Sequence Space $m(M, \phi)^F$ of Fuzzy Real Numbers Defined by Orlicz Functions with Fuzzy Metric

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Abstract. The sequence space $m(M, \phi)^F$ of fuzzy real numbers is introduced. Some properties of this sequence space like solidness, symmetricity, convergence-free etc. are studied. We obtain some inclusion relations involving this sequence space.

1. Introduction

The concept of fuzzy set theory was introduced by L.A. Zadeh in the year 1965. Later on different classes of sequences of fuzzy numbers have been investigated by Esi [2], Nuray and Savas [6], Syau [9], Tripathy and Baruah ([13], [14], [15]), Tripathy and Borgohain [16], Tripathy and Dutta ([17], [18]), Tripathy and Sarma [20] and many others.

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$.

If the convexity of the Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called as modulus function.

Remark. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

Sargent [8] introduced the crisp set sequence space $m(\phi)$ and studied some properties of this space. Later on it was studied from the sequence space point of view and some matrix classes were characterized with one member as $m(\phi)$ by Rath and Tripathy [7], Tripathy [10] and others. In this article we introduce the space $m(M, \phi)^F$ of fuzzy real numbers defined by Orlicz function.

Throughout the article $w^F$, $\ell^F$, $\ell_\infty^F$ represent the classes of all, absolutely...
2. Definitions and Background

**Definition 2.1.** A fuzzy real number $X$ is a fuzzy set on $\mathbb{R}$ i.e. a mapping $X : \mathbb{R} \rightarrow I(= [0, 1])$ associating each real number $t$ with its grade of membership $X(t)$.

**Definition 2.2.** A fuzzy real number $X$ is called convex if $X(t) \geq X(s) \land X(r) = \min(X(s), X(r))$, where $s < t < r$.

**Definition 2.3.** If there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$, then the fuzzy real number $X$ is called normal.

**Definition 2.4.** A fuzzy real number $X$ is said to be upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of $\mathbb{R}$.

The class of all upper semi-continuous, normal, convex fuzzy real numbers is denoted by $\mathbb{R}(I)$.

**Definition 2.5.** For $X \in \mathbb{R}(I)$, the $\alpha$-level set $X^\alpha$, for $0 < \alpha \leq 1$ is defined by, $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. The 0-level set of $X$ i.e. $X^0$ is the closure of strong 0-cut, i.e. $\text{cl}\{t \in R : X(t) > 0\}$.

**Definition 2.6.** The absolute value of $X \in \mathbb{R}(I)$ is defined by,

$$|X|(t) = \begin{cases} 
\max\{X(t), X(-t)\}, & \text{for } t \geq 0; \\
0 & \text{otherwise.}
\end{cases}$$

**Definition 2.7.** For $r \in \mathbb{R}$ and $\tau \in \mathbb{R}(I)$ is defined as,

$$\tau(t) = \begin{cases} 
1 & \text{if } t = r; \\
0 & \text{if } t \neq r.
\end{cases}$$

**Definition 2.8.** The additive and multiplicative identities of $\mathbb{R}(I)$ are denoted by $\overline{0}$ and $\overline{1}$.

**Definition 2.9.** Let $D$ be the set of all closed bounded intervals $X = [X^L, X^R]$. Define $d : D \times D \rightarrow \mathbb{R}$ by $d(X, Y) = \max\{|X^L - Y^L|, |X^R - Y^R|\}$. Then clearly $(D, d)$ is a complete metric space.

Define $\overline{d} : \mathbb{R}(I) \times \mathbb{R}(I) \rightarrow \mathbb{R}$ by $\overline{d}(X, Y) = \sup_{0<\alpha\leq1} d(X^\alpha, Y^\alpha)$, for $X, Y \in \mathbb{R}(I)$.

Then it is well known that $(\mathbb{R}(I), \overline{d})$ is a complete metric space.

**Definition 2.10.** A sequence $X = (X_k)$ of fuzzy real numbers is said to converge to the fuzzy number $X_0$, if for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\overline{d}(X_k, X_0) < \varepsilon$, for all $k \geq k_0$. 

summable and bounded sequences of fuzzy real numbers respectively.
Definition 2.11. A sequence space $E$ is said to be solid if $(Y_n) \in E$, whenever $(X_n) \in E$ and $|Y_n| \leq |X_n|$, for all $n \in N$.

Definition 2.12. Let $X = (X_n)$ be a sequence, then $S(X)$ denotes the set of all permutations of the elements of $(X_n)$ i.e. $S(X) = \{(X_{\pi(n)}) : \pi$ is a permutation of $N\}$. A sequence space $E$ is said to be symmetric if $S(X) \subseteq E$ for all $X \in E$.

Definition 2.13. A sequence space $E$ is said to be convergence-free if $(Y_n) \in E$ whenever $(X_n) \in E$ and $X_n = 0$ implies $Y_n = 0$.

Definition 2.14. A sequence space $E$ is said to be monotone if $E$ contains the canonical pre-images of all its step spaces.

Lemma 2.1. A sequence space $E$ is solid implies that $E$ is monotone.

Definition 2.15. Let $\mathcal{P}_s$ be the class of all subsets of $N$ those do not contain more than $s$ number of elements. Throughout $(\phi_n)$ is a non-decreasing sequence of positive real numbers such that $n\phi_{n+1} \leq (n + 1)\phi_n$ for all $n \in N$.

The space $m(\phi)$ introduced by Sargent [8] is defined by,

$$m(\phi) = \left\{ (x_k) \in w : \|x\|_{m(\phi)} = \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty \right\}.$$  

Afterwards different types of generalizations of the classes of sequences $m(\phi)$ was introduced and investigated by Rath and Tripathy [7], Tripathy ([10], [11]) and many others.

Definition 2.16. Lindenstrauss and Tzafriri [5] used the notion of Orlicz function and introduced the sequence space:

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right), \text{for some } \rho > 0 \right\}.$$  

The space $\ell_M$ with the norm,

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space, which is called an Orlicz sequence space. The space $\ell_M$ is closely related to the space $\ell_p$, which is an Orlicz sequence space with $M(x) = x^p$, for $1 \leq p \leq \infty$. In the later stage different classes of Orlicz sequence spaces were introduced and studied by Altin, Et and Tripathy [1], Esi [2], Tripathy, Altin and Et [12], Tripathy and Mahanta [19], Tripathy and Sarma ([20], [21]) and many others.
**Definition 2.17.** Let $d_F : R(I) \times R(I) \rightarrow R(I)$ be the fuzzy metric. Let the mappings $L, M : [0,1] \times [0,1] \rightarrow [0,1]$ be symmetric, non-decreasing in both arguments and satisfy, $L[0,0] = 0$ and $M[1,1] = 1$. i.e. $L = \min\{p,q\}$ and $M = \max\{p,q\}$, where $p,q \in [0,1]$.

Let $\lambda : R(I) \times R(I) \rightarrow R$ such that $\lambda(X,Y) = \sup_{0< \alpha \leq 1} \lambda_\alpha(X^\alpha, Y^\alpha)$, where $\lambda_\alpha : R \times R \rightarrow R$ and $\lambda_\alpha(X^\alpha, Y^\alpha) = \min\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}$.

Similarly, let $\rho : R(I) \times R(I) \rightarrow R$ be such that $\rho(X,Y) = \sup_{0< \alpha \leq 1} \rho_\alpha(X^\alpha, Y^\alpha)$, where $\rho_\alpha : R \times R \rightarrow R$ and $\rho_\alpha(X^\alpha, Y^\alpha) = \max\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}$.

Since the distance between two fuzzy numbers is again a fuzzy number, so the $\alpha$-level set of this distance $d_F$ between the fuzzy real numbers $X$ and $Y$ is denoted by,

$$[d(X,Y)]_\alpha = [\lambda_\alpha(X^\alpha, Y^\alpha), \rho_\alpha(X^\alpha, Y^\alpha)], 0 < \alpha \leq 1.$$ 

The quadruple $(R(I), d_F, M, N)$ is called a fuzzy metric space and $d_F$ is a fuzzy metric, if,

1. $d_F(X,Y) = \overline{0}$ if and only if $X = Y$.
2. $d_F(X,Y) = d_F(Y,X)$, for all $X,Y \in R(I)$.
3. For all $X,Y,Z \in R(I)$,
   (i) $d_F(X,Y)(s + t) \geq L(d_F(X,Z)(s), d_F(Z,Y)(t))$, whenever $s \leq \lambda_1(X,Z)$, $t \leq \lambda_1(Z,Y)$ and $s + t \leq \lambda_1(X,Y)$.
   (ii) $d_F(X,Y)(s + t) \leq R(d_F(X,Z)(s), d_F(Z,Y)(t))$, whenever $s \geq \lambda_1(X,Z)$, $t \geq \lambda_1(Z,Y)$ and $s + t \geq \lambda_1(X,Y)$.

Using the concept of Orlicz function and fuzzy metric, we introduce the following sequence spaces,

$$m(M, \phi)^F$$

$$= \left\{ (X_k) \in w^F : \sup_{s \geq 1, \sigma \in \rho_\phi} \frac{1}{\sigma} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, \overline{0})}{r} \right) : \sup_{s \geq 1, \sigma \in \rho_\phi} \frac{1}{\sigma} \sum_{k \in \sigma} M \left( \frac{\rho(X_k, \overline{0})}{r} \right) \right\},$$

for all $r > 0$

3. **Main Results**

**Theorem 3.1.** The sequence space $m(M, \phi)^F$ is a metric space with the metric defined by,

$$\overline{d}(X,Y)_M$$

$$= \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \rho_\phi} \frac{1}{\sigma} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, \overline{0})}{r} \right) \leq 1 ; \sup_{s \geq 1, \sigma \in \rho_\phi} \frac{1}{\sigma} \sum_{k \in \sigma} M \left( \frac{\rho(X_k, \overline{0})}{r} \right) \leq 1 \right\},$$

where $\lambda$ and $\rho$ are defined above.
for $X,Y \in m(M, \phi)^F$

Proof. Let $X,Y,Z \in m(M, \phi)^F$.

(i) $d(X,Y)_M = 0$.

This implies,

$$\lambda(X_k,Y_k) = 0 \quad \text{and} \quad \rho(X_k,Y_k) = 0, \quad \text{for all } k \in N. (\text{Since } M(0) = 0).$$

Which implies,

$$\sup_{0 < \alpha \leq 1} \lambda(a(X_k^\alpha,Y_k^\alpha)) = 0 \Rightarrow \lambda(a(X_k^\alpha,Y_k^\alpha)) = 0, \quad \text{for all } \alpha \in (0,1].$$

(3.1) $$\Rightarrow \min\{|X_{k1}^a - Y_{k1}^a|, |X_{k2}^a - Y_{k2}^a|\} = 0, \quad \text{for all } \alpha \in (0,1].$$

$$\sup_{0 < \alpha \leq 1} \rho(a(X_k^\alpha,Y_k^\alpha)) = 0 \Rightarrow \rho(a(X_k^\alpha,Y_k^\alpha)) = 0, \quad \text{for all } \alpha \in (0,1].$$

(3.2) $$\Rightarrow \max\{|X_{k1}^a - Y_{k1}^a|, |X_{k2}^a - Y_{k2}^a|\} = 0, \quad \text{for all } \alpha \in (0,1].$$

From (3.1) and (3.2), it follows that, $X_k = Y_k \Rightarrow X = Y$.

Conversely, assume that, $X = Y$. Then, using the definition of $\lambda$ and $\rho$, we get,

$$\lambda(a(X_k^\alpha,Y_k^\alpha)) = 0 \quad \text{and} \quad \rho(a(X_k^\alpha,Y_k^\alpha)) = 0, \quad \text{for all } k \in N, \alpha \in (0,1].$$

Which implies,

$$\sup_{0 < \alpha \leq 1} \lambda(a(X_k^\alpha,Y_k^\alpha)) = 0 \quad \text{and} \quad \sup_{0 < \alpha \leq 1} \rho(a(X_k^\alpha,Y_k^\alpha)) = 0, \quad \text{for all } k \in N.$$

It follows that, $\lambda(X_k,Y_k) = 0$ and $\rho(X_k,Y_k) = 0$.

Using the continuity of $M$, we get, $d(X,Y)_M = 0$. Which shows that, $d(X,Y)_M = 0$ if and only if $X = Y$.

(ii) $d(X,Y)_M$

$$= \inf \left\{ \varphi > 0 : \sup_{s \geq 1, \sigma \in \nu} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k,Y_k)}{\varphi} \right) \leq 1; \sup_{s \geq 1, \sigma \in \nu} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\rho(X_k,Y_k)}{\varphi} \right) \leq 1 \right\}.$$

From the definition of $\lambda$, it follows,

$$\lambda(X_k,Y_k) = \sup_{0 < \alpha \leq 1} \lambda(a(X_k^\alpha,Y_k^\alpha))$$

$$= \sup_{0 < \alpha \leq 1} \min\{|X_{k1}^a - Y_{k1}^a|, |X_{k2}^a - Y_{k2}^a|\}$$

$$= \sup_{0 < \alpha \leq 1} \min\{|Y_{k1}^a - X_{k1}^a|, |Y_{k2}^a - X_{k2}^a|\}$$

$$= \sup_{0 < \alpha \leq 1} \lambda(a(Y_k^\alpha,X_k^\alpha))$$

$$= \lambda(Y_k,X_k).$$
Proceeding in the same way, we get, $\rho(X_k, Y_k) = \rho(Y_k, X_k)$. Thus we get,

$$\inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, Y_k)}{r} \right) \leq 1; \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\rho(X_k, Y_k)}{r} \right) \leq 1 \right\}$$

$$= \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(Y_k, X_k)}{r} \right) \leq 1; \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\rho(Y_k, X_k)}{r} \right) \leq 1 \right\}$$

$$= \overline{d}(Y, X)_M$. Hence, $\overline{d}(X, Y)_M = \overline{d}(Y, X)_M$.

(iii) Let $r_1, r_2 > 0$ such that,

$$\sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, Z_k)}{r_1} \right) \leq 1.$$

$$\sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(Z_k, Y_k)}{r_2} \right) \leq 1.$$

Let $r = r_1 + r_2$, then we have,

$$\sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, Y_k)}{r} \right)$$

$$\leq \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, Z_k)}{r_1 + r_2} + \frac{\lambda(Z_k, Y_k)}{r_1 + r_2} \right)$$

$$\leq \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, Z_k)}{r_1 + r_2} + \frac{r_1}{r_1 + r_2} \left( \frac{\lambda(X_k, Z_k)}{r_1} \right) + \frac{r_2}{r_1 + r_2} \left( \frac{\lambda(Z_k, Y_k)}{r_2} \right) \right)$$

$$\leq \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, Z_k)}{r_1} \right) + \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(Z_k, Y_k)}{r_2} \right)$$

$$\leq 1.$$
Sequence Space of Fuzzy Real Numbers Defined by Orlicz Functions with Fuzzy Metric

\[ \leq \inf \left\{ r_1 > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \phi_s \sum_{k \in \sigma} M \left( \frac{\rho(X_k, Z_k)}{r_1} \right) \leq 1 \right\} \]

\[ + \inf \left\{ r_2 > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \phi_s \sum_{k \in \sigma} M \left( \frac{\rho(Z_k, Y_k)}{r_2} \right) \leq 1 \right\} \]

Thus we have,

\[ \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \phi_s \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, Y_k)}{r} \right) \leq 1; \right\} \]

\[ \leq \inf \left\{ r_1 > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \phi_s \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, Z_k)}{r_1} \right) \leq 1; \right\} \]

\[ + \inf \left\{ r_2 > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \phi_s \sum_{k \in \sigma} M \left( \frac{\lambda(Z_k, Y_k)}{r_2} \right) \leq 1; \right\} \]

\[ \Rightarrow \delta(X, Y)_M \leq \delta(X, Z)_M + \delta(Z, Y)_M. \]

This proves that \( m(M, \phi)^F \) is a metric space.

**Theorem 3.2.** The sequence space \( m(M, \phi)^F \) is a complete metric space with the metric defined by,

\[ \delta(X, Y)_M = \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \phi_s \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, Y_k)}{r} \right) \leq 1; \right\} \]

for \( X, Y \in m(M, \phi)^F \)

**Proof.** Let \( (X^{(i)}) \) be a Cauchy sequence in \( m(M, \phi)^F \) such that, \( X^{(i)} = (X_n^{(i)})_{n=1}^{\infty} \).

Let \( \varepsilon > 0 \) be given. For a fixed \( x_0 > 0 \), choose \( p > 0 \) such that \( M \left( \frac{px_0}{2} \right) \geq 1. \)

Then there exists a positive integer \( n_0 = n_0(\varepsilon) \) such that,

\[ \delta(X^{(i)}, X^{(j)})_M < \frac{\varepsilon}{px_0}, \text{ for all } i, j \geq n_0. \]
By the definition of $d_M$, we get;

\[
\begin{align*}
\inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq 1 ; \\
\sup_{s \geq 1, \sigma \in \wp_s} \sum_{k \in \sigma} M \left( \frac{\rho(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq 1 \right\} < \varepsilon
\end{align*}
\]

for all $i, j \geq n_0$. Which implies,

\[
\sup_{s \geq 1, \sigma \in \wp_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq 1
\]

\[
\sup_{s \geq 1, \sigma \in \wp_s} \sum_{k \in \sigma} M \left( \frac{\rho(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq 1
\]

From (3.4) we get,

\[
\sup_{s \geq 1, \sigma \in \wp_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq 1
\]

On taking $s = 1$ and varying $\sigma$ over $\wp_s$, we get,

\[
\sum_{k \in \sigma} M \left( \frac{\lambda(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq \phi_1, \text{ for all } i, j \geq n_0.
\]

\[
\Rightarrow M \left( \frac{\lambda(X_k^{(i)}, X_k^{(j)})}{d(X(i), X(j))} \right) \leq \phi_1 \leq M \left( \frac{px_0}{2} \right).
\]

Using the continuity of $M$, we get,

\[
\lambda_\alpha(X_k^{(i)}, X_k^{(j)}) \leq \frac{px_0}{2} \cdot \frac{\varepsilon}{px_0} = \frac{\varepsilon}{2},
\]

i.e $(X_k^{(i)})$ is a Cauchy sequence of $R(I)$. Since $R(I)$ is complete, so it follows that, $(X_k^{(i)})$ is also convergent.

Let, $\lim_{i} X_k^{(i)} = X_k$, for each $k \in N$. We have to prove that,

\[
\lim_{i} X^{(i)} = X \text{ and } X \in m(M, \phi)^	ext{F}.
\]

Since $M$ is continuous, so on taking $j \to \infty$ and fixing $i$, we get from (3.4);
Proceeding in the same way, we get from (3.5):
\[ \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\rho(X^{(i)}_k, X_k)}{r} \right) \leq 1, \text{ for some } r > 0 \text{ and } i \geq n_0. \]

Now on taking the infimum of such \( r \)'s together, we get from (3.3):
\[ \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X^{(i)}_k, X_k)}{r} \right) \right\} < \varepsilon, \]
for some \( r > 0 \) and \( i \geq n_0 \). Which shows, \( d(X^{(i)}_X, X) < \varepsilon \), for all \( i \geq n_0 \). i.e.
\[ \lim_{i \to \infty} X^{(i)}_X = X. \]

Now, to show that \( X \in m(M, \phi)^F \). We have,
\[ d(X, \theta) \leq d(X, X^{(i)}) + d(X^{(i)}, \theta) < \varepsilon + M, \text{ for all } i \geq n_0(\varepsilon). \]

i.e. \( d(X, \theta)_M \) is finite. Which implies \( x \in m(M, \phi)^F \). Hence \( m(M, \phi)^F \) is a complete metric space. This completes the proof of the theorem. Proofs are similar for other spaces also.

\textbf{Theorem 3.3.} The sequence space \( m(M, \phi)^F \) is solid.

\textit{Proof.} Let \( (X_k) \in m(M, \phi)^F \). Then we have, for some \( r > 0 \),
\[ \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, \theta)}{r} \right) < \infty; \quad \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\rho(X_k, \theta)}{r} \right) < \infty. \]

Let \( (Y_k) \) be a sequence of fuzzy numbers with,
\[ |d(Y_k, \theta)|_\alpha = [\lambda_\alpha(Y^\alpha_k, 0), \rho_\alpha(Y^\alpha_k, o)], \text{ for } 0 < \alpha \leq 1, \]
Such that, \( \lambda(Y_k, \theta) \leq \lambda(X_k, \theta) \) and \( \rho(Y_k, \theta) \leq \rho(X_k, \theta) \).

Since \( M \) is non-decreasing continuous function, so we get, for some \( r > 0 \),
\[ M \left( \frac{\lambda(Y_k, \theta)}{r} \right) \leq M \left( \frac{\lambda(X_k, \theta)}{r} \right) \text{ and } M \left( \frac{\rho(Y_k, \theta)}{r} \right) \leq M \left( \frac{\rho(X_k, \theta)}{r} \right). \]

Which implies,
\[ \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} M \left( \frac{\lambda(Y_k, \theta)}{r} \right) \leq \sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} M \left( \frac{\lambda(X_k, \theta)}{r} \right) < \infty, \text{ for some } r > 0. \]
\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left( \frac{\rho(Y_k, \bar{0})}{r} \right) \leq \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left( \frac{\rho(X_k, \bar{0})}{r} \right) < \infty, \text{ for some } r > 0.
\]

Which implies,
\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left( \frac{\lambda(Y_k, \bar{0})}{r} \right) < \infty, \text{ for some } r > 0.
\]
\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left( \frac{\rho(Y_k, \bar{0})}{r} \right) < \infty, \text{ for some } r > 0.
\]

Which shows, \((Y_k) \in m(M, \phi)^F\). Hence, \(m(M, \phi)^F\) is solid. This completes the proof.

\textbf{Theorem 3.4.} The sequence space \(m(M, \phi)^F\) is symmetric.

\textit{Proof.} Let \((X_k) \in m(M, \phi)^F\) and \((Y_k)\) be a rearrangement of \((X_k)\), such that,

\[X_k = Y_{m_k}, \text{ for each } k \in \mathbb{N}.\]

Then, we have, \(\lambda(X_k, \bar{0}) = \lambda(Y_{m_k}, \bar{0})\) and \(\rho(X_k, \bar{0}) = \rho(Y_{m_k}, \bar{0})\).

Using the continuity of \(M\), we get,
\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left( \frac{\lambda(Y_k, \bar{0})}{r} \right) = \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left( \frac{\lambda(Y_{m_k}, \bar{0})}{r} \right), \text{ for some } r > 0.
\]
\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left( \frac{\rho(X_k, \bar{0})}{r} \right) = \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left( \frac{\rho(Y_{m_k}, \bar{0})}{r} \right), \text{ for some } r > 0.
\]

Which implies,
\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left( \frac{\lambda(Y_{m_k}, \bar{0})}{r} \right) < \infty \text{ and } \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left( \frac{\rho(Y_{m_k}, \bar{0})}{r} \right) < \infty,
\]
for some \(r > 0\). Which shows, \((Y_k) \in m(M, \phi)^F\). Hence \(m(M, \phi)^F\) is symmetric. This completes the proof. \(\square\)

\textbf{Proposition 3.1.} The sequence space \(m(M, \phi)^F\) is not convergence-free.

\textit{Proof.} The result follows from the following example.

\textbf{Example 3.1.} Consider the sequence \((X_k)\) defined as follows:

\[X_k(t) = \begin{cases} 
1 + kt, & \text{ for } t \in [-\frac{1}{2}, 0] \\
1 - kt, & \text{ for } t \in [0, \frac{1}{2}] \\
0 & \text{ otherwise}
\end{cases}\]

Then we have, for some \(r > 0\),
\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, \bar{0})}{r} \right) < \infty
\]
and

\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\rho(X_k, \bar{0})}{r} \right) < \infty
\]

Which shows, \((X_k) \in m(M, \phi)^F\).

Now, let us take another sequence \((Y_k)\) such that,

\[
Y_k(t) = \begin{cases} 
1 + \frac{t}{k^2}, & \text{for } t \in [-k^2, 0] \\
1 - \frac{t}{k^2}, & \text{for } t \in [0, k^2]
\end{cases}
\]

But,

\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(Y_k, \bar{0})}{r} \right) = \infty \quad \text{and} \quad \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\rho(Y_k, \bar{0})}{r} \right) = \infty
\]

Thus, \((Y_k) \notin m(M, \phi)^F\). Thus \(m(M, \phi)^F\) is not convergence-free. This completes the proof.

\[\square\]

**Proposition 3.2.** \(m(M, \phi)^F \subseteq m(M, \phi, p)^F\), for \(1 \leq p < \infty\).

**Proof.** Let \(X \in m(M, \phi)^F\), then we have, for some \(r > 0\),

\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, \bar{0})}{r} \right) = K(< \infty)
\]

Hence, for each fixed \(s\), we have,

\[
\sum_{k \in \sigma} M \left( \frac{\lambda(X_k, \bar{0})}{r} \right) \leq K \phi_s, \quad \text{for } \sigma \in \wp_s.
\]

\[
\Rightarrow \left[ \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, \bar{0})}{r} \right) \right]^{\frac{1}{p}} \leq K^{\frac{1}{p}} \phi_s, \quad \text{for } \sigma \in \wp_s.
\]

\[
\Rightarrow \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \left[ \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, \bar{0})}{r} \right) \right]^{\frac{1}{p}} \leq K.
\]

\[
\Rightarrow \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \left[ \sum_{k \in \sigma} M \left( \frac{\lambda(X_k, \bar{0})}{r} \right) \right]^{\frac{1}{p}} < \infty.
\]

Proceeding in the same way, we get,

\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \left[ \sum_{k \in \sigma} M \left( \frac{\rho(X_k, \bar{0})}{r} \right) \right]^{\frac{1}{p}} < \infty.
\]

Which implies \(X \in m(M, \phi, p)^F\), for \(1 \leq p < \infty\).
This completes the proof. \qed

**Proposition 3.3.** \(m(M, \phi)^F \subseteq m(M, \psi)^F\) if and only if \(\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s}\right) < \infty\), for \(0 < p < \infty\).

**Proof.** Suppose, \(\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s}\right) = K(< \infty)\), then we have, \(\phi_s \leq K \psi_s\).

Now, if \((X_k) \in m(M, \phi)^F\), then,

\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r}\right) < \infty.
\]

\(\Rightarrow \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{K \psi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r}\right) \leq \infty.
\]

\(\Rightarrow (X_k) \in m(M, \psi)^F\).

Hence, \(m(M, \phi)^F \subseteq m(M, \psi)^F\).

Conversely, suppose that \(m(M, \phi)^F \subseteq m(M, \psi)^F\). To show that, \(\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s}\right) = \sup_{s \geq 1} (\eta_s) < \infty\).

Suppose, \(\sup_{s \geq 1} (\eta_s) = \infty\). Then there exists a subsequence \((\eta_{s_i})\) of \((\eta_s)\) such that,

\[
\lim_{i \to \infty} (\eta_{s_i}) = \infty.
\]

Then for \((X_k) \in m(M, \phi)^F\), we have,

\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\psi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r}\right) \geq \sup_{s_i \geq 1, \sigma \in \wp_{s_i}} \frac{\eta_{s_i}}{\phi_{s_i}} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r}\right) = \infty.
\]

\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\psi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r}\right) = \infty.
\]

Proceeding in the same way, we get,

\[
\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\psi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, \bar{0})}{r}\right) = \infty.
\]

Which implies that \((X_k) \notin m(M, \psi)^F\), a contradiction. This completes the proof. \(\square \)

**Corollary 3.1.** \(m(M, \phi)^F = m(M, \psi)^F\) if and only if \(\sup_{s \geq 1} (\eta_s) < \infty\) and \(\sup_{s \geq 1} (\eta^{-1}_s) < \infty\), where \(\eta_s = \frac{\phi_s}{\psi_s}\).
Theorem 3.5. \( \ell_p(M)^F \subseteq m(M, \phi, p)^F \subseteq \ell_\infty(M)^F \), for \( 1 \leq p < \infty \).

Proof. On taking \( M(x) = x^p \), for \( 1 \leq p < \infty \) and \( \phi_n = 1 \), for all \( n \in N \). We get, \( m(M, \phi, p)^F = \ell_p(M)^F \). So, the first inclusion is clear.

Next, suppose that, \( (X_k) \in m(M, \phi, p)^F \) that implies that,

\[
\sup_{s \geq 1, \sigma \in \mathcal{P}_s} \frac{1}{\phi_s} \left( \sum_{k \in \sigma} \left( M \left( \frac{\lambda(X_k, \overline{0})}{r} \right) \right)^p \right)^{\frac{1}{p}} = K(< \infty).
\]

For, \( s = 1, M \left( \frac{\lambda(X_k, \overline{0})}{r} \right) \leq K\phi_1, k \in \sigma \). Which implies that, \( \sup_{k \geq 1} \left( M \left( \frac{\lambda(X_k, \overline{0})}{r} \right) \right) < \infty \).

Following the same way, we get,

\[
\sup_{k \geq 1} M \left( \frac{\lambda(X_k, \overline{0})}{r} \right) < \infty.
\]

Which implies, \( (X_k) \in \ell_\infty(M)^F \). This completes the proof. \( \Box \)

Proposition 3.4. \( m(M, \phi, p)^F = \ell_p(M)^F \) if and only if \( \sup_{s \geq 1} \phi_s < \infty \) and \( \sup_{s \geq 1} (\phi_s^{-1}) < \infty \).

Putting \( \psi_n = 1 \), for all \( n \in N \), in Corollary 3.1, we get,

Corollary 3.2. \( m(M, \phi, p)^F = \ell_\infty(M)^F \) if \( \lim_{s \to \infty} \left( \frac{\phi_s}{s} \right) > 0 \), for \( 0 < p < \infty \).

References


