x_0^-}x_0^\pm_1x_0^\pm_2x_0^\pm_3\cdots x_0^\pm_{d-1}$$

on $D_n(a, b, c, r, s)$ induced by the cyclic sequence of $D_n(a, b, c, r, s)$, where

$$x_0^\pm_i = \begin{cases}
  x_0((\beta \alpha)^k(0)) & \text{if } (\beta \alpha)^k(0) = i \\
  x_0^-(\beta \alpha)^k(0) & \text{if } (\beta \alpha)^k(0) = i.
\end{cases}$$

If $(\beta \alpha)^k(0) = -i$ and $(\beta \alpha)^k(0) = i$, then $x_0^\pm_i(\beta \alpha)^k(0) = x_0^\mp_i$ and $x_0^\pm_i(\beta \alpha)^k(0) = x_0^\mp_i$, respectively. We can assume that $(\beta \alpha)^k(0) \geq 0$ under mod $n$ for all $1 \leq k \leq d$. For each $0 \leq i \leq n - 1$, we note that $\partial^i(w)$ has the starting point $(i, 1)$ and the final point $(ps + q + i, 1)$ with $(\beta \alpha)^d(i, 1) = (ps + q + i, 1)$. Thus the abelianized word of $w$ induces the Dunwoody polynomial

$$f_w^n(t) = t^0 \pm t^{(\beta \alpha)^2(0)} \pm \cdots \pm t^{(\beta \alpha)^{d-1}(0)}$$

by substituting $-t^i$ into $x_0^\mp_i$. If $|(\beta \alpha)^k(0)| < n$ for each $1 \leq k \leq d - 1$, then $w$ is the principal relation of the cyclic presentation of $D_n(a, b, c, r, s)$ and we have $\deg f_w^n(t) = M^+ - M^-$ where $M^+ = \max_{0 \leq k \leq d-1} \{(|\beta \alpha|^k(0))\}$ and $M^- = \min_{0 \leq k \leq d-1} \{(|\beta \alpha|^k(0))\}$.

Suppose that $D(a, b, c, r)$ is the Dunwoody $(1, 1)$-decomposition representing a $(1,1)$-knot $K$ in $S^3$. Since $p = \pm 1$ and $ps + q = 0$, $s = \mp q$. For each $n > 1$, there exists the Dunwoody 3-manifold represented by $D_n(a, b, c, r, s)$, which is the $n$-fold (strongly-) cyclic covering of $S^3$ branched over $K$. By Lemma 3.4, the cyclic sequence of $D_n(a, b, c, r, s)$ has $(\beta \alpha)^d(0, 1) = (0, 1)$. Since each $\{C_i, \hat{C}_{i+s}\}$
On the Polynomial of the Dunwoody (1, 1)-knots

in $D_n(a, b, c, r, s)$ is connected by $c$ edges, the relation $w$ is independent of $n$ if $n > M^+ - M^- + |s|$. Therefore $f_n^w(t)$ is the Alexander polynomial of the Dunwoody (1, 1)-knot $K$ in $S^3$ if $n > M^+ - M^- + |s|$. We remark that there is a natural way to obtain the Dunwoody polynomial $f_n^w(t) \in \mathbb{Z}[t]/(t^n - 1)$ associated to the Dunwoody (1, 1)-knot $K$ in $S^3$ because of the uniqueness of $s$ in $\mathbb{Z}_n$ with $ps + q \equiv 0 \mod n$.

Summarizing, we have proved the following.

**Theorem 3.5.** Let $D(a, b, c, r)$ be the Dunwoody (1, 1)-decomposition of $(S^3, K)$, and $\alpha$ and $\beta$ be two transpositions defined in Theorem 2.1. Let $w(x_0, \ldots, x_{n-1})$ be a relation induced by the cyclic sequence of $D_n(a, b, c, r, \pm q)$ for $n > 1$, $M^+ = \max_{0 \leq k \leq d - 1} \{(\beta \alpha)^k(0)\}$ and $M^- = \min_{0 \leq k \leq d - 1} \{(\beta \alpha)^k(0)\}$. Then $f_n^w(t)$ is the Alexander polynomial of $K$ if $n > M^+ - M^- + |s|$.

We give two canonical examples as follows.

**Example 3.**

\[
\begin{array}{ccc}
(0, 1) & \xrightarrow{a} & (1, 9) \\
(1, -4) & \xrightarrow{\beta} & (8, 7) \\
(8, -2) & \xrightarrow{a} & (9, -8) \\
(9, 4) & \xrightarrow{\beta} & (3, -6) \\
(3, 2) & \xrightarrow{a} & (4, 8) \\
(4, -3) & \xrightarrow{\beta} & (11, 6) \\
(11, -1) & \xrightarrow{a} & (12, -9) \\
(12, 5) & \xrightarrow{\beta} & (6, -7) \\
(6, 3) & \xrightarrow{a} & (0, -5) \\
(0, 1). 
\end{array}
\]

Let $D(2, 3, 2, 5)$ be a Dunwoody (1, 1)-decomposition. Then we obtain $p = 1$ and $q = 7$ from an oriented curve on $D(2, 3, 2, 5)$ defined in section 2. Since $p = 1$, it is representing a Dunwoody (1,1)-knot $K(2, 3, 2, 5)$ in $S^3$. Since $q = 7$, we have $s = -7$. Therefore, for all $n > 1$, there exists a Dunwoody 3-manifold represented by $D_n(2, 3, 2, 5, -7)$. In order to show a principal relation on $D_n(2, 3, 2, 5, -7)$ we need a cyclic sequence as above. For a Dunwoody (1, 1)-decomposition $D(2, 3, 2, 5)$, setting $A^+ = \{1, 2\}$, $B^+ = \{3, 4, 5\}$, $C^+ = \{6, 7\}$, $E^+ = \{8, 9\}$, and $A^- = \{-1, -2\}$, $C^- = \{-3, -4\}$, $B^- = \{-5, -6, -7\}$, and $E^- = \{-8, -9\}$, then we have $A^+ \cup B^+ \cup$
$C^+ \cup E^+ = X^+$ and $A^- \cup C^- \cup B^- \cup E^- = X^-$. Let $(0, 1)$ be the starting point on $D_n(2, 3, 2, 5, -7)$. Then we have a cyclic sequence by applying (3.1), (3.2) and (3.3) as above. Thus the relation $w$ induced by the above cyclic sequence is

$$w = x_0 x_1^{-1} x_8^{-1} x_9 x_3 x_4^{-1} x_{11}^{-1} x_{12} x_6$$

and the Dunwoody polynomial $f_w^n(t)$ is

$$f_w^n(t) = 1 - t + t^3 - t^4 + t^6 - t^8 + t^9 - t^{11} + t^{12}.$$  

Since $M^+ = 12$ and $M^- = 0$, the relation $w$ on $D_n(2, 3, 2, 5, -7)$ is principal for $n > 19$. In fact, $f_w^n(t)$ is the Alexander polynomial of $K(2, 3, 2, 5)$ representing $T(3, 7)$, which can be obtained by considering the principal cyclic presentation of $D_{20}(2, 3, 2, 5, -7)$.

**Example 4.**

$$
\begin{align*}
(0, \ 1) & \xrightarrow{\alpha} (1, \ 9) \\
(1, \ -3) & \xrightarrow{\beta} (-2, \ 4) \\
(-2, \ -7) & \xrightarrow{\beta} (3, \ 1) \\
(-6, \ -6) & \xrightarrow{\beta} (9, \ 7) \\
(-9, \ -1) & \xrightarrow{\beta} (-8, \ -9) \\
(-8, \ 6) & \xrightarrow{\beta} (-3, \ 5) \\
(-5, \ 2) & \xrightarrow{\beta} (-4, \ 8) \\
(-4, \ -2) & \xrightarrow{\beta} (-3, \ -8) \\
(-3, \ 5) & \xrightarrow{\beta} (0, \ -4) \\
(0, \ 1) & \xrightarrow{\beta}
\end{align*}
$$

Let $D(2, 1, 4, 6)$ be a Dunwoody $(1, 1)$-decomposition with $p = -1$ and $q = 3$. Since $ps + q = 0$, there exists a Dunwoody 3-manifold represented by $D_n(2, 1, 4, 6, 3)$ for all $n > 1$. To obtain a principal relation for $D_n(2, 1, 4, 6, 3)$, we need a cyclic sequence as above. For $D(2, 1, 4, 6)$, setting $A^+ = \{1, 2\}$, $B^+ = \{3\}$, $C^+ = \{4, 5, 6, 7\}$, $E^+ = \{8, 9\}$, and $A^- = \{-1, -2\}$, $C^- = \{-3, -4, -5, -6\}$, $B^- = \{-7\}$, and $E^- = \{-8, -9\}$, then we have $A^+ \cup B^+ \cup C^+ \cup E^+ = X^+$ and $A^- \cup C^- \cup B^- \cup E^- = X^-$. By applying (3.1), (3.2) and (3.3), we have a cyclic sequence as above. Thus we have a relation

$$w = x_1 x_{-2} x_{-6} x_{-9} x_{-5}^{-1} x_{-4}^{-1} x_{-3}^{-1} x_0^{-1}.$$
Hence the Dunwoody polynomial is
\[ f_w^n(t) = t^{-9}(1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10}). \]

Since \( M^+ = 1 \) and \( M^- = -9 \), the relation \( w \) on \( D_n(2,1,4,6,3) \) is principal for \( n > 13 \), so \( f_w^n(t) \) is the Alexander polynomial of \( K(2, 1, 4, 6) \) representing the pretzel knot \( P(-2, 3, 7) \), which can be obtained by considering the principal cyclic presentation of \( D_{14}(2,1,4,6,3) \).

The next corollaries are immediate consequences of the previous considerations.

**Corollary 3.6.** Let \( D(a,b,c,r) \) be the Dunwoody \((1,1)\)-decomposition of \((S^3, K)\). Suppose that \( \alpha \) and \( \beta \) are two permutations defined by Theorem 2.1. Let \( w(x_0, \cdots, x_k) \) be a relation induced by the cyclic sequence of \( D_n(a,b,c,r,q) \) for each \( n > 1 \), \( M^+ = \max_{0 \leq k \leq d-1} \{(\beta \alpha)^k(0)\} \) and \( M^- = \min_{0 \leq k \leq d-1} \{(\beta \alpha)^k(0)\} \), where \( d = 2a + b + c \). Then \( f_w^n(t) \) is the Alexander polynomial of \( K \) if \( n \) is the smallest positive integer \( n_0 \) such that \( n_0 > M^+ - M^- + |s| \).

Let \( T(i, j) \) be the torus knot such that \( 2 \leq i \leq j \) and \( j = j + ik \) for some \( k \in \mathbb{Z} \). Let \( j = \pm 1 \), then the following \( (*) \) is the families of the Dunwoody 3-manifolds and their branched sets \( T(i, j) \), where \( i \geq 3 \) and \( k \geq 1 \). Note that these are different with the families introduced in [4]. As applications of Theorem 3.5, for the Dunwoody \((1,1)\)-knots representing \( T(k_1, k_2) \) satisfying \( k_2 \equiv \pm 1 \mod k_1 \) as

\[
T(i, ki + 1) \leftrightarrow D_n(1, i - 2, (i - 1) + (k - 1)(2i - 2), (i - 1) + (k - 1)(2i - 2), i)
\]

\[
T(i, (k + 1)i - 1) \leftrightarrow D_n(1, i - 2, (3i - 5) + (k - 1)(2i - 2), 3i - 4, -i) \quad (\ast)
\]

and \( k_2 \equiv \pm 2 \mod k_1 \) as \((2.1)\) or \((2.2)\), we show their Alexander polynomial and certain invariant. In our case Corollary 3.3 can be modified as follows.

**Corollary 3.7.** Let \( K = K(a,b,c,r) \) be the Dunwoody \((1,1)\)-knot as in \((\ast)\) and \( n > 1 \). Then \( f_w^n(t) \equiv \Delta_K(t) \) if \( n > M^+ - M^- + |s| \), where \( \equiv \) means equal up to units.

For the Dunwoody \((1,1)\)-knots satisfying \((2.1)\) or \((2.2)\), we have the following.

**Corollary 3.8.** Let \( K = K(a,b,c,r) \) be the Dunwoody \((1,1)\)-knot representing \( T(k_1, k_2) \) with \( k_2 \equiv \pm 2 \mod k_1 \) as \((2.1)\) or \((2.2)\). Then \( f_w^n(t) \equiv \Delta_K(t) \) if \( n > M^+ - M^- + |s| \), where \( \equiv \) means equal up to units.

We suppose that \( K(a,b,c,r) \) is the Dunwoody \((1,1)\)-knot representing \( T(k_1, k_2) \) satisfying \( k_2 \equiv \pm 1 \) or \( \pm 2 \mod k_1 \) as \((\ast)\) or \((2.1)\) and \((2.2)\). Then the following shows that \( d = 2a + b + c \) is an invariant for \( K(a,b,c,r) \).
Theorem 3.9. Let \( T(k_1, k_2) \) be the torus knot with \( k_2 \equiv \pm 2 \mod k_1 \) as in (2.1) or (2.2). Then \( d \) is an invariant of \( T(k_1, k_2) \), where

\[
d = \begin{cases} 
  k_1 + \frac{(k_1^2 - 1)(k_2^2 - 2)}{2k_1} & \text{if } k_2 \equiv 2 \mod k_1 \\
  k_1 + \frac{k_2(k_2^2 - 2)(k_2 + 2)}{2k_1} & \text{if } k_2 \equiv -2 \mod k_1
\end{cases}
\]

Proof. We suppose that \( T(k_1, k_2) \) be the torus knot with \( k_2 \equiv \pm 2 \mod k_1 \). Then the Dunwoody 3-manifold represented by \( D_n(a, b, c, r, s) \) satisfies (2.1) or (2.2). Let \( n > M^+ - M^- + |s| \). On \( D_n(a, b, c, r, s) \), we have a principal relation \( w \) from a cyclic sequence by applying (3.1), (3.2) and (3.3). Thus the Dunwoody polynomial \( f_w(t) \) of degree \( M^+ - M^- \) is the Alexander polynomial of \( T(k_1, k_2) \), and \( M^+ - M^- = (k_1 - 1)(k_2 - 1). \) Since the length of \( w \) is \( d \), the number of terms of \( \triangle(k_1, k_2) \) is \( d = 2a + b + c \). Therefore \( d \) is an invariant of \( T(k_1, k_2) \).

For (s), the similar argument can be applied as the following.

Corollary 3.10. Let \( T(i, j) \) be the torus knot with \( 3 \leq i \leq j \) and \( j = ki \pm 1 \) for some \( k \geq 1 \) in \( \mathbb{Z} \). Then \( d \) is an invariant of \( T(i, j) \), where

\[
d = \begin{cases} 
  \frac{(2i - 1) + (k - 1)(2i - 2)}{2} & \text{if } j = ki + 1 \\
  \frac{(4i - 5) + (k - 1)(2i - 2)}{2} & \text{if } j = ki - 1
\end{cases}
\]

We recall that if \( \triangle_n^K(t) \in \mathbb{Z}[t]/(t^n - 1) \) is the projection of the Alexander polynomial of \( K = K(a, b, c, r) \), then there is a connection between \( f_w^n(t) \) and \( \triangle_n^K(t) \), which follows from the result of Theorem 4.1 in [3].

Corollary 3.11. Let \( K = K(a, b, c, r) \) be a \((1, 1)\)-knot in the lens space \( L(p, q') \) and \( H_1(L(p, q') - K) = \mathbb{Z} \oplus \mathbb{Z}_d \), where \( d = \gcd(p, q) \). Then for each \( n > 1 \) such that \( \gcd(n, p) = 1 \), we have

\[
f_w^n(t^{p/d}) = \triangle_n^K(t) \cdot \frac{(t^{p/d} - 1)}{(t - 1)}
\]

up to units of \( \mathbb{Z}[t]/(t^n - 1) \).

In Corollary 3.11, the cyclotomic polynomial

\[
\frac{(t^{p/d} - 1)}{(t - 1)} = 1 + t + t^2 + \ldots + t^{p/d - 1}
\]

is irreducible polynomial if \( p/d \) is prime. Let \( n > 1 \) and \( \gcd(p, n) = 1 \). Then the following example explains one way to obtain the Alexander polynomial of \( K(a, b, c, r) \) in \( L(p, q') \) from the Dunwoody polynomial on \( D_n(a, b, c, r, s) \) with \( ps + q \equiv 0 \mod n \).

Example 5. Let \( D(1, 5, 0, 6) \) be a \((1, 1)\)-decomposition with \( p = 5 \) and \( q = 7 \) and
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Let \( K = K(1, 5, 0, 6) \) a (1, 1)-knot in the lens space \( L(5, 1) \). Then there is a unique \( s \in \mathbb{Z}_{12} \) such that \( 5s + 7 \equiv 0 \mod 12 \). On \( D_{12}(1, 5, 0, 6, 1) \) we have a principal relation

\[
w = x_0 x_1^{-1} x_2 x_4 x_6 x_8 x_{10}
\]

induced by a cyclic sequence as follow:

\[
\begin{align*}
(0, 1) & \xrightarrow{\alpha} (1, 7) \\
(1, -1) & \xrightarrow{\beta} (2, -7) \\
(2, 6) & \xrightarrow{\alpha} (4, -6) \\
(4, 5) & \xrightarrow{\beta} (6, -5) \\
(6, 4) & \xrightarrow{\alpha} (8, -4) \\
(8, 3) & \xrightarrow{\beta} (10, -3) \\
(10, 2) & \xrightarrow{\alpha} (12, -2) \\
(12, 1) & \xrightarrow{\beta}
\end{align*}
\]

Thus we obtain the Dunwoody polynomial

\[
f_w^{12}(t) = 1 + t^2 + t^4 + t^6 + t^8 + t^{10} - t^{-1}
\]

By Corollary 3.11, putting \( t^{p/d} = t^5 \), we have

\[
f_w^{12}(t^5) = 1 + t^{10} + t^{20} + t^{30} + t^{40} + t^{50} - t^{55} \\
= t^{-10}(1 + t^2 - t^3 + t^4 - t^5 + t^6) \\
= (1 - t + t^2 - t^3 + t^4 - t^5 + t^6) (t^5 - 1) \\
\]

and so \( \Delta_K^{12} = 1 - t + t^2 - t^3 + t^4 - t^5 + t^6 \) is the Alexander polynomial of the (1, 1)-knot \( K(1, 5, 0, 6) \), where the multiplication for \( t^{10} \) requires condition \( n > 10 \).

Indeed we have the same result from \( D_{22}(1, 5, 0, 6, 3) \). However the Dunwoody polynomial induced by \( D_7(1, 5, 0, 6, 0) \) does not give the Alexander polynomial of the (1, 1)-knot \( K(1, 5, 0, 6) \) because of \( 7 < 10 \). In other words, the Dunwoody polynomial induced by \( D_7(1, 5, 0, 6, 0) \) is not the Alexander polynomial of the (1, 1)-knot \( K(1, 5, 0, 6) \). From this example, we may have the possibility that the Alexander
polynomial of the Dunwoody (1, 1)-knot in a lens space can be obtained from the
Dunwoody polynomial.

References


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