Characterizations of Several Modules Relative to the Class of $B(M,X)$

YAHYA TALEBI* AND MEHRAB HOSSEINPOUR
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
e-mail: talebi@umz.ac.ir and m.hpour@umz.ac.ir

ABSTRACT. Let $M$ and $X$ be right $R$-modules. We introduce several modules relative to the class of $B(M,X)$ and we investigate relation among these modules. In this note, we show if $M$ is $X$-$\oplus$-supplemented such that $M = M_1 \oplus M_2$ implies $M_1$ and $M_2$ are relatively $B$-projective, then $M$ is an $X$-$H$-supplemented module.

1. Introduction

Throughout this paper, $R$ will be an associative ring with identity, and all modules are unitary right $R$-modules. A submodule $K$ of $M$ is denoted by $K \leq M$. The notation $N \leq \oplus M$ means that $N$ is a direct summand of $M$. A submodule $K$ of $M$ is called essential (or large) in $M$ (denoted by $K \leq_e M$), if $K \cap L \neq 0$ for every nonzero submodule $L$ of $M$, and a submodule $K$ of $M$ is called small in $M$ (denoted by $K \ll M$), if $N + K \neq M$ for any proper submodule $N$ of $M$. A module $M$ is called hollow if every proper submodule of $M$ is small in $M$. Let $N$ be a submodule of $M$, a supplement of $N$ in $M$ is a submodule $K$ of $M$ minimal with respect to the property $M = N + K$, equivalently, $M = N + K$ and $N \cap K \ll K$. Following [14], $M$ is called supplemented if every submodule of $M$ has a supplement in $M$. $M$ is called a lifting module or $(D_1)$-module if for every submodule $A$ of $M$ there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \ll M_2$. Following [11], $M$ is called $\oplus$-supplemented if every submodule of $M$ has a supplement that is a direct summand of $M$ and $M$ is called $H$-supplemented if for every submodule $A$ of $M$ there is a direct summand $D$ of $M$ such that $M = A + X$ holds if and only if $M = D + X$. $H$-supplemented modules are $\oplus$-supplemented [11, A.2]. Suppose $N \subseteq K$ are submodules of $M$, $N$ is said to be a cosmall submodule of $K$ in $M$ if $K/N \ll M/N$ (denoted by $N \leq^{cs} K$). A submodule $N$ of $M$ is coclosed in $M$ if it has no proper cosmall submodules in $M$ (denoted by $N \leq^{cc} M$). $N$ is called a coclosure of $K$ in $M$, if $N \leq^{cs} K$ and $N \leq^{cc} M$.

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Recall that a module $M$ has the \textit{Summand Intersection Property (SIP)} if the intersection of any two direct summands of $M$ is again a direct summand (see [6]) and $M$ has the \textit{Summand Sum Property (SSP)} if the sum of any two direct summands of $M$ is again a direct summand (see [5]). Let $M$ be a module, a submodule $N$ of $M$ is called \textit{fully invariant} if for every $h \in \text{End}_R(M)$, $h(N) \subseteq N$.

Supplemented and lifting modules are worthy of study in module theory since they are dual of complemented and extending modules, and there has been a great deal of work on lifting modules by many authors. Supplemented modules, lifting modules are also studied in [11] and [14].

Let $M$ and $X$ be modules. Lopez-Permouth, Oshiro and Tariq Rizvi in [10], defined the family

$$A(M, X) = \{ A \leq M \mid \exists Y \leq X, \exists f \in \text{Hom}(Y, M), f(Y) \leq_e A \}$$

They studied extending, quasi-continuous, or continuous modules relative to this class.

In [8], D. Keskin and A. Harmanci dualized the class $A(X, M)$ and defined the family

$$B(M, X) = \{ A \leq M \mid \exists Y \leq X, \exists f \in \text{Hom}(M, X/Y), \text{Ker}f/A \ll M/A \}$$

They considered the following conditions:

- \textit{B(M,X)-(D1):} For every submodule $A \in B(M, X)$, there exists a direct summand $A^* \leq \oplus M$ such that $A/A^* \ll M/A^*$.
- \textit{B(M,X)-(D2):} For any $A \in B(M, X)$, if $B \leq \oplus M$ and $M/A \cong B$ implies $A \leq \oplus M$.
- \textit{B(M,X)-(D3):} For any $A \in B(M, X)$ and $B \leq \oplus M$, if $A \leq \oplus M$ and $M = A + B$ then $A \cap B \leq \oplus M$.

They call $M$ is \textit{X-lifting, X-quasi-discrete and X-discrete}, respectively, if $M$ satisfies $B(M, X)-(D1)$, $B(M, X)-(D1)$ and $B(M, X)-(D2)$.

Let $\{X_\lambda \mid \lambda \in \Lambda \}$ be a family of submodules of a module $M$ with $X_\lambda \in B(M, X)$, $\Sigma_{\lambda \in \Lambda}X_\lambda$ is called an \textit{X-local summand} of $M$, if $\Sigma_{\lambda \in \Lambda}X_\lambda$ is direct and $\Sigma_{\lambda \in F}X_\lambda \leq \oplus M$ for every finite subset $F \subseteq \Lambda$.

Let $X$ and $M$ be $R$-modules. Following [8], an $R$-module $N$ is called $B(M, X)$-\textit{projective} if for any submodule $A$ of $M$ with $A \in B(M, X)$, any homomorphism $\phi : N \rightarrow M/A$ can be lifted to a homomorphism $\phi : N \rightarrow M$. Two $R$-modules $M_1$ and $M_2$ are called \textit{relatively B-projective} if $M_1$ is $B(M_2, X)$-projective and $M_2$ is $B(M_1, X)$-projective.

Let $A$ and $P$ be submodules of $M$ with $P \in B(M, X)$. $P$ is called an \textit{X-supplement} of $A$ if $M = A + P$ and $A \cap P \ll P$. The module $M$ is called \textit{X-supplemented} if every submodule $N$ of $M$ with $N \in B(M, X)$ has a $X$-supplement in $M$. Let $X$ be an $R$-module. A non-zero module $M$ is \textit{X-hollow}, if for any proper submodule $K$ of $M$ with $K \in B(M, X)$, $K \ll M$. 

In this paper, we consider $H$-supplemented and $\oplus$-supplemented relative to this class. Therefore we define $X$-$H$-supplemented, $X$-$\oplus$-supplemented, and $X$-FI-lifting modules. $M$ is called $X$-$H$-supplemented if for any $A \in B(M, X)$ there exists a direct summand $D$ of $M$ such that $M = A + Y$ if and only if $M = D + Y$. $M$ is called $X$-$\oplus$-supplemented if every $N \in B(M, X)$ has an $X$-supplement that is a direct summand of $M$. A module $M$ is called $X$-FI-lifting if for every fully invariant submodule $A$ with $A \in B(M, X)$ there exists a direct summand $N$ of $M$ such that $A/N \ll M/N$. It is easy to see that $M$ is $H$-supplemented($\oplus$-supplemented) if and only if $M$ is $M$-$H$-supplemented($M$-$\oplus$-supplemented) if and only if $M$ is $X$-$H$-supplemented($X$-$\oplus$-supplemented) for every module $X$. Clearly $X$-hollow modules are $X$-$\oplus$-supplemented and $X$-$\oplus$-supplemented modules are $X$-supplemented.

In Section 2, we will give some properties of $X$-$\oplus$-supplemented and $X$-$H$-supplemented modules. We investigate general properties of this modules, relation of them with other modules. We give a condition for an $X$-$\oplus$-supplemented module to be $X$-$H$-supplemented (see Proposition 2.18).

In Section 3, we define and investigate a generalization of $X$-$H$-supplemented modules.

2. Main results

A module $M$ is called $X$-supplement bounded, if it is $X$-supplemented and every proper $X$-supplement submodule of $M$ is contained in a nontrivial fully invariant submodule belongs to the class $B(M, X)$.

**Lemma 2.1.** Let $M$ be an $X$-supplemented module, then for every submodule $K$ of $M$, $M/K$ is $X$-supplemented.

*Proof.* Simple to check. \[ \square \]

**Proposition 2.2.** Let $M$ be an $X$-supplemented module such that $B(M, X)$ is closed under taking arbitrary intersection. Then $M$ is $X$-supplement bounded if and only if every proper coclosed submodule $K$ of $M$ is cosmall in a fully invariant submodule $Y$ of $M$ with $Y \in B(M, X)$.

*Proof.* Assume $M$ is $X$-supplement bounded. Let $K \leq_{co} M$ be proper. Let $Y$ be the intersection of fully invariant submodules in $B(M, X)$ containing $K$. Then $Y \in B(M, X)$ is a fully invariant submodule of $M$. By Lemma 2.1, let $L/K$ be the $X$-supplement of $Y/K$ in $M/K$. Then $L + Y = M$ and $L/K \cap Y/K \ll M/K$. Suppose $L \neq M$, by [8, Lemma 2.2], $L \in B(M, X)$. Since $L$ is an $X$-supplement submodule of $M$, then there exists a fully invariant submodule $S \in B(M, X)$ such that $S \neq M$ and $L \subseteq S$. So $L + Y \subseteq Y \neq M$, a contradiction. Therefore $L = M$ and hence $K \leq_{co} Y$ in $M$. The converse is trivial. \[ \square \]

**Proposition 2.3.** Let $M$ be $X$-supplement bounded such that $B(M, X)$ is closed under taking arbitrary intersection. If every submodule of $M$ has a coclosure, then
Proof. The necessity is clear. For the sufficiency assume \( M \) is \( X\)-\( FI \)-lifting. Let \( Y \leq M \) and \( Y \neq M \). Since \( Y \) has a coclosure, there exists a submodule \( K \) of \( M \) such that \( K \subseteq Y \), \( Y/K \ll M/K \) and \( K \leq \text{cc} \ M \). Since \( M \) is \( X \)-supplement bounded there exists a fully invariant submodule \( B \in B(M, X) \) with \( K \leq B \) and \( B/K \ll M/K \) by Proposition 2.2. Since \( M \) is \( X\)-\( FI \)-lifting, there exists a direct summand \( D \) of \( M \) such that \( D \leq B \) and \( B/D \ll M/D \). Let \( M = Y + L \) for some \( L \leq M \). Then \( M/K = Y/K + (L + K)/K = (L + K)/K \) implies that \( M = L + K \). Then \( M = L + B \) and hence \( M/D = (L + D)/D + B/D = (L + D)/D \). Thus \( M = L + D \). Conversely assume that \( M = L + D \). Then \( M = L + B \). Now \( M/K = (L + K)/K + B/K \) implies that \( M = L + K \) and hence \( M = L + Y \). \( \square \)

By analogy with the proof of [13, Proposition 2.5], we have the following proposition.

**Proposition 2.4.** The following are equivalent for a module \( M \):

1. \( M \) is \( X\)-\( FI \)-lifting.
2. Every fully invariant submodule \( N \) of \( M \) with \( N \in B(M, X) \) has a supplement which is a direct summand.

Now we consider the \( X\)-\( \oplus \)-supplemented module;

**Proposition 2.5.** Let \( M \) be a nonzero module and let \( U \) be a fully invariant submodule of \( M \) with \( U \in B(M, X) \) such that \( M = U \oplus V \). If \( M \) is \( X\)-\( \oplus \)-supplemented, then \( V \) is \( X\)-\( \oplus \)-supplemented.

**Proof.** Suppose that \( M \) is \( X\)-\( \oplus \)-supplemented. Let \( L \in B(M, X) \) be a submodule of \( M \) which contains \( U \). There exist submodules \( N \) and \( N' \) of \( M \) such that \( M = N \oplus N' \), \( N = L + N \), and \( L \cap N \) is small in \( N \) and \( N \in B(M, X) \). By [8, Lemma 2.2], \( L/U \in B(M/U, X) \) and it is clear that \( (N + U)/U \) is a \( X \)-supplement of \( L/U \) in \( M/U \) and by [8, Lemma 3.5], \( (N + U)/U \in B(M/U, X) \).

Since \( U \) be a fully invariant submodule of \( M \), \( U = (U \cap N) \oplus (U \cap N') \). Thus, \( (N + U) \cap (N' + U) \leq (N + U + N') \cap U + (N + U + U) \cap N' \). Hence, \( (N + U) \cap (N' + U) \leq U + (N + U + N + U \cap N') \cap N' \). It follows that \( (N+U)\cap(N'+U) \leq U \) and \( ((N+U)/U) \oplus ((N'+U)/U) = M/U \). Then \( (N+U)/U \) is a direct summand of \( M/U \). Consequently, \( M/U \) is \( X\)-\( \oplus \)-supplemented. \( \square \)

**Theorem 2.6.** Any finite direct sum of \( X\)-\( \oplus \)-supplemented modules is \( X\)-\( \oplus \)-supplemented.

**Proof.** Let \( M = M_1 \oplus M_2 \) where \( M_1 \) and \( M_2 \) are two \( X\)-\( \oplus \)-supplemented modules. Let \( N \in B(M, X) \), we have \( N + M_2 = M_2 \oplus [(N + M_2) \cap M_1] \) and \( (N + M_2) \cap M_1 \) is a submodule of \( M_1 \). Since \( N \in B(M, X) \), \( N + M_2 \in B(M, X) \) by [8, Lemma 3.5 and 2.2]. By [12, Lemma 3.1], \( (N + M_2) \cap M_1 \in B(M_1, X) \). Since \( M_1 \) is \( X\)-\( \oplus \)-supplemented, there exists a direct summand \( K_1 \) of \( M_1 \) with \( K_1 \in B(M_1, X) \) such that \( [(N + M_2) \cap M_1] + K_1 = M_1 \) and \( (N + M_2) \cap K_1 \ll K_1 \). By [8, Lemma 3.5 and 2.2] and [12, Lemma 3.1], \( (N + K_1) \cap M_2 \) is a submodule of \( M_2 \) such that

\[ M \text{ is } X\-\text{supplemented if and only if } M \text{ is } X\-\text{FI-lifting.} \]
Proposition 2.7. Let $M$ be an $X$-supplemented module with $B(M,X)$-(D$_3$). Then $M$ is completely $X$-supplemented.

Proof. Let $N \leq M$ and $A \leq N$ such that $N \in B(M,X)$ and $A \in B(N,X)$. We show that $A$ has an $X$-supplement in $N$ that is a direct summand of $N$. We have $M = N \oplus N'$ for some submodule $N'$ of $M$. Let $\pi : M \to N$ be projection along $N'$. Since $A \in B(N,X)$, by [8, Lemma 2.2(4)], $A \oplus N' = \pi^{-1}(A) \in B(M,X)$. Since $M = A + N + N'$, by [12, Lemma 3.1], $A = (A \oplus N') \cap N \in B(M,X)$. Since $M$ is $X$-supplemented, there exists a direct summand $B$ of $M$ with $B \in B(M,X)$ such that $M = A + B$ and $A \cap B \leq B$. Then $N = A + (N \cap B)$. Again by Lemma [12, Lemma 3.1], $N \cap B \in B(M,X)$. Furthermore $N \cap B \leq M$ because $M$ has $B(M,X)$-(D$_3$). Then $A \cap (N \cap B) = A \cap B$ is small in $N \cap B$ and by [12, Lemma 3.1], $N \cap B \in B(N,X)$.

Proposition 2.8. Let $M$ be an indecomposable module. Then $M$ is X-hollow if and only if $M$ is completely $X$-supplemented.

Proof. Let $M$ be completely $X$-supplemented. If $N \in B(M,X)$ is a proper submodule of $M$ then there exists an $X$-supplement $A$ of $N$ such that $A$ is a direct summand of $M$. By hypothesis we have $A = M$. Thus $N = N \cap M = N \cap A \leq M$. Therefore $M$ is $X$-hollow. Conversely, if $M$ is $X$-hollow and $N \in B(M,X)$ then $N \leq M$. Since $M \in B(M,X)$, so $M$ is an $X$-supplement of $N$ in $M$.

Let $M$ be any module. $M$ is called a (D$_3$)-module if whenever $M_1$ and $M_2$ are direct summands of $M$ with $M = M_1 + M_2$, $M_1 \cap M_2$ is also a direct summand of $M$. Clearly (D$_3$) is $B(M,X)$-(D$_3$).

In the set $B(M,X)$, if we take $X = M$, then $B(M,X)$ coincides with the set of all submodules of $M$. Therefore we obtain the following corollaries:

Corollary 2.9. Any finite direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented.

Proof. See [7, Theorem 1.4].

Corollary 2.10. Let $M$ be a $\oplus$-supplemented module with (D$_3$). Then $M$ is com-
Corollary 2.11. Let $M$ be an indecomposable module. Then $M$ is hollow if and only if $M$ is completely $\oplus$-supplemented.

Proof. See [7, Proposition 2.3].

Example 2.12. (1) $\mathbb{Z}_{p^\infty}$ is a lifting $\mathbb{Z}$-module and so an $X$-lifting $\mathbb{Z}$-module for every $\mathbb{Z}$-module $X$.

(2) Clearly $B(M, 0) = M$ for any module $M$. Therefore every module $M$ is $0$-$\oplus$-supplemented (completely $0$-$\oplus$-supplemented), this means that the $\mathbb{Z}$-module $\mathbb{Z}_I$ is completely $0$-$\oplus$-supplemented. But by Proposition 2.8, for every nonzero module $X$, it is not completely $X$-$\oplus$-supplemented.

(3) Let $X$ be simple projective module and $M$ any module. Then for any $A \in B(M, X)$, $A$ is direct summand of $M$. Therefore $M$ is $X$-$H$-supplemented module.

(4) If $M$ is a divisible $\mathbb{Z}$-module, then $B(M, \mathbb{Z}) = \emptyset$, since $\text{Hom}(M, \mathbb{Z}_n) = 0$.

Lemma 2.15. If every $X$-local summand of a module $M$ is a direct summand, then $M$ has an indecomposable decomposition.

Proof. See [3, Lemma 3.2]}

Theorem 2.16. Let $M$ be an $X$-$\oplus$-supplemented module with (SSP), $(D_3)$ and every $X$-local summand $Y$ of $M$ such that $Y \in B(M, X)$. Then $M$ is a direct sum of $X$-hollow modules.

Proof. By Lemma 2.13, Lemma 2.15, $M$ is a direct sum of indecomposable modules and since $M$ has $(D_3)$, therefore $M$ has $B(M, X)$-$(D_3)$, so by Proposition 2.7, every direct summand of $M$ is $X$-$\oplus$-supplemented. Therefore $M$ is a direct sum of indecomposable $X$-$\oplus$-supplemented modules, which are $X$-hollow.

Remark 2.17. Let $M$ be an $X$-$H$-supplemented module such that for every direct summand $A$ of $M$ with $A \in B(M, X)$. Then $M$ is $X$-$\oplus$-supplemented.

Proposition 2.18. Assume that $M$ is $X$-$\oplus$-supplemented such that whenever
$M = M_1 \oplus M_2$ then $M_1$ and $M_2$ are relatively $B$-projective. Then $M$ is an $X$-$H$-supplemented module.

**Proof.** Let $N \in B(M, X)$, since $M$ is $X-\oplus$-supplemented, there exists a decomposition $M = M_1 \oplus M_2$ such that $M = N + M_2$ and $N \cap M_2 \ll M_2$ such that $M_2 \in B(M, X)$. By hypothesis, $M_1$ is $B(M_2, X)$-projective, by [8, Proposition 2.5], we obtain $M = A \oplus M_2$ for some submodule $A$ of $M$ such that $A \leq N$. Then $N = A \oplus (M_2 \cap N)$. Let $Y \leq M$ with $M = N + Y$. Then $M = A + (M_2 \cap N) + Y$. Since $M_2 \cap N$ is small in $M_2$ and so is small in $M$, $M = A + Y$. Hence $M = N + Y$ if and only if $M = A + Y$. Thus $M$ is $X$-$H$-supplemented module. □

**Lemma 2.19.** For a submodule $U$ of $M$, the following are equivalent:

1. there is a direct summand $Y$ of $M$ with $Y \subseteq U$ and $U/Y \ll M/Y$;
2. there is a direct summand $Y \subseteq M$ and a submodule $N$ of $M$ with $Y \subseteq U$, $U = Y + L$ and $L \ll M$;

**Proof.** See [4, 22.1]. □

Let $X$ and $M_2$ be $R$-modules. Following [12], an $R$-module $M_1$ is called $B(M_2, X)$-cojective if for any submodule $A$ of $M_2$ with $A \in B(M_2, X)$, any homomorphism $\phi : M_1 \to M_2/A$, there exist decompositions $M_1 = M_1' \oplus M_1''$, $M_2 = M_2' \oplus M_2''$, and homomorphisms $\phi_1 : M_1' \to M_2'$, $\phi_2 : M_1'' \to M_2''$ such that $\phi_2$ is onto, $\pi M_1 = \phi(M_1')$ and $\phi \phi_1 = \pi | M_1''$, where $\pi : M_2 \to M_2/A$ is the natural epimorphism. Two $R$-modules $M_1$ and $M_2$ are called relatively cojective if $M_1$ is $B(M_2, X)$-cojective and $M_2$ is $B(M_1, X)$-cojective.

**Proposition 2.20.** Assume that $M$ is $X-\oplus$-supplemented such that whenever $M = M_1 \oplus M_2$ then $M_1$ and $M_2$ are relatively cojective. Then $M$ is $X$-lifting.

**Proof.** Let $A \in B(M, X)$. Then $A$ has an $X$-supplement $M_2$ which is a direct summand of $M$, $M = M_1 \oplus M_2$. Then by hypothesis, $M_1$ is $B(M_2, X)$-cojective, since $M = A + M_2$, by [12, Proposition 3.2], we have $M = A' \oplus M_1' \oplus M_2' = A' + M_2$, $A' \leq A$, $M_1' \subseteq M_1$, $M_2' \subseteq M_2$. Then $A = A' + (A \cap M_2)$. Thus, since $A \cap M_2 \ll M_2$, Now from Lemma 2.19, $M$ is an $X$-lifting module. □

3. $X$-$H$-cofinitely supplemented

A submodule $N$ of $M$ is called cofinite in $M$ if the factor module $M/N$ is finitely generated. A module $M$ is called $H$-cofinitely supplemented if for every cofinite submodule $A$ of $M$, there exists a direct summand $D$ of $M$ such that $M = A + X$ holds if and only if $M = D + X$. Clearly $H$-supplemented modules are $H$-cofinitely supplemented. On the other hand, every finitely generated $H$-cofinitely supplemented module is $H$-supplemented.

We call $M$ is called $X$-$H$-cofinitely supplemented if for cofinite $A \in B(M, X)$ there exists a direct summand $D$ of $M$ such that $M = A + Y$ if and only if $M = D + Y$. $\mathbb{Z}$-module $\mathbb{Q}$ has no proper cofinite submodule, so it is $X$-$H$-cofinitely-supplemented. By definition every $X$-$H$-supplemented is $X$-$H$-cofinitely supple-
Theorem 3.3. \[ \text{second part is the same.} \]

The module \( M \) is called \textit{dual} module, if every submodule of \( M \) is fully invariant. \( M \) is called \textit{distributive} if \( N \cap (L + K) = (N \cap L) + (N \cap K) \) and \( N + (L \cap K) = (N + L) \cap (N + K) \) for every submodules \( N,K,L \) of \( M \).

Example 3.1. Let \( p \) be any prime number. Let \( M \) denote the \( \mathbb{Z} \)-module \( \mathbb{Q} \oplus (\mathbb{Z}/\mathbb{Z}p) \). Let \( L \) be any cofinite submodule of \( M \). Hence \( \mathbb{Q} / (\mathbb{Q} \cap L) \) is finitely generated. Thus \( \mathbb{Q} \leq L \). It follows that \( L = \mathbb{Q} \oplus L \cap (\mathbb{Z}/\mathbb{Z}p) \). Then \( L = \mathbb{Q} \) or \( L = M \).

So, \( M \) is \( H \)-cofinitely supplemented, then \( M \) is \( X-H \)-cofinitely supplemented.

Now we consider the \( X-H \)-cofinitely supplemented module;

Theorem 3.2. Let \( M \) be a module. The following are equivalent:

1. \( M \) is \( X-H \)-cofinitely supplemented module;
2. For each cofinite submodule \( Y \in B(M, X) \) there exists a direct summand \( D \) of \( M \) such that \( (Y + D)/D \ll M/D \) and \( (Y + D)/Y \ll M/Y \);
3. For each cofinite submodule \( Y \in B(M, X) \) there exists \( L \leq M \) and a direct summand \( D \) of \( M \) such that \( L/Y \ll M/Y, L/D \ll M/D \).

Proof. (1) \( \Rightarrow \) (2) Let \( Y \leq M \) be cofinite. By assumption there exists a direct summand \( D \) of \( M \) such that \( M = Y + L \) holds if and only if \( M = D + L \). Let \( (Y + D)/D \ll M/D \) for some submodule \( L \) of \( M \) containing \( D \). So, \( Y + L = M \) and hence \( D + L = M \), if follows that \( L = M \). Thus \( (Y + D)/D \ll M/D \). The second part is the same.

(2) \( \Rightarrow \) (3) Let \( Y \in B(M, X) \) be cofinite. Then there exists a direct summand \( D \) of \( M \) such that \( (Y + D)/D \ll M/D \) and \( (Y + D)/Y \ll M/Y \). Now take \( L = Y + D \).

(3) \( \Rightarrow \) (1) Let \( Y \in B(M, X) \) be cofinite. Then there exist a submodule \( L \) of \( M \) and a direct summand \( D \) of \( M \) such that both \( Y \) and \( D \) are cosmall submodules of \( L \) in \( M \). It is easy to see that \( M = A + D \) if and only if \( M = A + Y \) for all \( A \leq M \).

Thus \( M \) is \( X-H \)-cofinitely supplemented. \( \Box \)

Theorem 3.3.

1. Let \( M \) be an \( X-H \)-cofinitely supplemented module and \( L \) a submodule of \( M \). If for every direct summand \( K \) of \( M \), \( (L + K)/L \) is a direct summand of \( M/L \) then \( M/L \) is \( X-H \)-cofinitely supplemented.
2. Let \( M \) be an \( X-H \)-cofinitely supplemented module with the (SSP). Then every direct summand of \( M \) is \( X-H \)-cofinitely supplemented module.
3. Let \( M \) be an \( X-H \)-cofinitely supplemented distributive module. Then \( M/N \) is \( X-H \)-cofinitely supplemented for every submodule \( N \) of \( M \).

Proof. (1) Let \( N/L \in B(M/L, X) \) be cofinite where \( N \) is a cofinite submodule of \( M \) and \( L \subseteq N \), then by [8, Lemma 2.2], \( N \in B(M, X) \). Since \( M \) is \( X-H \)-cofinitely supplemented, for every cofinite \( N \in B(M, X) \), there exists a direct summand \( D \) of \( M \) such that \( M = N + Y \) if and only if \( M = D + Y \). By hypothesis,
\((D + L)/L\) is a direct summand of \(M/L\). Then \(M/L = N/L + A/L\) if and only if \(M/L = (D + L)/L + A/L\) for every \(A/L \leq M/L\) so \(M/L\) is \(X\)-\(H\)-cofinitely supplemented.

(2) Assume that \(M\) is \(X\)-\(H\)-cofinitely supplemented and \(M\) has the summand sum property. Let \(N\) be a direct summand of \(M\). We show that \(N\) is \(X\)-\(H\)-cofinitely supplemented. Let \(M = N \oplus K\) for some submodule \(K\) of \(M\). Assume that \(A\) is a direct summand of \(M\). Since \(M\) has the summand sum property, \(A + K\) is a direct summand of \(M\). Let \(M = (A + K) \oplus B\) for some submodule \(B\) of \(M\). Then \(M/K = (A + K)/K \oplus (B + K)/K\). Hence \(M/K\) is \(X\)-\(H\)-cofinitely supplemented by (1) and so \(N\) is \(X\)-\(H\)-cofinitely supplemented.

(3) Let \(D\) be a direct summand of \(M\). Then \(M = D \oplus D'\) for some submodule \(D'\) of \(M\). Now \(M/N = [(D + N)/N] \oplus [(D' + N)/N]\). Note that \(N = N + (D \cap D') = (N + D) \cap (N + D')\) by distributive of \(M\). So \(M/N = [(D + N)/N] \oplus [(D' + N)/N]\). By (1), \(M/N\) is \(X\)-\(H\)-cofinitely supplemented.

**Theorem 3.4.** Let \(M\) be a duo module. Then \(M\) has the \((SIP)\) and the \((SSP)\).

*Proof.* See [1, Theorem 3.5]. \(\Box\)

As a result of Theorem 3.3 and Theorem 3.4, we can obtain the following corollary:

**Corollary 3.5.** Let \(M\) be an \(X\)-\(H\)-cofinitely supplemented duo module. Then every direct summand of \(M\) is \(X\)-\(H\)-cofinitely supplemented module.

**Theorem 3.6.** Let \(M = M_1 \oplus M_2\) be a duo module and for any \(A \in B(M, X)\), \(M = A + M_i\) \((i = 1, 2)\). If \(M_1\) and \(M_2\) are \(X\)-\(H\)-cofinitely supplemented modules, then \(M\) is \(X\)-\(H\)-cofinitely supplemented.

*Proof.* Assume \(M_1\) and \(M_2\) are \(X\)-\(H\)-cofinitely supplemented modules. Let \(L \in B(M, X)\) be cofinite. \(L = (L \cap M_1) \oplus (L \cap M_2)\). Clearly, \(L \cap M_1\) and \(L \cap M_2\) are cofinite submodules of \(M_1\) and \(M_2\), then by [12, Lemma 3.1], \(L \cap M_1 \in B(M_1, X)\) and \(L \cap M_2 \in B(M_2, X)\). Since \(M_1, M_2\) are \(X\)-\(H\)-cofinitely supplemented, there exists a direct summand \(A_1, A_2\) of \(M_1, M_2\) such that \(M_1 = A_1 + Y\) if and only if \(M_1 = (L \cap M_1) + Y\) for any submodule \(Y\) of \(M_1\) that \(L \cap M_1 \in B(M_1, X)\) and also \(M_2 = A_2 + Y\) if and only if \(M_2 = (L \cap M_2) + Y\) for any submodule \(Y\) of \(M_2\) that \(L \cap M_2 \in B(M_2, X)\). It is clear until that show that \(M = (A_1 \oplus A_2) + Z\) if and only if \(M = L + Z\) for any submodule \(Z\) of \(M\). \(\Box\)

**Corollary 3.7.** Let \(M = \bigoplus_{i=1}^n M_i\) be a finite direct sum of duo modules and for any \(A \in B(M, X)\), \(M = A + M_i\) \((i = 1, ..., n)\). If every \(M_i\) is \(X\)-\(H\)-cofinitely supplemented modules, then \(M\) is \(X\)-\(H\)-cofinitely supplemented.

Finally, we get the following results as corollaries of Theorem 3.3, Corollary 3.5 and Corollary 3.7.

**Corollary 3.8** ([9, Theorem 2.1]). (1) Let \(M\) be an \(H\)-cofinitely supplemented module and \(L\) a submodule of \(M\). If for every direct summand \(K\) of \(M\), \((L + K)/L\) is
a direct summand of $M/L$ then $M/L$ is $H$-cofinitely supplemented.

(2) Let $M$ be an $H$-cofinitely supplemented module with the (SSP). Then every direct summand of $M$ is $H$-cofinitely supplemented module.

(3) Let $M$ be an $H$-cofinitely supplemented distributive module. Then $M/N$ is $H$-cofinitely supplemented for every submodule $N$ of $M$.

**Corollary 3.9** [9, Corollary 2.3]. Let $M$ be an $H$-cofinitely supplemented duo module. Then every direct summand of $M$ is $H$-cofinitely supplemented module.

**Corollary 3.10.** Let $M = \bigoplus_{i=1}^{n} M_i$ be a finite direct sum of duo modules. If every $M_i$ is $H$-cofinitely supplemented modules, then $M$ is $H$-cofinitely supplemented.

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**References**


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