Edge-Maximal $\theta_{2k+1}$-Edge Disjoint Free Graphs

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Abstract. For two positive integers $r$ and $s$, $S(n;r;\theta_s)$ denotes to the class of graphs on $n$ vertices containing no $r$ of edge disjoint $\theta_s$-graphs and $f(n;r;\theta_s) = \max\{E(G) : G \in S(n;r;\theta_s)\}$. In this paper, for integers $r$, $k \geq 2$, we determine $f(n;r;\theta_{2k+1})$ and characterize the edge maximal members in $S(n;r;\theta_{2k+1})$.

1. Introduction

The graphs considered in this paper are finite, undirected and have no loops or multiple edges. Most of the notations that follow can be found in [6]. For a given graph $G$, we denote the vertex set of a graph $G$ by $V(G)$ and the edge set by $E(G)$. The cardinalities of these sets are denoted by $|V(G)|$ and $|E(G)|$, respectively. The cycle on $n$ vertices is denoted by $C_n$. A theta graph $\theta_n$ is defined to be a cycle $C_n$ to which we add a new edge that joins two non-adjacent vertices. We would like to mention that the method used in this paper follows the same lines used in [2] for the same authors.

Let $G_1$ and $G_2$ be graphs. The union of $G_1$ and $G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. $G_1$ and $G_2$ are disjoint if and only if $V(G_1) \cap V(G_2) = \emptyset$; $G_1$ and $G_2$ are edge disjoint if $E(G_1) \cap E(G_2) = \emptyset$. If $G_1$ and $G_2$ are disjoint, we denote their union by $G_1 + G_2$. The intersection $G_1 \cap G_2$ of graphs $G_1$ and $G_2$ is defined similarly, but in this case we need to assume that $V(G_1) \cap V(G_2) \neq \emptyset$. The join $G \vee H$ of two disjoint graphs $G$ and $H$ is the graph obtained from $G + H$ by joining each vertex of $G$ to each vertex of $H$. For two vertex disjoint subgraphs $H_1$ and $H_2$ of $G$, we let $E_G(H_1,H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $E_G(H_1,H_2) = |E_G(H_1,H_2)|$.

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In this paper we consider the Turán-type extremal problem with the odd edge-disjoint theta graphs being the forbidden subgraph. Since a bipartite graph contains no odd theta graph, the non-bipartite graphs have been considered by some authors. First, we recall some notations and terminologies. For a positive integer $n$ and a set of graphs $F$, let $G(n; F)$ denote the class of non-bipartite $F$-free graphs on $n$ vertices, and $$f(n; F) = \max\{E(G) : G \in G(n; F)\}.$$ An important problem in extremal graph theory is determine the values of the function $f(n; F)$. Further, an additional goal is to characterize the extremal graphs $G(n; F)$ where $f(n; F)$ is attained. This problem has been studied extensively by a number of authors [4, 5, 7, 8, 9]. In 1998, Jia proved that $E(G) \leq \lfloor (n - 2)^2/4 \rfloor + 3$ for $G \in G(n; C_5)$ and $n \geq 10$. Furthermore, equality holds if and only if $G \in G^*(n)$ where $G^*(n)$ is the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor (n - 2)/2 \rfloor, \lceil (n - 2)/2 \rceil}$. In 2007, Bataineh established the following result: Let $k \geq 3$ be a positive integer and $G \in G(n; C_{2k+1})$. Then for large value of $n$, $E(G) \leq \lfloor (n - 2)^2/4 \rfloor + 3$. Furthermore, equality holds if and only if $G \in G^*(n)$ where $G^*(n)$ is as above.

Let $G(n; r; \theta_s)$ denote to class of graphs on $n$ vertices containing no $r$ edge-disjoint $\theta_s$-graphs and $$f(n; r; \theta_s) = \max\{E(G) : G \in G(n; r; \theta_s)\}.$$ Note that $$G(n; 1; \theta_s) \subseteq G(n; 2; \theta_s) \subseteq G(n; 3; \theta_s) \subseteq \cdots \subseteq G(n; r; \theta_s).$$ Let $\Omega(n, r)$ denote the class of graphs obtained by adding $r - 1$ edges to the complete bipartite graphs $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Figure 1 displays a member of $\Omega(n, 2)$.

The Turán-type extremal problem with $r$ odd edge-disjoint cycles being the forbidden subgraph, was studied by Bataineh and Jaradat [2]. In fact, they proved that for $G \in G(n; r; C_{2k+1}), k \geq 2$ and large value of $n$, $f(n; r; C_{2k+1}) \leq \lfloor n^2/4 \rfloor + r - 1$. Furthermore, equality holds if and only if $G \in \Omega(n, r)$. Recently, Bataineh et al [3] and Jaradat et al [10], proved the following results:

**Theorem 1.1** (Bataineh et al). For $n \geq 9$, $$f(n; \theta_3) \leq \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + 1.$$ Furthermore, the bound is best possible.

**Theorem 1.2** (Jaradat et al). Let $k \geq 3$ be a positive integer and $G \in G(n; \theta_{2k+1})$. Then for large $n$, $$E(G) \leq \left\lfloor \frac{(n - 2)^2}{4} \right\rfloor + 3.$$
Furthermore, the bound is best possible.

**Theorem 1.3** (Jaradat et al). Let $k \geq 3$ be a positive integer and $G$ be a graph on $n$ vertices that contains no $\theta_{2k+1}$ graph as a subgraph. Then for large value of $n$,

$$E(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$  

Furthermore, equality holds if and only if $G$ is the complete bipartite graph $K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$.

We continue the work initiated in [3] and [10] by generalizing and extending the above theorem. In fact, we determine $f(n; r; \theta_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n; r; \theta_{2k+1})$ for $k, r \geq 2$.

In the rest of this paper, $N_G(u)$ stands for the set of neighbors of $u$ in the graph $G$. Moreover, $G[X]$ denotes the induced subgraph of $X$ in $G$.

2. Edge-Maximal $\theta_{2k+1}$ - Disjoint Free Graphs

In this section, we determine $f(n; r; \theta_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n; r; \theta_{2k+1})$ for $k, r \geq 2$. Observe that $\Omega(n, r) \subseteq \mathcal{G}(n; r; \theta_{2k+1})$ and every graph in $\Omega(n, r)$ contains $\left\lceil n^2/4 \right\rceil + r - 1$ edges. Thus, we have established that

$$f(n; r, \theta_{2k+1}) \geq \left\lceil n^2/4 \right\rceil + r - 1.$$  

In the following theorem, we establish that equality holds. Further, we characterize the edge maximal members in $\mathcal{G}(n; r; \theta_{2k+1})$.
Theorem 2.1. Let $k, r \geq 2$ be two positive integers and $G \in S(n; r; \theta_{2k+1})$. For large value of $n$,

$$f(n; r; \theta_{2k+1}) \leq \lfloor \frac{n^2}{4} \rfloor + r - 1.$$ 

Furthermore, equality holds if and only if $G \in \Omega(n, r)$.

Proof. We prove this theorem using induction on $r$.

**Step 1:** We show the result for $r = 2$ and $k \geq 2$. Let $G \in S(n, 2; \theta_{2k+1})$. If $G$ contains no $\theta_{2k+1}$ as a subgraph, then by Theorem 1.3, $\mathcal{E}(G) \leq \lfloor \frac{n^2}{4} \rfloor$. Thus, $\mathcal{E}(G) < \lfloor \frac{n^2}{4} \rfloor + 1$. So, we need to consider the case when $G$ has $\theta_{2k+1}$ as a subgraph. Assume $x_1x_2 \ldots x_{2k+1}x_1$ be a $\theta_{2k+1}$ in $G$ for some $3 \leq t \leq 2k$. Consider $H = G - \{e_1 = x_1x_2, e_2 = x_2x_3, \ldots, e_{2k+1} = x_{2k+1}x_1, e_{2k+2} = x_1x_t\}$. Observe that $H$ cannot have $\theta_{2k+1}$ as otherwise $G$ would have two edge-disjoint $\theta_{2k+1}$ as a subgraph.

**Subcase 1.1.** $k = 2$. Then by Theorem 1.1

$$\mathcal{E}(H) \leq \lfloor (n - 1)^2/4 \rfloor + 1.$$

Now,

$$\mathcal{E}(G) = \mathcal{E}(H) + 2k + 2 \leq \lfloor (n - 1)^2/4 \rfloor + 2k + 3 < \lfloor \frac{n^2}{4} \rfloor + 1$$

for $n \geq 4k + 7$.

**Subcase 1.2.** $k \geq 3$. Then by Theorem 1.2

$$\mathcal{E}(H) \leq \lfloor (n - 2)^2/4 \rfloor + 3.$$  

Now,

$$\mathcal{E}(G) = \mathcal{E}(H) + 2k + 2 \leq \lfloor (n - 2)^2/4 \rfloor + 2k + 5 \leq \lfloor \frac{n^2}{4} \rfloor - n + 2k + 6,$$

for $n \geq 2k + 6$, we have

$$\mathcal{E}(G) < \lfloor \frac{n^2}{4} \rfloor + 1.$$
Case 2: $H$ is a bipartite graph. Let $X$ and $Y$ be the partition of $V(H)$. Thus, $\mathcal{E}(H) \leq |X||Y|$. Observe $|X| + |Y| = n$. The maximum of the above is obtained when $|X| = \left\lceil \frac{n}{2} \right\rceil$ and $|Y| = \left\lfloor \frac{n}{2} \right\rfloor$. Thus, $\mathcal{E}(H) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. Restore the edges of the $\theta_{2k+1}$-graph. We now consider the following subcases:

**Subcase 2.1:** One of $X$ and $Y$ contains two edges of the $\theta_{2k+1}$-graph, say $e_i$ and $e_j$ in $X$. Let $y_1, y_2, \ldots, y_{k-1}$ be a set of vertices in $X - \{x_i, x_{i+1}, x_j, x_{j+1}\}$. We split this subcase into two subsubcases:

**Subsubcase 2.1.1:** $i$ and $j$ are not consecutive. Then $|N_Y(x_i) \cap N_Y(x_{i+1}) \cap N_Y(x_j) \cap N_Y(x_{j+1}) \cap N_Y(y_1) \cap \ldots \cap N_Y(y_{k-1})| \leq k + 2$, as otherwise $G$ contains two edge-disjoint $\theta_{2k+1}$-graph. Thus,

$$\mathcal{E}_G(x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}, Y) \leq (k + 2)|Y| + k + 2.$$

So,

$$\mathcal{E}(G) = \mathcal{E}_G(X - \{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}\}, Y) + \mathcal{E}_G(x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}, Y) + \mathcal{E}(G[X]) + \mathcal{E}(G[Y])$$

$$\leq (|X| - k - 3)|Y| + (k + 2)|Y| + k + 2 + 2k + 2$$

$$\leq |X||Y| - |Y| + 3k + 4$$

$$\leq (|X| - 1)|Y| + 3k + 4$$

Observe that $|X| + |Y| = n$. The maximum of the above equation is when $|Y| = \left\lceil \frac{n - 1}{2} \right\rceil$ and $|X| - 1 = \left\lfloor \frac{n - 1}{2} \right\rfloor$. Thus,

$$\mathcal{E}(G) \leq \left\lceil \frac{n - 1}{2} \right\rceil + 3k + 4.$$

Hence, for $n \geq 6k + 9$,

$$\mathcal{E}(G) < \left\lceil \frac{n^2}{4} \right\rceil + 1.$$  

**Subsubcase 2.1.2:** $i$ and $j$ are consecutive, say $j = i + 1$. Then by following the same arguments as in subsubcase 2.1.1 and by taking into the account that $|N_Y(x_i) \cap N_Y(x_{i+1}) \cap N_Y(x_{i+2}) \cap N_Y(y_1) \cap \ldots \cap N_Y(y_{k-1})| \leq k + 1$ and so $\mathcal{E}_G(x_i, x_{i+1}, x_{i+2}, y_1, y_2, \ldots, y_{k-1}, Y) \leq (k + 1)|Y| + k + 1$, we get the same inequality.

**Subcase 2.2:** $\mathcal{E}(G[X]) = 1$ and $\mathcal{E}(G[Y]) = 0$ or $\mathcal{E}(G[X]) = 0$ and $\mathcal{E}(G[Y]) = 1$. Then

$$\mathcal{E}(G) \leq \mathcal{E}(H) + 1$$

$$\leq \left\lceil \frac{n^2}{4} \right\rceil + 1.$$
One can observe from the above arguments that for $r = 2$ only time when we have equality is when $G$ is obtained by adding an edge to the complete bipartite graph $K_{\lceil n/2 \rceil \lceil n/2 \rceil}$. This leads to the class $\Omega(n, 2)$.

**Step 2:** Assume that the result is true for $r - 1$.

**Step 3:** We show the result is true for $r \geq 3$. To accomplish that we use similar arguments to those in Step 1. Let $G \in \mathcal{G}(n; r; 2k+1)$. If $G$ contains no $r - 1$ edge-disjoint of $\theta_{2k+1}$ graphs, then by the inductive step $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor + r - 2$.

Thus, $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + r - 1$. So, we need to consider the case when $G$ has $r - 1$ edge-disjoint of $\theta_{2k+1}$-graphs. Assume that $\{\theta^{(i)} = x_{i1}x_{i2} \ldots x_{i(2k+1)}x_{it}\}^{r-1}_{i=1}$ be the set of $r - 1 \theta_{2k+1}$-graphs. Consider $H = G - \bigcup_{i=1}^{r-1} E(\theta^{(i)})$. Observe that $H$ cannot have $\theta_{2k+1}$-graphs as otherwise $G$ would have $r$ edge-disjoint $\theta_{2k+1}$-graphs.

As in Step 1, we consider two cases:

**Case I:** $H$ is not a bipartite graph. Then we consider two subcases

**Subcase 1.1.** $k = 2$. Then by Theorem 1.1

$$\mathcal{E}(H) \leq \lfloor (n - 1)^2/4 \rfloor + 1.$$ 

Now,

$$\mathcal{E}(G) = \mathcal{E}(H) + (r - 1)(2k + 2) \leq \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + (r - 1)(2k + 2) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1$$

for $n \geq 2(2k + 2)(r - 1) + 1$.

**Subcase 1.2.** $k \geq 3$. Then by Theorem 1.2

$$\mathcal{E}(H) \leq \lfloor (n - 2)^2/4 \rfloor + 3.$$ 

Thus,

$$\mathcal{E}(G) = \mathcal{E}(H) + (r - 1)(2k + 2) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + (r - 1) - n + 4 + (2k + 1)(r - 1),$$

for $n > 4 + (2k + 1)(r - 1)$,

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1.$$ 

**Case II:** $H$ is a bipartite graph. Let $X$ and $Y$ be the partition of $V(H)$. Thus, $\mathcal{E}(H) \leq |X||Y|$. Observe that $|X| + |Y| = n$. The maximum of the above is
obtained when $|X| = \left\lfloor \frac{n}{2} \right\rfloor$ and $|Y| = \left\lceil \frac{n}{2} \right\rceil$. Thus, $E(H) \leq \left\lceil \frac{n^2}{4} \right\rceil$. Now, we consider the following two subcases:

**Subcase II.1:** There is $1 \leq m \leq r - 1$ such that $\theta^{(m)}$ contains at least two edges, say $e_i = x_{m_i} x_{m(i+1)}$ and $e_j = x_{m_j} x_{m(j+1)}$, joining vertices of one of $X$ and $Y$, say $X$. Let $y_1, y_2, \ldots, y_{k-1}$ be a set of vertices in $X \setminus \{x_{m_i}, x_{m(i+1)}, x_{m_j}, x_{m(j+1)}\}$. To this end we have two subsubcases:

**Subsubcase II.1.i:** $i$ and $j$ are consecutive. Then $|N_Y(x_{m_i}) \cap N_Y(x_{m(i+1)}) \cap N_Y(x_{m_j}) \cap N_Y(x_{m(j+1)}) \cap N_Y(y_1) \cap \ldots \cap N_Y(y_{k-1})| \leq k + 2$, as otherwise $H \cup \{e_i, e_j\}$ contains two edge-disjoint $\theta_{2k+1}$-graphs and so $G$ contains $r$ edge-disjoint $\theta_{2k+1}$-graphs. Thus, as in Subsubcase 2.1.1 of Step 1,

$$E_H(X \setminus \{x_{m_i}, x_{m(i+1)}, x_{m_j}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) \leq (k + 2)|Y| + k + 2.$$  

And so,

$$E(G) = E(H) + \left| \bigcup_{i=1}^{r-1} E(\theta^i) \right|$$

$$= E_H(X \setminus \{x_{m_i}, x_{m(i+1)}, x_{m_j}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) + E_H(X \setminus \{x_{m_i}, x_{m(i+1)}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) + \left| \bigcup_{i=1}^{r-1} E(\theta^i) \right|$$

$$\leq (|X| - k - 3)|Y| + (k + 2)|Y| + k + 2 + (r - 1)(2k + 2)$$

Moreover, the maximum of the above inequality is obtained when $|Y| = \left\lceil \frac{n}{2} \right\rceil$ and $|X| - 1 = \left\lfloor \frac{n}{2} \right\rfloor$. Thus,

$$E(G) \leq \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + k + 2 + (r - 1)(2k + 2)$$

For $n \geq (6k + 2)(r - 1) + 7$, we have

$$E(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + (r - 1).$$

**Subsubcase II.1.ii:** $i$ and $j$ are not consecutive. Then $|N_Y(x_{m_i}) \cap N_Y(x_{m(i+1)}) \cap N_Y(x_{m_j}) \cap N_Y(x_{m(j+1)}) \cap N_Y(y_1) \cap \ldots \cap N_Y(y_{k-1})| \leq k + 2$, as otherwise $H \cup \{e_i, e_j\}$ contains two edge-disjoint $\theta_{2k+1}$-graphs and so $G$ contains $r$ edge-disjoint $\theta_{2k+1}$-graphs. Thus, as in Subsubcase 2.1.1 of Step 1,

$$E_H(X \setminus \{x_{m_i}, x_{m(i+1)}, x_{m_j}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) \leq (k + 2)|Y| + k + 2.$$  

And so,

$$E(G) = E(H) + \left| \bigcup_{i=1}^{r-1} E(\theta^i) \right|$$

$$= E_H(X \setminus \{x_{m_i}, x_{m(i+1)}, x_{m_j}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) + E_H(X \setminus \{x_{m_i}, x_{m(i+1)}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) + \left| \bigcup_{i=1}^{r-1} E(\theta^i) \right|$$

$$\leq (|X| - k - 3)|Y| + (k + 2)|Y| + k + 2 + (r - 1)(2k + 2)$$

Moreover, the maximum of the above inequality is obtained when $|Y| = \left\lceil \frac{n}{2} \right\rceil$ and $|X| - 1 = \left\lfloor \frac{n}{2} \right\rfloor$. Thus,

$$E(G) \leq \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + k + 2 + (r - 1)(2k + 2)$$

For $n \geq (6k + 2)(r - 1) + 7$, we have

$$E(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + (r - 1).$$

**Subcase II.ii:** Each $1 \leq m \leq r - 1$, $\theta^{(m)}$ has exactly one edge belonging to one of $X$ and $Y$. Let $e$ be the edge of $\theta^{(1)}$ that belongs to one of $X$ and $Y$. Then
$G - e \in \Omega(n, r - 1) \subseteq \mathcal{O}(n; r - 1; \theta_{2k+1})$ and so by the inductive step,

$$\mathcal{E}(G) = \mathcal{E}(G - e) + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + r - 2 + 1 = \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1.$$  

This completes the proof of the theorem. \hfill \Box

We can now characterize the extremal graphs. Throughout the proof, we noticed that the only time when we had equality was in the case when $G$ was obtained by adding $r - 1$ edges to the complete bipartite graph $K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$. This leads rise to the class $\Omega(n, r)$.

References


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