On Generalizations of the Hadamard Inequality for $(\alpha, m)$-Convex Functions

Erhan Set
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
e-mail: erhanset@yahoo.com

Maryam Sardari
Institute for Advanced Studies in Basic Sciences, P. O. Box 45195-1159, Zanjan, Iran
e-mail: m_sardari@iasbs.ac.ir

Muhamet Emin Özdemir
Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey
e-mail: emos@atauni.edu.tr

Jamal Rooin
Institute for Advanced Studies in Basic Sciences, P. O. Box 45195-1159, Zanjan, Iran
e-mail: rooin@iasbs.ac.ir

Abstract. In this paper we establish several Hadamard-type integral inequalities for $(\alpha, m)$–convex functions.

1. Introduction

One of the most important integral inequalities for convex functions is the Hadamard inequality (or the Hermite-Hadamard inequality). The following double inequality is well known as the Hadamard inequality in the literature.

Theorem 1. If $f$ is convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$
Proof. See [1].

If the function \( f \) is concave, (1.1) can be written as following:

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x)dx \leq f \left( \frac{a+b}{2} \right).
\]

For recent results related to the Hadamard inequality are given in [9], [10] and [17].

In the literature, the concepts of \( m \)-convexity and \((\alpha, m)\)-convexity are well known. The concept of \( m \)-convexity was first introduced by G. Toader in [18] (see also [5], [6]) and it is defined as follows:

The function \( f : [0, b] \to \mathbb{R} \) is said to be \( m \)-convex, where \( m \in [0, 1] \), if for every \( x, y \in [0, b] \) and \( t \in [0, 1] \), we have:

\[
f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).
\]

The class of \((\alpha, m)\)-convex functions was also first introduced in [8] and it is defined as follows:

The function \( f : [0, b] \to \mathbb{R}, \ b > 0, \) is said to be \((\alpha, m)\)-convex , where \((\alpha, m) \in [0, 1]^2, \) if we have

\[
f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)
\]

for all \( x, y \in [0, b] \) and \( t \in [0, 1] \).

It can be easily seen that for \((\alpha, m) \in \{(0, 0), (1, 1) (1, m)\} \) one obtains the following classes of functions: increasing, convex and \( m \)-convex functions respectively. The interested reader can find more about partial ordering of convexity in [15, P. 8,280]. For many papers connected with \( m \)-convex and \((\alpha, m)\)-convex functions see ([2], [3], [6], [11], [12], [13], [14], [19]) and the references therein. There are similar inequalities for \( s \)-convex and \( h \)-convex functions in [7] and [16], respectively.

In [6], S. S. Dragomir and G. Toader proved the following Hadamard type inequality for \( m \)-convex functions.

**Theorem 2.** Let \( f : [0, \infty) \to \mathbb{R} \) be a \( m \)-convex function with \( m \in (0, 1] \). If \( 0 \leq a < b < \infty \) and \( f \in L_1[a, b] \), then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b f(x)dx \leq \min \left\{ \frac{f(a) + mf \left( \frac{b}{m} \right)}{2}, \frac{f(b) + mf \left( \frac{a}{m} \right)}{2} \right\}.
\]

Some generalizations of this result can be found in [2], [3].

In [4] S. S. Dragomir established two new Hadamard-type inequalities for \( m \)-convex functions. They are given in the following Theorems.
Theorem 3. Let \( f : [0, \infty) \to \mathbb{R} \) be a \( m \)–convex function with \( m \in (0, 1] \). If \( 0 \leq a < b < \infty \) and \( f \in L_1[a, b] \cap L_1 \left[ \frac{a}{m}, \frac{b}{m} \right] \), then the following inequality holds:

\[
(1.5) \quad f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) + \frac{m f \left( \frac{x}{m} \right)}{2} \, dx.
\]

Theorem 4. Let \( f : [0, \infty) \to \mathbb{R} \) be a \( m \)–convex function with \( m \in (0, 1] \). If \( f \in L_1[am, b] \) where \( 0 \leq a < b \), then the following inequality holds:

\[
(1.6) \quad \frac{1}{m + 1} \left[ \frac{1}{mb - a} \int_a^b f(x) \, dx + \frac{1}{b - ma} \int_{ma}^b f(x) \, dx \right] \leq \frac{f(a) + f(b)}{2}.
\]

The goal of this paper is to obtain new inequalities like those given in Theorems 1, 2, 3, 4, but now for the class of \((\alpha, m)\)–convex functions.

2. Inequalities for \((\alpha, m)\)-convex functions

The following theorem is a generalization of the Hadamard inequality.

Theorem 5. Let \( f : [0, \infty) \to \mathbb{R} \) be an \((\alpha, m)\)–convex function with \( 0 \leq a < b \) and \((\alpha, m) \in [0, 1] \times (0, 1] \). If \( f \in L_1 \left[ m^2a, (2 - m)b \right] \cap L_1 \left[ ma, \frac{(2-m)b}{m} \right] \), then the following inequalities hold:

\[
(2.1) \quad f \left( \frac{2-m}{2} b + \frac{m}{2} (ma) \right) \\
\leq \frac{1}{2^\alpha b(2-m) - m^2a} \left\{ \int_{m^2a}^{(2-m)b} \left[ f(x) + (2^\alpha - 1) m \right. \right. \right. \\
\left. \times f \left( \frac{(2-m)b}{m} \left( 1 - \frac{x - m^2a}{2b - mb - m^2a} \right) + m \frac{x - m^2a}{2b - mb - m^2a} \right) \right\} \, dx \\
\leq \frac{1}{2^\alpha \alpha^2 (2-m)} \left[ m f (am) + (2^\alpha - 1) \alpha m^2 f \left( \frac{(2-m)b}{m^2} \right) \right].
\]

Proof. Let \( U_1 = t (2 - m) b + (1 - t) m^2a \) and \( U_2 = (1 - t) (2 - m) b + tm^2a \), where \( t \in [0, 1] \) is arbitrary. Then we get

\[
f \left( \frac{U_1 + U_2}{2} \right) = f \left( \frac{2-m}{2} b + \frac{m}{2} (ma) \right).
\]
By the \((\alpha, m)\)–convexity of \(f\) we can write the following inequality:

\[
\begin{align*}
    f \left( \frac{2-m}{2} b + \frac{m}{2} (ma) \right) &= f \left( \frac{U_1 + U_2}{2} \right) \\
    &\leq \frac{1}{2^\alpha} f(U_1) + (1 - \frac{1}{2^\alpha}) m f \left( \frac{U_2}{m} \right) \\
    &= \frac{1}{2^\alpha} \left[ f(U_1) + (2^\alpha - 1) m f \left( \frac{U_2}{m} \right) \right],
\end{align*}
\]

or

\[
\begin{align*}
    f \left( \frac{2-m}{2} b + \frac{m}{2} (ma) \right) &\leq \frac{1}{2^\alpha} \left[ f \left( t (2 - m) b + (1 - t) m^2 a \right) \\
    &\quad + (2^\alpha - 1) m f \left( \frac{(1-t)(2-m)b}{m} + tma \right) \right].
\end{align*}
\]

Integrating over \(t \in [0, 1]\), we get

\[
\begin{align*}
    f \left( \frac{2-m}{2} b + \frac{m}{2} (ma) \right) &\leq \frac{1}{2^\alpha} \int_0^1 f \left( t (2 - m) b + (1 - t) m^2 a \right) \\
    &\quad + (2^\alpha - 1) m f \left( \frac{(1-t)(2-m)b}{m} + tma \right) \\
    &\quad + \frac{1}{2^\alpha} \int_0^1 \left\{ f \left( \frac{(2-m)b}{m} \left( 1 - \frac{x-m^2a}{2b-mb-m^2a} \right) + m \frac{x-m^2a}{2b-mb-m^2a} \right) \right\} dx
\end{align*}
\]

where we used the change of the variable \(x = t (2 - m) b + (1 - t) m^2 a\) or \(t = \frac{x-m^2a}{2b-mb-m^2a}\) and so

\[
\int_0^1 f \left( t (2 - m) b + (1 - t) m^2 a \right) dt = \frac{1}{(2-m)b-m^2a} \int_{m^2a}^{(2-m)b} f(x) \, dx
\]

and

\[
\int_0^1 f \left( \frac{(1-t)(2-m)b}{m} + tma \right) dt = \frac{1}{(2-m)b-m^2a} \int_{m^2a}^{(2-m)b} f \left( \frac{(2-m)b}{m} \left( 1 - \frac{x-m^2a}{2b-mb-m^2a} \right) + m \frac{x-m^2a}{2b-mb-m^2a} \right) dx.
\]

This completes the proof of the first inequality in (2.1).

Next, by the \((\alpha, m)\)–convexity of \(f\), we also have

\[
\begin{align*}
    f \left( t (2 - m) b + (1 - t) m^2 a \right) &= f \left( t (2 - m) b + m (1 - t) ma \right) \\
    &\leq t^\alpha f \left( (2-m)b \right) + m (1-t^\alpha) f \left( ma \right)
\end{align*}
\]
and
\[ f \left( \frac{(1-t)(2-m)b}{m} + tma \right) = f \left( t(ma) + m(1-t) \left( \frac{(2-m)b}{m^a} \right) \right) \leq t^\alpha f(ma) + m(1-t^\alpha) f \left( \frac{(2-m)b}{m^a} \right). \]

So
\[ \frac{1}{b-a} \left[ f \left( t(2-m)b + (1-t)m^2a \right) + (2^\alpha - 1) m f \left( \frac{(1-t)(2-m)b}{m} + tma \right) \right] \]
\[ \leq \frac{1}{b-a} \left\{ t^\alpha f \left( (2-m)b \right) + m(1-t^\alpha) f(ma) \right\} \]
\[ + (2^\alpha - 1) m \left[ t^\alpha f(ma) + m(1-t^\alpha) f \left( \frac{(2-m)b}{m^a} \right) \right]. \]

Integrating (2.3) over \( t \) on \([0,1]\), we get
\[ \frac{1}{b-a} \int_{m^2a}^{(2-m)b} \left\{ f \left( \frac{(2-m)b}{m} \left( 1 - \frac{x-m^2a}{2b-mb-m^2a} \right) + m \frac{x-m^2a}{2b-mb-m^2a} \right) \right\} dx \]
\[ \leq \frac{1}{2^{\alpha(\alpha+1)}} \left\{ \int (2-m)b \right\} \]
\[ + (\alpha + (2^\alpha - 1)) m f (ma) + (2^\alpha - 1) \alpha m^2 f \left( \frac{(2-m)b}{m^a} \right). \]

This completes the proof of the second inequality in (2.1). □

**Remark 1.** Choosing \((\alpha, m) = (1, 1)\) in (2.1), from the first and the second inequalities of (2.1), respectively, we obtain

\[ f \left( \frac{a+b}{2} \right) \leq \frac{1}{2} \frac{1}{b-a} \left[ \int_a^b f(x) + f(a+b-x) \right] dx \]
\[ = \frac{1}{2} \frac{1}{b-a} \left[ \int_a^b f(x) dx + \int_a^b f(x) dx \right] \]
\[ = \frac{1}{b-a} \int_a^b f(x) dx \]
\[ \leq \frac{1}{b} \int_a^b f(x) dx \leq \frac{1}{4} [f(b) + 2f(a) + f(b)] \]
\[ = \frac{f(a)+f(b)}{2}. \]

Note that, we used
\[ \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \]
and so
\[ \int_a^b [f(x) + f(a + b - x)] \, dx = 2 \int_a^b f(x) \, dx. \]

Clearly, we can drop the assumption \( f \in L_1 [m^2a, (2 - m)b] \cap L_1 [ma, \frac{(2-m)b}{m}] \) = \( L_1 [a, b] \), and in this case (2.1) exactly becomes the Hermite-Hadamard inequalities for \( (\alpha, m) = (1, 1) \).

**Theorem 6.** Let \( f : [0, \infty) \to \mathbb{R} \) be an \((\alpha, m)\)-convex function with \((\alpha, m) \in (0, 1]^2 \). If \( 0 \leq a < b < \infty \) and \( f \in L_1 [a, b] \), then the following inequality holds:

\[
(2.4) \quad \frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + \alpha m f \left( \frac{b}{m} \right)}{\alpha + 1}, \frac{f(b) + \alpha m f \left( \frac{a}{m} \right)}{\alpha + 1} \right\}.
\]

**Proof.** Since \( f \) is \((\alpha, m)\)-convex, we have

\[
f(ta + (1-t)b) \leq t^\alpha f(a) + m(1-t^\alpha) f(b)
\]

for all \( x, y \geq 0 \), which gives:

\[
f(ta + (1-t)b) \leq t^\alpha f(a) + m(1-t^\alpha) f \left( \frac{b}{m} \right)
\]

and

\[
f(tb + (1-t)a) \leq t^\alpha f(b) + m(1-t^\alpha) f \left( \frac{a}{m} \right)
\]

for all \( t \in [0,1] \). Integrating on \([0,1]\), we obtain

\[
\int_0^1 f(ta + (1-t)b) \, dt \leq \frac{f(a) + \alpha m f \left( \frac{b}{m} \right)}{\alpha + 1}
\]

and

\[
\int_0^1 f(tb + (1-t)a) \, dt \leq \frac{f(b) + \alpha m f \left( \frac{a}{m} \right)}{\alpha + 1}.
\]

However,

\[
\int_0^1 f(ta + (1-t)b) \, dt = \int_0^1 f(tb + (1-t)a) \, dt = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

and the inequality (2.4) is obtained. \( \square \)

**Remark 2.** The inequality (2.4) yields inequality (1.4) for \( \alpha = 1 \).

**Theorem 7.** Let \( f : [0, \infty) \to \mathbb{R} \) be an \((\alpha, m)\)-convex function with \((\alpha, m) \in (0, 1]^2 \).
On Generalizations of the Hadamard Inequality for \((\alpha, m)\)-Convex Functions

If \(0 \leq a < b < \infty\) and \(f \in L_1 [a, b] \cap L_1 \left[ \frac{a}{m}, \frac{b}{m} \right]\), then the following inequalities hold:

\[
\begin{align*}
\frac{f \left( \frac{a+b}{2} \right)}{2} & \leq \frac{1}{2^{\alpha(b-a)}} \int_a^b \left[ f(x) + m \left( 2^\alpha - 1 \right) f \left( \frac{a}{m} \right) \right] dx \\
& \leq \frac{1}{2^{\alpha+1}} \left[ (f(a) + f(b)) \\
+ m (\alpha + 2^\alpha - 1) \left( f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right) \right) \\
+ \alpha m^2 (2^\alpha - 1) \left( f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right) \right) \right].
\end{align*}
\]

(2.5)

Proof. By the \((\alpha, m)\)-convexity of \(f\), we have

\[
f \left( \frac{x+y}{2} \right) = f \left( \frac{x}{2} + m \frac{y}{2m} \right)
\leq \frac{1}{2^\alpha} f(x) + m \left( 1 - \frac{1}{2^\alpha} \right) f \left( \frac{y}{m} \right)
= \frac{1}{2^\alpha} \left[ f(x) - mf \left( \frac{y}{m} \right) \right] + mf \left( \frac{y}{m} \right)
\]
for all \(x, y \in [0, \infty)\).

Now, if we choose \(x = ta + (1-t) b\) and \(y = (1-t) a + tb\), we deduce

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{2^\alpha} \left[ f(ta + (1-t) b) - mf \left( \frac{(1-t)a + tb}{m} \right) \right] + mf \left( \frac{(1-t) a + tb}{m} \right)
= \frac{1}{2^\alpha} \left[ f(ta + (1-t) b) + m (2^\alpha - 1) f \left( 1-t \frac{a}{m} + t \frac{b}{m} \right) \right]
\]
for all \(t \in [0, 1]\).

Integrating over \(t \in [0, 1]\), we get

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{2^\alpha} \left[ \int_0^1 f(ta + (1-t) b) \, dt + m (2^\alpha - 1) \int_0^1 f \left( 1-t \frac{a}{m} + t \frac{b}{m} \right) \, dt \right]
\]
(2.6)

Taking into account that

\[
\int_0^1 f(ta + (1-t) b) \, dt = \frac{1}{b-a} \int_a^b f(x) \, dx
\]
and

\[
\int_0^1 f \left( 1-t \frac{a}{m} + t \frac{b}{m} \right) \, dt = \frac{1}{b-a} \int_a^b f \left( \frac{x}{m} \right) \, dx,
\]

(0, 1]^2.
we deduce from (2.6) the first inequality in (2.5). Next, by the \((\alpha, m)\) -convexity of \(f\), we also have

\[
\frac{1}{2^\alpha} \left[ f \left( t(a + (1 - t)b) + m (2^\alpha - 1) f \left( \frac{b}{m} \right) \right) \right. \\
\left. + m (2^\alpha - 1) \left( t^\alpha f \left( \frac{b}{m} \right) + m (1 - t^\alpha) f \left( \frac{a}{m^2} \right) \right) \right].
\]

Integrating over \(t\) on \([0, 1]\), we get

\[
\frac{1}{2^\alpha} \int_a^b \left( f(x) + m (2^\alpha - 1) f \left( \frac{x}{m} \right) \right) dx \\
\leq \frac{1}{2^\alpha} \left[ f(a) \int_0^1 t^\alpha dt + mf \left( \frac{b}{m} \right) \int_0^1 (1 - t^\alpha) dt + \\
m (2^\alpha - 1) f \left( \frac{b}{m^2} \right) \int_0^1 (1 - t^\alpha) dt \right. \\
\left. + m^2 (2^\alpha - 1) f \left( \frac{a}{m^2} \right) \int_0^1 (1 - t^\alpha) dt \right]
\]

\[
= \frac{1}{2^\alpha (\alpha + 1)} \left[ f(a) + m (\alpha + 2^\alpha - 1) f \left( \frac{b}{m^2} \right) \right]
\]

Similarly, changing the roles of \(a\) and \(b\), we get

\[
\frac{1}{2^\alpha (\alpha + 1)} \left[ f(b) + m (\alpha + 2^\alpha - 1) f \left( \frac{a}{m^2} \right) \right]
\]

Now adding (2.8) and (2.9) with each other, we obtain the second inequality in (2.5).

**Remark 3.** Choosing \(\alpha = 1\) in the first part of (2.5), we get (1.5).

**Remark 4.** The inequality (2.5) yields the Hadamard inequality (1.1) for \(\alpha = 1\) and \(m = 1\).

**Theorem 8.** Let \(f : [0, \infty) \to \mathbb{R}\) be an \((\alpha, m)\) -convex function with \((\alpha, m) \in (0, 1]^2\). If \(0 \leq a < b < \infty\) and \(f \in L_1 [a, b]\), then the following inequality holds:

\[
\int_a^b f(x) dx \leq \frac{1}{\alpha + 1} \left[ f(a) + f(b) + \alpha m \left( f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right) \right) \right]
\]
Proof. By the \((\alpha, m)\)-convexity of \(f\), we can write
\[
f(ta + (1 - t)b) \leq t^\alpha f(a) + m (1 - t^\alpha) f \left( \frac{b}{m} \right)
\]
and
\[
f(tb + (1 - t)a) \leq t^\alpha f(b) + m (1 - t^\alpha) f \left( \frac{a}{m} \right)
\]
for all \(t \in [0, 1]\).

Adding the above inequalities, we get
\[
f(ta + (1 - t)b) + f(tb + (1 - t)a) \leq t^\alpha f(a) + m (1 - t^\alpha) f \left( \frac{b}{m} \right) + t^\alpha f(b) + m (1 - t^\alpha) f \left( \frac{a}{m} \right).
\]

Integrating over \(t \in [0, 1]\), we obtain
\[
\int_0^1 f(ta + (1 - t)b) \, dt + \int_0^1 f(tb + (1 - t)a) \, dt \leq \int_0^1 t^\alpha \left( f(a) + f(b) \right) \, dt + \int_0^1 m (1 - t^\alpha) \left( f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right) \right) \, dt
\]
\[
= \frac{f(a) + f(b)}{\alpha + 1} + \frac{ma}{\alpha + 1} \left( f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right) \right)
\]
\[
= \frac{f(a) + f(b) + ma \left( f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right) \right)}{\alpha + 1}.
\]

As it is easy to see that
\[
\int_0^1 f(ta + (1 - t)b) \, dt = \int_0^1 f(tb + (1 - t)a) \, dt = \frac{1}{b-a} \int_a^b f(x) \, dx,
\]
from (2.11) we deduce the desired result, namely, the inequality (2.10).

\[\square\]

Remark 5. The inequality (2.10) yields the right side of the Hadamard inequality (1.1) for \(\alpha = 1\) and \(m = 1\).

Theorem 9. Let \(f : [0, \infty) \to \mathbb{R}\) be an \((\alpha, m)\)-convex function with \((\alpha, m) \in (0, 1]^2\). If \(f \in L_1[am, b]\) where \(0 \leq a < b\), then the following inequality holds:
\[
\frac{1}{mb - a} \int_a^b f(x) \, dx + \frac{1}{b - ma} \int_{ma}^b f(x) \, dx \leq \frac{1}{\alpha + 1} \left[ (f(a) + f(b)) (1 + ma) \right].
\]
Proof. By \((\alpha, m) -\)convexity of \(f\), for all \(t \in [0, 1]\), we can write:

\[
\begin{align*}
    f (ta + m (1-t) b) & \leq t^\alpha f (a) + m (1-t^\alpha) f (b) , \\
    f (tb + m (1-t) a) & \leq t^\alpha f (b) + m (1-t^\alpha) f (a) , \\
    f ((1-t) a + mtb) & \leq (1-t)^\alpha f (a) + m (1 - (1-t)^\alpha) f (b) , \\
    f ((1-t) b + mta) & \leq (1-t)^\alpha f (b) + m (1 - (1-t)^\alpha) f (a) .
\end{align*}
\]

Adding the above inequalities with each other, we get:

\[
\begin{align*}
    f (ta + m (1-t) b) + f (tb + m (1-t) a) \\
    + f ((1-t) a + mtb) + f ((1-t) b + mta) \\
    & \leq [t^\alpha + m (1 - t^\alpha) + (1-t)^\alpha + m (1 - (1-t)^\alpha)] (f(a) + f(b)) .
\end{align*}
\]

Now integrating over \(t \in [0, 1]\) and taking into account that:

\[
\begin{align*}
    \int_0^1 f (ta + m (1-t) b) dt = \int_0^1 f ((1-t) a + mtb) dt = \frac{1}{mb-a} \int_a^{mb} f(x)dx \\
    \int_0^1 f (tb + m (1-t) a) dt = \int_0^1 f ((1-t) b + mta) dt = \frac{1}{b-ma} \int_{ma}^b f(x)dx,
\end{align*}
\]

we obtain the inequality (2.12). \(\square\)

**Remark 7.** Choosing \(\alpha = 1\) in (2.12), we obtain (1.6).

References


On Generalizations of the Hadamard Inequality for \((\alpha, m)\)-Convex Functions


