Monodromy Groups on Knot Surgery 4-manifolds

Ki-Heon Yun
Department of Mathematics, Sungshin Women’s University, Seoul 136-742, Republic of Korea
e-mail: kyun@sungshin.ac.kr

Abstract. In the article we show that nondiffeomorphic symplectic 4-manifolds which admit marked Lefschetz fibrations can share the same monodromy group. Explicitly we prove that, for each integer \( g \geq 0 \), every knot surgery 4-manifold in a family \( \{ E(2)\_K \mid K \text{ is a fibered 2-bridge knot of genus } g \text{ in } S^3 \} \) admits a marked Lefschetz fibration structure which has the same monodromy group.

1. Introduction

Seiberg-Witten invariants are one of the most powerful invariants in the classification of smooth 4-manifolds and Fintushel-Stern’s knot surgery method is one of the most effective methods to modify smooth structures on a given 4-manifold. But Seiberg-Witten invariants are not complete invariants and there are known examples of nondiffeomorphic symplectic 4-manifolds which share the same Seiberg-Witten invariants [3, 15].

R. Fintushel and R. Stern showed that Seiberg-Witten invariants of knot surgery 4-manifold
\[
E(2)\_K = E(2)\_\sharp F = m\_K \times S^1 (M\_K \times S^1)
\]
can be computed by using the Alexander polynomial of the related knot \( K \) [2]. If we restrict our attention to a fibered knot \( K \), then \( E(2)\_K \) naturally has a symplectic structure by a result of R. Gompf [6]. Since there are infinitely many fibered knots of genus \( g \geq 2 \) which share the same Alexander polynomial, we could have an infinite family of symplectic 4-manifolds which share the same Seiberg-Witten invariants [3, 15].

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On the one hand, by a result of S. Donaldson and R. Gompf [7], a symplectic 4-manifold is characterized by its Lefschetz pencil or Lefschetz fibration structure. It is also well known that a Lefschetz fibration over \( S^2 \) with fiber genus bigger than one is characterized by its monodromy factorization [12]. Moreover, R. Fintushel
and R. Stern [4] figured out a Lefschetz fibration structure on $E(2)K$ and its explicit monodromy factorization was known [17]. So it is very natural to ask whether one can define an invariant coming from monodromy factorization. A conjugacy class of a monodromy group, which is a subgroup of the mapping class group generated by each single letter in monodromy factorization, is a well-defined invariant of a Lefschetz fibration up to Lefschetz fibration isomorphism. By using this fact, we could give a family of simply connected symplectic 4-manifolds which have more than one inequivalent Lefschetz fibration structures [13, 14].

In this article, we show that this conjugacy class of a monodromy group is a very rough invariant and there is a family of knot surgery 4-manifolds which share the same conjugacy class of monodromy group even though they are not diffeomorphic each other.

**Theorem 1.1.** For each positive integer $g > 0$, every knot surgery 4-manifold lying in $\{E(2)_K | K$ is a fibered 2-bridge knot of genus $g$ in $S^3\}$ admits a marked Lefschetz fibration structure which shares the same monodromy group.

In [15], we constructed a family of smooth 4-manifolds which share the same Seiberg-Witten invariants even though they are not diffeomorphic each other. Such examples come from a family of fibered 2-bridge knots $K(n, i)$ with the same Alexander polynomial (Definition 3.4.) and they are distinguished by using a covering method. So it may also be possible to distinguish

$$\{E(2)_{K(n, i)} | n \in \mathbb{Z}_+, \ i = 0, 1, \cdots, 2^n - 1\}$$

in smooth category even though we don’t know the answer yet.

**Corollary 1.2.** For each integer $n \geq 1$ and $i = 0, 1, 2, \cdots, 2^n - 1$, every knot surgery 4-manifold $E(2)_{K(n, i)}$ admits a marked monodromy factorization which shares the same monodromy group.

One the other hand, R. Fintushel and R. Stern [4] constructed various families of 4-manifolds which share the same Seiberg-Witten invariants even though they are not diffeomorphic each other. Such examples are constructed by using a pair of Kanenobu’s knots. In [14], we proved that smooth structure of $Y(2 : K, K')$ does not determine the knot type of $K$ and $K'$. Such examples are constructed by using a pair of Kanenobu’s knots. In this article, we also show that $Y(2 : K, K')$ are diffeomorphic each other for any given pair of fibered 2-bridge knots $K$ and $K'$ of same genus.

**Corollary 1.3.** For any fibered 2-bridge knots $K$ and $K'$ of the same genus $g > 0$, $Y(2 : K, K') = E(2)_{K \# \Sigma_{2g+1}}^{\ K'}$ are all diffeomorphic to each other.
2. Lefschetz Fibration and Its Monodromy Factorization

In the section we will briefly review some well-known facts about symplectic Lefschetz fibration and its monodromy factorization.

**Definition 2.1.** Let $X$ be a compact smooth oriented 4-manifold and $B$ be a compact oriented smooth two manifold. A smooth map $f : M \to B$ is a Lefschetz fibration of genus $g$ if

1. $f^{-1}(\partial B) = \partial X$
2. $f$ has finitely many and nonempty set of critical values $\{b_1, b_2, \ldots, b_n\} \subset \text{Int}(B)$ and $f$ is a smooth genus $g$ fiber bundle over $B - \{b_1, b_2, \ldots, b_n\}$
3. there is unique critical point $p_i$ in $f^{-1}(b_i)$ and $f$ is locally written as $f(z_1, z_2) = z_1^2 + z_2^2$
4. we also assume that there is no $-1$ sphere on each $f^{-1}(b_i)$.

**Definition 2.2.** For a given Lefschetz fibration $f : X \to S^2$ with $n$ singular values, we can consider $X$ as $(F \times D^2) \cup (h_1^2 \cup h_2^2 \cup \cdots \cup h_n^2) \cup (F \times D^2)$ where $F$ is a closed oriented Riemann surface and $h_i^2$ is a 4-dimensional two handle $D^2 \times D^2$ whose attaching sphere is a simple closed curve $c_i$ on $F$ so that

$$t_{c_n} t_{c_{n-1}} \cdots t_{c_2} t_{c_1}$$

is the identity element in the surface mapping class group $\text{Mod}(F)$. This ordered sequence of right handed Dehn twists is called monodromy factorization of the Lefschetz fibration and we denote it by $t_{c_n} \cdot t_{c_{n-1}} \cdots \cdot t_{c_2} \cdot t_{c_1}$.

It is known that a Lefschetz fibration is characterized by its monodromy factorization, an ordered sequence of right handed Dehn twists [9, 12]. A right handed Dehn twist along a simple closed curve $c$ is denoted by $t_c$. In the article we use usual function notation, i.e. $t_c t_d$ means that we apply $t_d$ first and then apply $t_c$.

For any element $f \in \text{Mod}(F)$ and a simple closed curve $c$ on $F$,

$$f(t_c) = ft_c f^{-1} = t_{f(c)}.$$  

A monodromy factorization is well defined up to Hurwitz equivalences which come from a choice of Hurwitz system and simultaneous conjugation equivalences which come from a choice of generic fiber of the Lefschetz fibration.

**Definition 2.3.** Two monodromy factorizations $W_1$ and $W_2$ are Hurwitz equivalence, denoted by $W_1 \sim W_2$, if $W_1$ can be changed to $W_2$ in finitely many steps by using the following two operations:
and it is denoted by

\[
\{\text{the same.}
\]

of Hurwitz equivalences. So in the case two corresponding monodromy groups are

The simultaneous conjugation equivalence of two monodromy factorizations is given by

\[
t_{e_n} \cdot t_{e_{n-1}} \cdot \ldots \cdot t_{e_2} \cdot t_{e_1} \equiv f(t_{e_n}) \cdot f(t_{e_{n-1}}) \cdot \ldots \cdot f(t_{e_2}) \cdot f(t_{e_1})
\]

for some \( f \in \text{Mod}(F) \). We will consider \( f(w_1, \ldots, w_2, w_1) \) as \( f(w_1) \cdot \ldots \cdot f(w_2) \cdot f(w_1) \).

**Definition 2.4.** [12] Two Lefschetz fibrations \( f : M \to B, f' : M' \to B' \) are isomorphic if there are orientation preserving diffeomorphisms \( H : M \to M' \) and \( h : B \to B' \) such that

\[
\begin{array}{ccc}
M & \xrightarrow{H} & M' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{h} & B'
\end{array}
\]

commutes i.e \( f' \circ H = h \circ f \).

**Theorem 2.5.** [9] [12] Let \( X_i \to \mathbb{C}P^1, i = 1, 2 \), be Lefschetz fibrations of genus \( g \) with monodromy factorization \( W_i \) corresponding to a fixed generic fiber \( F_i \). Then the two Lefschetz fibrations are isomorphic if and only if \( W_1 \) can be changed to \( W_2 \) by a finite sequence of Hurwitz equivalences and simultaneous conjugation equivalences.

From the above theorem, we can define a map from the set of all isomorphic class of genus \( g \) Lefschetz fibration over \( S^2 \) to the set of all conjugacy classes of subgroups of mapping class group \( \text{Mod}(\Sigma_g) \) of oriented closed surface of genus \( g \).

**Definition 2.6.** For a given Lefschetz fibration \( f : X \to S^2 \) with \( n \) singular values and generic fiber \( F \), let us fix an identification of generic fiber \( F \) with oriented closed Riemann surface \( \Sigma_g \) (which is called marked Lefschetz fibration), then its monodromy factorization is given by \( t_{e_n} \cdot t_{e_{n-1}} \cdot \ldots \cdot t_{e_2} \cdot t_{e_1} \) by using isotopy class of simple closed curves \( \{c_1, c_2, \ldots, c_n\} \) on \( \Sigma_g \). The subgroup of \( \text{Mod}(\Sigma_g) \) generated by \( \{t_{c_1}, t_{c_2}, \ldots, t_{c_n}\} \) is called the monodromy group of the marked Lefschetz fibration and it is denoted by

\[
G_F(t_{e_n} \cdot t_{e_{n-1}} \cdot \ldots \cdot t_{e_2} \cdot t_{e_1})
\]

**Remark 2.7.** Hurwitz move and inverse Hurwitz move does not change monodromy group. Two marked Lefschetz fibrations are isomorphic if one monodromy factorization can be changed to the other monodromy factorization by a finite sequence of Hurwitz equivalences. So in the case two corresponding monodromy groups are the same.

Now we will briefly explain how to consider knot surgery 4-manifold \( E(n)_K \) with fibered knot \( K \) can be considered as a Lefschetz fibration.
Definition 2.8. Let $M(n,g)$ be the desingularization of the two fold covering of $\Sigma_g \times S^2$ with branch set

$$(\Sigma_g \times \{2 \text{ pts}\}) \cup (\{2n \text{ pts}\} \times S^2).$$

Then this manifold is diffeomorphic to $(\Sigma_g \times S^2)^{\#4n\mathbb{C}P^2}$.

If we consider it as a singular genus $(2g + n - 1)$ fibration over $S^2$ with two singular fibers, then after a local perturbation we get a Lefschetz fibration with $2(2g + 4n - 2)$ singular values by computing the Euler number and signature of $(\Sigma_g \times S^2)^{\#4n\mathbb{C}P^2}$. It is known to various authors [12, 11, 8, 16] that the corresponding involution can be written as a product of right handed Dehn twists.

Figure 1: Simple closed curves for Korkmaz word: $g = 2$ and $n = 2$ case

Theorem 2.9. [11] Monodromy of $M(2, g)$ is given by $\eta_g^2$ where

$$\eta_g = t_{B_0} \cdot t_{B_1} \cdot \cdots \cdot t_{B_{2g+1}} \cdot t_{b_{g+1}}^2 \cdot t_{b'_g}^2.$$

Theorem 2.10. [4, 17] Let $K$ be a fibered knot of genus $g$ such that

$$S^3 \setminus \nu(K) = ([0, 1] \times \Sigma_g^1)/(1, x) \sim (0, \phi_K(x))$$

and let

$$\Phi_K = \phi_K \circ id \circ id : \Sigma_g \# \Sigma_1 \# \Sigma_g \rightarrow \Sigma_g \# \Sigma_1 \# \Sigma_g,$$

then monodromy factorization of $E(2)K$ is given by $\Phi_K(\eta_g^2) \cdot \eta_g^2$.

3. Monodromy Group

Assume that $p$ and $q$ are relatively prime integers with $p$ odd. Let us consider a 2-bridge knot $b(p, q)$ which is defined as follows:
Definition 3.1. A 2-bridge knot $b(p,q)$ is of the form

$$C(n_1, -n_2, n_3, -n_4, \ldots, (-1)^{k-1}n_k)$$

as in Figure 2, where

$$\frac{q}{p} = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{\ddots + \frac{1}{n_{k-1} + \frac{1}{n_k}}}}} = [n_1, n_2, \ldots, n_k].$$

\[\begin{align*}
&= \begin{cases}
  n_k \text{ half twists if } n_k \geq 0 \\
  |n_k| \text{ half twists if } n_k \leq 0
\end{cases}
\end{align*}\]

Figure 2: A 2-bridge knot $C(n_1, n_2, \ldots, n_k)$

It is a 4-plat whose defining braid is

$$\sigma_2^{n_1} \sigma_1^{-n_2} \sigma_2^{n_3} \sigma_1^{-n_4} \cdots \sigma_1^{-n_k} \text{ if } k \text{ is even,}$$

$$\sigma_2^{n_1} \sigma_1^{-n_2} \sigma_2^{n_3} \sigma_1^{-n_4} \cdots \sigma_2^{n_k} \text{ if } k \text{ is odd.}$$

Here $\sigma_i$ is a standard braid generator as in Figure 10.3 of [1]. We now denote

$$D(n_1, n_2, \ldots, n_k) = C(2n_1, 2n_2, \ldots, 2n_k).$$

Theorem 3.2. Let $K$ be any fibered 2-bridge knot of genus $g > 0$ and

$$K_g = D(-1, -1, \ldots, -1).$$
Then $E(2)K$ and $E(2)K_r$ admit marked monodromy factorizations whose monodromy groups are the same.

Proof. It is known \textsuperscript{[5, 10]} that any fibered 2-bridge knot of genus $g$ is of the form

$$D(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2g-1}, \epsilon_{2g})$$

where each $\epsilon_i \in \{+1, -1\}$. Since a Seifert surface of $D(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2g-1}, \epsilon_{2g})$ with $\epsilon_i \in \{+1, -1\}$ can be obtained by a sequence of plumblings of positive or negative Hopf band corresponding to $\epsilon_i = +1$ or $\epsilon_i = -1$ and since a positive Hopf band corresponds to a right handed Dehn twist along the core circle of Hopf band and negative Hopf band corresponds to a left handed Dehn twist, we get

$$\Phi(D(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2g-1}, \epsilon_{2g}) = t_{2g}^2 t_{2g-2}^{-1} t_{2g-4}^{-1} \cdots t_{2}^{-1}$$

where simple closed curves $c_i$ are as in Figure 3.

Figure 3: Simple closed curves for monodromy of 2-bridge knot

Let $H$ be the subgroup of $\text{Mod}(\Sigma_{2g+1})$ which is generated by

$$\{ t_{B_i}, t_{c_j}, t_{b_{g+1}}, t_{b_{g+1}}^{-1} | i = 0, 1, 2, \cdots, 2g+1, j = 1, 2, \cdots, 2g \}$$

by using the notation as in Figure 4.

Let us first show that $G_F(\Phi_K(\eta^2_g) \cdot \eta^2_g) = H$.

Since $\Phi_K = t_{c_{2g+1}}^{-1} t_{c_{2g-3}}^{-1} \cdots t_{c_1}^{-1} \in H$, we get

$$\Phi_K(t_{B_i}) = \Phi_K(t_{B_j} \Phi_K^{-1} \in H$$

and $\Phi_K(t_{b_{g+1}}) = t_{b_{g+1}} \Phi_K(t_{b_{g+1}}) = t_{b_{g+1}}^{-1} \cdot \Phi_K(t_{b_{g+1}}). \cdot \eta^2_g$. So $G_F(\Phi_K(\eta^2_g) \cdot \eta^2_g) \leq H$.

Now we will prove that $H \leq G_F(\Phi_K(\eta^2_g) \cdot \eta^2_g)$. To do this, we need to check that

$$t_{c_j} \in G_F(\Phi_K(\eta^2_g) \cdot \eta^2_g)$$

for each $j = 1, 2, \cdots, 2g$. Let us observe from Figure 4 that

$$\Phi_K(B_j) = t_{c_{2g+1}}^{-1} t_{c_{2g-1}}^{-1} \cdots t_{c_{2g-3}}^{-1} \cdots t_{c_1}^{-1} (B_j)$$

$$= t_{c_{2g}}^{-1} \cdots t_{c_{j+1}}^{-1} (B_j)$$

because $c_i \cap B_j = \emptyset$ for $i < j$

$$= t_{c_{2g}}^{-1} \cdots t_{c_{j+1}}^{-1} (t_{c_j}^{-1} (B_j))$$

$$= t_{c_j}^{-1} (B_j)$$

because $t_{c_j}^{-1} (B_j) \cap c_i = \emptyset$ for $j < i$. 

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Therefore we get
\[
(3.2) \quad c_j = t_{B_j}^{-1}(t_{c_j}^{-1}(B_j)) = t_{B_j}^{-1}(\Phi_{K_\#}(B_j))
\]
as in Figure 5. Since \( t_{B_j}^{\pm 1} \Phi_{K_\#}(B_j) \in GF(\Phi_{K_\#}(\eta_2^g) \cdot \eta_2^g) \), it implies
\[
(3.3) \quad t_{c_j} = t_{B_j}^{-1}(\Phi_{K_\#}(t_{B_j})) = t_{B_j}^{-1}(\Phi_{K_\#}(t_{B_j}))t_{B_j} \in GF(\Phi_{K_\#}(\eta_2^g) \cdot \eta_2^g)
\]
for each \( j = 1, 2, \ldots, 2g \). So we get
\[
H \leq GF(\Phi_{K_\#}(\eta_2^g) \cdot \eta_2^g).
\]

Now we will show that
\[
GF(\Phi_{K}(\eta_2^g) \cdot \eta_2^g) = H
\]
for any fibered 2-bridge knot \( K = D(\epsilon_1, \epsilon_2, \ldots, \epsilon_{2g-1}, \epsilon_{2g}) \).

If \( \epsilon_i = -1 \), then by the same method of equation \((3.1)\) we get \( \Phi_{K}(B_i) = t_{c_i}^{-1}(B_i) \) and \( c_i = t_{B_i}^{-1}(\Phi_{K}(B_i)) \). So by the same way as in equation \((3.3)\) we get
\[
(3.4) \quad t_{c_i} \in GF(\Phi_{K}(\eta_2^g) \cdot \eta_2^g)
\]
Figure 5: $t_{c_j}^{-1}(B_2^i)$ is isotopic to $c_j$ for $j = 2i$ (same for $j = 2i + 1$)

whenever $\epsilon_i = -1$.

Now let us consider the case $\epsilon_{i_0} = +1$ and $\epsilon_j = -1$ for each $i_0 + 1 \leq j \leq 2g$. Then $t_{c_j} \in G_F(\Phi_K(\eta_2^g) \cdot \eta_2^g)$ for each $j = i_0 + 1, i_0 + 2, \ldots, 2g$ by equation (3.4). Therefore

$$(t_{c_{j_0}} t_{c_{j_0-1}} \cdots t_{c_2} t_{c_1})(t_{B_i}) \in G_F(\Phi_K(\eta_2^g) \cdot \eta_2^g)$$

for each $\ell = 0, 1, \ldots, 2g + 1$. Because $c_j \cap B_{i_0} = \emptyset$ for each $j = 1, 2, \ldots, i_0 - 1$, we get

$$(t_{c_{j_0}} t_{c_{j_0-1}} \cdots t_{c_2} t_{c_1})(B_{i_0}) = t_{c_{j_0}}(B_{i_0})$$

and

$c_{j_0} = t_{B_{i_0}}(t_{c_{j_0}}(B_{i_0})) = t_{B_{i_0}}(t_{c_{j_0}} t_{c_{j_0-1}} \cdots t_{c_2} t_{c_1}(B_{i_0}))$

as in Figure 6.

Therefore

(3.5) \quad $t_{c_{j_0}} \in G_F(\Phi_K(\eta_2^g) \cdot \eta_2^g)$

Now we will use mathematical induction argument. Suppose that $\epsilon_i = +1$ for $i = i_0, i_1, \ldots, i_N$ which satisfies $1 \leq i_0 < \cdots < i_1 < i_0 \leq 2g$ and all other $\epsilon_i = -1$. Then by Equations (3.4) and (3.5),

$t_{c_i} \in G_F(\Phi_K(\eta_2^g) \cdot \eta_2^g)$

for each $j = i_1 + 1, i_1 + 2, \ldots, 2g$. So we can apply the same method as before and we get $t_{c_{i_1}} \in G_F(\Phi_K(\eta_2^g) \cdot \eta_2^g)$.

By repeating the same method, we get

(3.6) \quad $t_{c_{i_j}} \in G_F(\Phi_K(\eta_2^g) \cdot \eta_2^g)$
Figure 6: $t_{B_j}(t_{c_j}(B_j))$ is isotopic to $c_j$ for $j = 2i + 1$ (same for $j = 2i$)

for each $j = 0, 1, 2, \cdots, N$ at which $\epsilon_{i_j} = +1$.

So Equations (3.4) and (3.6) imply that $t_{c_i} \in G_F(\Phi_K(\eta_2^2) \cdot \eta_g^2)$ for each $i = 1, 2, \cdots, 2g$. Therefore we get

$$H = G_F(\Phi_D(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2g-1}, \epsilon_{2g})(\eta_2^2) \cdot \eta_g^2).$$

\[ \square \]

**Remark 3.3.** Note that smooth 4-manifolds with the same Seiberg-Witten invariants are very hard to prove whether they are diffeomorphic or not in general. Regarding this direction, R. Fintushel and R. Stern first constructed a pair of non-diffeomorphic 4-manifolds which share the same Seiberg-Witten invariants \[3\] by using covering method at the price of big fundamental group and recently we extended such family of examples \[15\]. A special family of 2-bridge knots in Definition 3.4. are the main ingredient when we constructed such examples.

**Definition 3.4.** \[15\] Let us define inductively a family of 2-bridge knots as follows:

(a) Set $W(0, 0) = 1, 1$ and $K(0, 0) = D(W(0, 0))$.

(b) For each integer $n > 0$ and $i = \sum_{j=0}^{n-1} \epsilon_j 2^j$ with $\epsilon_j \in \{0, 1\}$, define a list $W(n, i)$ by

$$W(n-1, \sum_{j=0}^{n-2} \epsilon_j 2^j), (-1)^{\epsilon_{n-1}+1}, -W(n-1, \sum_{j=0}^{n-2} \epsilon_j 2^j), (-1)^{\epsilon_{n-1}+1}, W(n-1, \sum_{j=0}^{n-2} \epsilon_j 2^j)$$

and $K(n, i) = D(W(n, i))$. 
Corollary 3.5. For each integer $n \geq 1$ and $i = 0, 1, \cdots, 2^n - 1$, every knot surgery $4$-manifold $E(2)_K(n, i)$ admits a marked monodromy factorization which shares the same monodromy group.

Proof. Since each $K(n, i)$ is a fibered 2-bridge knot, we get the result directly from Theorem 3.2.

Remark 3.6. Even though we could not distinguish $E(2)_K(n, i)$ by using monodromy group, we expect that these 4-manifolds can be distinguished in smooth category by using other new invariants.

Corollary 3.7. For any fibered 2-bridge knots $K$ and $K'$ of the same genus $g > 0$, $Y(2; K, K') = E(2)_K \# \Sigma_{2g+1} E(2)_{K'}$ are all diffeomorphic to each other.

Proof. It is shown in [17] that if $\varphi \in \mathcal{G}(\Phi_K(\eta^2_g) \cdot \eta^{2^k}_g)$, then

$$\Phi_K(\eta^2_g) \cdot \eta^2_g \cdot \Phi_{K'}(\eta^2_g) \cdot \eta^2_g \sim \Phi_K(\eta^2_g) \cdot \eta^2_g \cdot (\varphi^{2^k} \Psi)(\eta^2_g) \cdot \eta^2_g$$

for any $\Psi \in \text{Mod}(\Sigma_{2g+1})$. Theorem 3.2 implies that $\Phi_{K_s} \cdot \Phi_K \cdot \Phi_{K'} \in \mathcal{H} = \mathcal{G}(\Phi_K(\eta^2_g) \cdot \eta^{2^k}_g) = \mathcal{G}(\Phi_{K_s}(\eta^2_g) \cdot \eta^{2^k}_g)$ for any fibered 2-bridge knot $K$ and $K'$, so we get

$$\Phi_K(\eta^2_g) \cdot \eta^2_g \cdot \Phi_{K'}(\eta^2_g) \cdot \eta^2_g \sim \Phi_K(\eta^2_g) \cdot \eta^2_g \cdot \Phi_K(\eta^2_g) \cdot \eta^2_g \sim \eta^2_g \cdot \eta^2_g \cdot \Phi_{K_s}(\eta^2_g) \cdot \eta^2_g \sim \Phi_{K_s}(\eta^2_g) \cdot \eta^2_g \cdot \eta^2_g \cdot \eta^2_g.$$

If two 4-manifolds have isomorphic Lefschetz fibration structures, then they are diffeomorphic because they are related by a sequence of 2-handle moves. Therefore we get the conclusion.

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