Differential Subordination Properties of Sokół-Stankiewicz Starlike Functions

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Abstract. Let $p(z)$ be an analytic function defined on the open unit disk $D$ and $p(0) = 1$. Condition $\beta$ in terms of complex numbers $D$ and real $E$ with $-1 < E < 1$ and $|D| \leq 1$ is determined such that $1 + \beta z p'(z) < 1 + D z + Ez$ implies $p(z) < \sqrt{1 + z}$. Furthermore, the expression $1 + \frac{\beta z p'(z)}{p(z)}$ and $1 + \frac{\beta z p'(z)}{p^2(z)}$ are considered in obtaining similar results.

1. Introduction

Let $A$ denote the class of all analytic functions $f$ in the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ and normalised by $f(0) = 0, f'(0) = 1$. An analytic function $f$ is subordinate to an analytic function $g$, written $f(z) \prec g(z)$ ($z \in D$), if there exists an analytic function $w$ in $D$ such that $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. In particular, if $g$ is univalent in $D$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$.

Sokół and Stankiewicz [6] introduced the class $SL^*$ consisting of normalised analytic functions $f$ in $D$ satisfying the condition $\left| \left[ \frac{zf'(z)}{f(z)} \right]^2 - 1 \right| < 1, z \in D$. Geometrically, a function $f \in SL^*$ if $\frac{zf'(z)}{f(z)}$ is in the interior of the right half of the lemniscate of Bernoulli $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$. A function in the class $SL^*$ is

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called Sokól-Stankiewicz starlike function. Alternatively, we can also write
\[ f \in SL^* \iff \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}. \]

Properties of functions in \( SL^* \) have intensively been studied by authors in [4], [7], [8], [9] and [10].

Next, we denote \( S^*[A, B] \) as the class of Janowski starlike functions introduced by Janowski [1] and it consists of functions \( f \in A \) satisfying
\[ \frac{zf'(z)}{f(z)} \prec 1 + Az \frac{1 + Bz}{1 - 1/B} \quad (-1 \leq B < A \leq 1). \]

For analytic function \( p(z) \) in \( D \) with \( p(0) = 1 \), Nunokawa et. al. [3] investigated and established the relation \( 1 + zp'(z) \prec 1 + z \) implies \( p(z) \prec 1 + z \). Ali et. al. [5] extended this result and obtained conditions for which \( 1 + zp'(z) \prec \frac{1 + Dz}{1 + Ez} \) implies \( p(z) \prec \sqrt{1+z} \). Recently, in [4], condition for which \( 1 + zp'(z) \prec \sqrt{1+z} \) were determined. Motivated by these studies, this paper considers ascertaining condition so that \( 1 + zp'(z) \prec \frac{1 + Dz}{1 + Ez} \) implies \( p(z) \prec \sqrt{1+z} \). Other results involving the expression \( 1 + \frac{\beta zp'(z)}{p(z)} \) and \( 1 + \frac{\beta zp'(z)}{p^2(z)} \) were also looked at.

2. Main Results

In proving our results, the following lemma proved by Miller and Mocanu is used.

**Lemma 2.1** ([2], p. 135. Let \( q \) be univalent in \( D \) and let \( \varphi \) be analytic in a domain containing \( q(D) \). Let \( zq'(z)\varphi[q(z)] \) be starlike. If \( p \) is analytic in \( D, p(0) = q(0) \) and satisfies \( zp'(z)\varphi[p(z)] \prec zq'(z)\varphi[q(z)] \) then \( p \prec q \) and \( q \) is the best dominant.

Our first result is as follows:

**Theorem 2.1.** Let \( p \) be an analytic function on \( D \) and \( p(0) = 1 \).

Let \( \beta \geq \beta_0, \beta_0 = \frac{2\sqrt{|D-E|}}{(1-|E|)} \) where \(-1 < E < 1 \) and \( |D| \leq 1 \).

If
\[ 1 + \beta zp'(z) \prec \frac{1 + Dz}{1 + Ez} , \]
then
\[ p(z) \prec \sqrt{1+z} . \]

**Proof.** Let \( q(z) = \sqrt{1+z} \) with \( q(0) = 1, q : D \to C \). Since \( q(D) \) is a convex set thus \( q \) is a convex function which implies \( zq'(z) \) is starlike with respect to 0.
Lemma 2.1 suggests

\[ 1 + \beta z p'(z) \prec 1 + \beta z q'(z) \Rightarrow p(z) \prec q(z), \]

so to prove our result, it is suffice to show

\[ s(z) = \frac{1 + Dz}{1 + Ez} \prec 1 + \beta z q'(z) = 1 + \frac{\beta z}{2\sqrt{1 + z}} = h(z). \]

Since \( s^{-1}(w) = \frac{w - 1}{D - Ez} \), then

\[ s^{-1}[h(z)] = \frac{\beta z}{2\sqrt{1 + z}(D - E) - \beta Ez}. \]

For \( z = e^{i\theta}, \theta \in [-\pi, \pi] \),

\[ |s^{-1}[h(z)]| = |s^{-1}[h(e^{i\theta})]| \]

\[ = \frac{\beta}{|2\sqrt{1 + e^{i\theta}}(D - E) - \beta E e^{i\theta}|} \]

\[ \geq \frac{\beta}{2\sqrt{1 + e^{i\theta}}|D - E| + \beta |E|} \]

\[ = \frac{\beta}{2 \sqrt{2|\cos \frac{\theta}{2}|(D - E)| + \beta |E|}} \]

It can be shown that the above expression is minimum when \( \theta = 0 \).

Thus

\[ |s^{-1}[h(z)]| \geq \frac{\beta}{2\sqrt{|D - E|} + |E|} \geq 1 \]

for \( \beta \geq \frac{2\sqrt{|D - E|}}{1 + |E|} \). Therefore \( D \subset s^{-1}[h(D)] \) or \( s(D) \subset h(D) \) implies \( s(z) \prec h(z) \).

Hence, the result is proven. \( \square \)

**Corollary 2.1.** Let \( \beta \geq \beta_0 \), \( \beta_0 = \frac{2\sqrt{|D - E|}}{|1 - |E||} \) where \(-1 < E < 1, \ |D| \leq 1, \) and \( f \in A \).

i) If \( f \) satisfies the following

\[ 1 + \beta \frac{zf''(z)}{f(z)} \left( \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)} + 1 \right) \prec \frac{1 + Dz}{1 + Ez} \]
then \( f \in SL^\star \).

ii) If \( 1 + \beta zf''(z) < \frac{1 + Dz}{1 + Ez} \) then \( f'(z) < \sqrt{1 + z} \).

**Proof.** Define \( p(z) = \frac{zf'(z)}{f(z)} \) and using Theorem 2.1, the first part of Corollary 2.1 is proved. The second part of our results in Corollary 2.1 can be derived by letting \( p(z) = f'(z) \).

**Theorem 2.2.** Let \( p \) be an analytic function in \( D \) and \( p(0) = 1 \). Let \( \beta \geq \beta_0, \beta_0 = \frac{4|D - E|}{(1 - |E|)} \), \(-1 < E < 1 \) and \( |D| \leq 1 \).

\[
1 + \beta \frac{zp'(z)}{p(z)} \prec 1 + \frac{Dz}{1 + Ez} \Rightarrow p(z) \prec \sqrt{1 + z}.
\]

**Proof.** Let \( q(z) = \sqrt{1 + z} \), \( q(0) = 1 \). Elementary calculation will show that \( \frac{\beta zq'(z)}{q(z)} = \frac{\beta z}{2(1 + z)} \) is starlike. Thus, Lemma 2.1 can be applied as

\[
1 + \beta \frac{zp'(z)}{p(z)} \prec 1 + \beta \frac{zq'(z)}{q(z)} \Rightarrow p(z) \prec q(z).
\]

Next, we prove the subordination

\[
s(z) = \frac{1 + Dz}{1 + Ez} \prec 1 + \beta \frac{zq'(z)}{q(z)} = 1 + \frac{\beta z}{2(1 + z)} = h(z).
\]

\[
s^{-1}[h(z)] = \frac{\beta z}{2(1 + z)(D - E) - \beta Ez}.
\]

For \( z = e^{i\theta}, \theta \in [-\pi, \pi] \),

\[
|s^{-1}[h(z)]| = |s^{-1}[h(e^{i\theta})]| = \frac{\beta}{|2(1 + e^{i\theta})(D - E) - \beta Ee^{i\theta}|} \geq \frac{\beta}{|2(1 + e^{i\theta})||D - E| + \beta|E|} = \frac{4|\cos \frac{\theta}{2}||D - E| + \beta|E|}{\beta}.
\]

A straight forward computation verifies that the above expression is minimum when \( \theta = 0 \).

Then

\[
|s^{-1}[h(z)]| \geq \frac{\beta}{4|D - E| + \beta|E|} \geq 1
\]

for \( \beta \geq \frac{4|D - E|}{(1 - |E|)} \). Hence \( s(D) \subset h(D) \) implies \( s(z) \prec h(z) \). \( \square \)
Corollary 2.2. Let $\beta \geq \beta_0$, $\beta_0 = \frac{4|D-E|}{|1-E|}$, $-1 < E < 1$ and $|D| \leq 1$.

i) $1 + \beta \left[ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right] < 1 + \frac{Dz}{1+Ez} \Rightarrow f \in SL^*$.

ii) $1 + \beta \left[ \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right] < 1 + \frac{Dz}{1+Ez} \Rightarrow \frac{z^2f'(z)}{f^2(z)} < \sqrt{1+z}$.

Proof. Letting $p(z) = \frac{zf'(z)}{f(z)}$ in (i) and $p(z) = \frac{z^2f'(z)}{f^2(z)}$ in (ii) and applying Theorem 2.2 proves the results. \qed

Theorem 2.3. Let $\beta \geq \beta_0$, $\beta_0 = \frac{4\sqrt{2}|D-E|}{|1-E|}$, $-1 < E < 1$ and $|D| \leq 1$.

$1 + \beta \frac{zp'(z)}{p^2(z)} < 1 + \frac{Dz}{1+Ez} \Rightarrow p(z) < \sqrt{1+z}$.

Proof. Let $q(z) = \sqrt{1+z}$, which implies $\frac{q(z)}{q(z)}$ is starlike. Using Lemma 2.1,

$1 + \beta \frac{zp'(z)}{p^2(z)} < 1 + \beta \frac{zq'(z)}{q^2(z)} \Rightarrow p(z) < q(z)$.

Next, let $h(z) = 1 + \beta \frac{zq'(z)}{q^2(z)} = 1 + \frac{\beta z}{2(1+z)^2}$

$s^{-1}[h(z)] = \frac{\beta z}{2(1+z)^2(D-E) - \beta Ez}$.

For $z = e^{i\theta}, \theta \in [-\pi, \pi]$,

$|s^{-1}[h(z)]| = |s^{-1}[h(e^{i\theta})]|$

$\geq \frac{\beta}{|2(1+e^{i\theta})^2(D-E) - \beta E e^{i\theta}|}$

$\geq \frac{\beta}{|2(1+e^{i\theta})^2||(D-E)| + \beta |E|}$

As in previous case, the above expression is minimum when $\theta = 0$.

Then

$|s^{-1}[h(z)]| \geq \frac{\beta}{4\sqrt{2}||D-E|| + \beta |E|} \geq 1$.
for \( \beta \geq \frac{4\sqrt{2}|D-E|}{(1-|E|)} \). Hence \( D \subset s^{-1}[h(D)] \) implies \( s(z) \prec h(z) \).

**Corollary 2.3.** Let \( \beta \geq \beta_0, \beta_0 = \frac{4\sqrt{2}|D-E|}{(1-|E|)} \), \( -1 < E < 1, |D| \leq 1 \) and \( f \in A \),

\[
1 - \beta + \beta \left[ \frac{1 + zf''(z)}{f'(z)} \right] \prec \frac{1 + Dz}{1 + Ez} \Rightarrow f \in SL^*.
\]

*Proof.* The result is obtained by taking \( p(z) = \frac{zf''(z)}{f'(z)} \) in Theorem 2.3. \( \square \)

**Theorem 2.4.** Let \( p \) be an analytic function in \( D \) and \( p(0) = 1 \).

Let \( \beta \geq \beta_0, 0 < \alpha \leq 1, \beta_0 = \frac{|1+B||D-E|}{\alpha|(A-B)(1-|E|)|}, -1 < E < 1, |D| \leq 1 \) and \(-1 \leq B < A \leq 1\).

\[
1 + \beta \frac{zp'(z)}{p(z)} \prec \frac{1 + Dz}{1 + Ez} \Rightarrow p(z) \prec \left( \frac{1 + Az}{1 + Bz} \right)^\alpha.
\]

*Proof.* Let \( q(z) = \left( \frac{1 + Az}{1 + Bz} \right)^\alpha \), Then

\[
\frac{\beta zq'(z)}{q(z)} = \frac{\beta \alpha z(A-B)}{(1 + Az)(1 + Bz)} = Q(z)
\]

It can easily be verified that \( Q(z) \) is starlike. Lemma 2.1, we prove the subordination

\[
s(z) = \frac{1 + Dz}{1 + Ez} \prec 1 + \beta \frac{zp'(z)}{p(z)} = 1 + \frac{\beta \alpha z(A-B)}{(1 + Az)(1 + Bz)} = h(z)
\]

Since \( s^{-1}(w) = \frac{w}{D-E} \) then

\[
|s^{-1}[h(z)]| = \frac{\beta \alpha z(A-B)}{|(1 + Az)(1 + Bz)(D-E) - \beta \alpha zE(A-B)|} \geq \frac{|\beta \alpha z(A-B)|}{||(1 + Az)(1 + Bz)(D-E)|| + |\beta \alpha zE(A-B)|}.
\]

For \( z = e^{i\theta}, \theta \in [-\pi, \pi] \),

\[
|s^{-1}[h(e^{i\theta})]| \geq \frac{\beta \alpha |(A-B)|}{||(1 + Aei\theta)(1 + Be^{i\theta})(D-E)|| + \beta \alpha |E(A-B)|}
\]

with minimum value being attained at \( \theta = 0 \).

Hence

\[
|s^{-1}[h(e^{i\theta})]| \geq \frac{\beta \alpha |(A-B)|}{||(1 + A)(1 + B)(D-E)|| + \beta \alpha |E(A-B)|} \geq 1
\]
for $\beta \geq \frac{||(1+A)(1+B)(D-E)||}{|A-B||1-E||}$ implies $s(z) < h(z)$ and the result is obtained. \(\Box\)

**Remark.** Theorem 2.4 is reduced to Theorem 2.2 when $\alpha = \frac{1}{2}$, $A = 1$ and $B = 0$.

Finally, we state the next obvious result.

**Corollary 2.4.** Let $\beta_0 = \frac{1+A|1+B||D-E|}{|A-B||1-E||}$, $-1 < E < 1$, $|D| \leq 1$ and $-1 \leq B < A \leq 1$.

\[
1 + \beta \left[ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right] < \frac{1+Dz}{1+Ez} \Rightarrow \frac{zf'(z)}{f(z)} < \left( \frac{1+Az}{1+Bz} \right)^{\alpha}.
\]

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**References**


