Some Difference Paranormed Sequence Spaces over $n$-normed Spaces Defined by a Musielak-Orlicz Function

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ABSTRACT. In the present paper we introduce difference paranormed sequence spaces $c_0(M, \Delta^n, p, u, ||\cdot, \cdot||)$, $c(M, \Delta^n, p, u, ||\cdot, \cdot||)$ and $l_1(M, \Delta^n, p, u, ||\cdot, \cdot||)$ defined by a Musielak-Orlicz function $M = (M_k)$ over $n$-normed spaces. We also study some topological properties and some inclusion relations between these spaces.

1. Introduction and Preliminaries

Let $w$, $l_\infty$, $c$ and $c_0$ denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with real or complex entries respectively. The zero sequence is denoted by $\theta = (0, 0, \ldots)$. The notion of difference sequence spaces was introduced by Kızmaz [9], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let $m$, $n$ be non-negative integers, then for $Z = l_\infty$, $c$ and $c_0$ we have sequence spaces,

$$Z(\Delta^n_m) = \{x = (x_k) \in w : (\Delta^n_m x_k) \in Z\}$$

where $\Delta^n_m x = (\Delta^n_m x_k) = (\Delta_{m}^{n-1} x_k - \Delta_{m}^{n-1} x_{k+m})$ and $\Delta^0_m x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n_m x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}.$$ 

Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [6].

Let $X$ be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

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1. \( p(x) \geq 0 \), for all \( x \in X \);
2. \( p(-x) = p(x) \), for all \( x \in X \);
3. \( p(x + y) \leq p(x) + p(y) \), for all \( x, y \in X \);
4. if \( (\sigma_n) \) is a sequence of scalars with \( \sigma_n \to \sigma \) as \( n \to \infty \) and \( (x_n) \) is a sequence of vectors with \( p(x_n - x) \to 0 \) as \( n \to \infty \), then \( p(\sigma_n x_n - \sigma x) \to 0 \) as \( n \to \infty \).

A paranorm \( p \) for which \( p(x) = 0 \) implies \( x = 0 \) is called total paranorm and the pair \((X, p)\) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [18], Theorem 10.4.2, P-183). For more details about sequence spaces (see [1], [2], [3], [14], [15], [16], [17]) and references therein.

An Orlicz function \( M : [0, \infty) \to [0, \infty) \) is a continuous and convex with \( M(0) = 0 \), \( M(x) > 0 \) for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \).

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space,
\[
\ell_{M} = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}
\]
which is called as an Orlicz sequence space. Also \( \ell_{M} \) is a Banach space with the norm
\[
\| (x_k) \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}.
\]
Also, it was shown in [10] that every Orlicz sequence space \( \ell_{M} \) contains a subspace isomorphic to \( \ell_{p}(p \geq 1) \). An Orlicz function \( M \) satisfies \( \Delta_2 \)-condition if and only if for any constant \( L > 1 \) there exists a constant \( K(L) \) such that \( M(Lu) \leq K(L)M(u) \) for all values of \( u \geq 0 \). An Orlicz function \( M \) can always be represented in the following integral form
\[
M(x) = \int_{0}^{x} \eta(t)dt
\]
where \( \eta \) is known as the kernel of \( M \), is right differentiable for \( t \geq 0 \), \( \eta(0) = 0 \), \( \eta(t) > 0 \), \( \eta \) is non-decreasing and \( \eta(t) \to \infty \) as \( t \to \infty \).

A sequence \( M = (M_k) \) of Orlicz functions is called a Musielak-Orlicz function (see [11], [13]). A sequence \( N = (N_k) \) defined by
\[
N_k(u) = \sup\{ |v| : M_k(u) \geq 0 \}, \quad k = 1, 2, \cdots
\]
is called the complementary function of a Musielak-Orlicz function \( M \). For a given Musielak-Orlicz function \( M \), the Musielak-Orlicz sequence space \( t_{M} \) and its subspace \( h_{M} \) are defined as follows
\[
t_{M} = \left\{ x \in w : I_{M}(cx) < \infty \text{ for some } c > 0 \right\},
\]
and
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\[ h_M = \left\{ x \in w : I_M(cx) < \infty \text{ for all } c > 0 \right\}, \]

where \( I_M \) is a convex modular defined by

\[ I_M(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_M. \]

We consider \( t_M \) equipped with the Luxemburg norm

\[ ||x|| = \inf \left\{ k > 0 : I_M\left(\frac{x}{k}\right) \leq 1 \right\} \]

or equipped with the Orlicz norm

\[ ||x||^0 = \inf \left\{ \frac{1}{k}(1 + I_M(kx)) : k > 0 \right\}. \]

The concept of 2-normed spaces was initially developed by Gähler [5] in the mid of 1960’s, while that of \( n \)-normed spaces one can see in Misiak[9]. Since then, many others have studied this concept and obtained various results, see Gunawan ([6], [7]) and Gunawan and Mashadi [8]. Let \( n \in \mathbb{N} \) and \( X \) be a linear space over the field \( \mathbb{K} \), where \( \mathbb{K} \) is the field of real or complex numbers of dimension \( d \), where \( d \geq n \geq 2 \). A real valued function \( ||\cdot, \cdot, \cdot|| \) on \( X^n \) which satisfies the following four conditions:

1. \( ||x_1, x_2, \cdots, x_n|| = 0 \) if and only if \( x_1, x_2, \cdots, x_n \) are linearly dependent in \( X \);
2. \( ||x_1, x_2, \cdots, x_n|| \) is invariant under permutation;
3. \( ||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \cdot ||x_1, x_2, \cdots, x_n|| \) for any \( \alpha \in \mathbb{K} \), and
4. \( ||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n|| \)

is called a \( n \)-norm on \( X \) and the pair \((X, ||\cdot, \cdot, \cdot||)\) is called a \( n \)-normed space over the field \( \mathbb{K} \).

For example, we may take \( X = \mathbb{R}^n \) being equipped with the \( n \)-norm \( ||x_1, x_2, \cdots, x_n||_E = \) the volume of the \( n \)-dimensional parallelopiped spanned by the vectors \( x_1, x_2, \cdots, x_n \) which may be given explicitly by the formula

\[ ||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|, \]

where \( x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, \cdots, n \) and script \( E \) denotes the Euclidean norm. Let \((X, ||\cdot, \cdot, \cdot||)\) be a \( n \)-normed space of dimension \( d \geq n \geq 2 \) and \( \{a_1, a_2, \cdots, a_n\} \) be linearly independent set in \( X \). Then the following function \( ||\cdot, \cdot, \cdot||_\infty \) on \( \mathbb{R}^{n-1} \) defined by

\[ ||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{|x_1, x_2, \cdots, x_{n-1}, a_i| : i = 1, 2, \cdots, n\} \]

defines an \((n - 1)\)-norm on \( X \) with respect to \( \{a_1, a_2, \cdots, a_n\} \).
A sequence \((x_k)\) in a \(n\)-normed space \((X, |.|, \cdots, |.|)\) is said to converge to some \(L \in X\) if
\[
\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \quad \text{for every } z_1, \cdots, z_{n-1} \in X.
\]

A sequence \((x_k)\) in a \(n\)-normed space \((X, |.|, \cdots, |.|)\) is said to be Cauchy if
\[
\lim_{k, p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \quad \text{for every } z_1, \cdots, z_{n-1} \in X.
\]

If every Cauchy sequence in \(X\) converges to some \(L \in X\), then \(X\) is said to be complete with respect to the \(n\)-norm. Any complete \(n\)-normed space is said to be \(n\)-Banach space.

Let \(M = (M_k)\) be a Musielak-Orlicz function, \(p = (p_k)\) be a bounded sequence of positive real numbers and \(u = (u_k)\) be a sequence of positive reals such that \(u_k \neq 0\) for all \(k\), then we define the following classes of sequences in the present paper:
\[
c_0(M, \Delta^u_m, p, u, |.|, \cdots, |.|) = \left\{ x = (x_k) \in w : \lim_{k \to \infty} u_k M_k \left( \left| \frac{\Delta^u_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right)^{p_k} = 0, \quad \text{for some } \rho > 0 \right\},
\]
\[
c(M, \Delta^u_m, p, u, |.|, \cdots, |.|) = \left\{ x = (x_k) \in w : \lim_{k \to \infty} u_k M_k \left( \left| \frac{\Delta^u_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right)^{p_k} = 0, \quad \text{for some } \rho > 0 \text{ and } L \in X \right\},
\]
and
\[
l_\infty(M, \Delta^u_m, p, u, |.|, \cdots, |.|) = \left\{ x = (x_k) \in w : \sup_{k \geq 1} u_k M_k \left( \left| \frac{\Delta^u_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right)^{p_k} < \infty, \quad \text{for some } \rho > 0 \right\}.
\]

For \(p_k = 1\), for all \(k\)
\[
c_0(M, \Delta^u_m, u, |.|, \cdots, |.|) = \left\{ x = (x_k) \in w : \lim_{k \to \infty} u_k M_k \left( \left| \frac{\Delta^u_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right) = 0, \quad \text{for some } \rho > 0 \right\},
\]
\[
c(M, \Delta^u_m, u, |.|, \cdots, |.|) = \left\{ x = (x_k) \in w : \lim_{k \to \infty} u_k M_k \left( \left| \frac{\Delta^u_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right) = 0, \quad \text{for some } \rho > 0 \text{ and } L \in X \right\},
\]
and
\[
l_\infty(M, \Delta^u_m, u, |.|, \cdots, |.|) = \left\{ x = (x_k) \in w : \sup_{k \geq 1} u_k M_k \left( \left| \frac{\Delta^u_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right) < \infty, \quad \text{for some } \rho > 0 \right\}.
\]
For $\mathcal{M}(x) = x$, we have
\[
c_0(\Delta_m^n, p, u, ||\cdot||, \ldots, ||\cdot||) = \left\{ x = (x_k) \in w : \lim_{k \to \infty} u_k \left( \left\| \frac{\Delta_m^n x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} = 0, \right. \hspace{1cm} \text{for some } \rho > 0 \right\},
\]
\[
c(\Delta_m^n, p, u, ||\cdot||, \ldots, ||\cdot||) = \left\{ x = (x_k) \in w : \lim_{k \to \infty} u_k \left( \left\| \frac{\Delta_m^n x_k - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} = 0, \text{ for some } \rho > 0 \text{ and } L \in X \right\},
\]
and
\[
l_\infty(\Delta_m^n, p, u, ||\cdot||, \ldots, ||\cdot||) = \left\{ x = (x_k) \in w : \sup_{k \geq 1} u_k \left( \left\| \frac{\Delta_m^n x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} < \infty, \right. \hspace{1cm} \text{for some } \rho > 0 \right\}.
\]
The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = G$, $K = \max(1, 2^{G-1})$ then
\[
(1.1) \quad |a_k + b_k|^{p_k} \leq K \{ |a_k|^{p_k} + |b_k|^{p_k} \}
\]
for all $k$ and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

The aim of this paper is to study some difference sequence spaces in more general setting i.e. over $n$-normed spaces defined by a Musielak-Orlicz function.

2. Main Results

In this section, we study some topological properties and inclusion relation between the spaces $c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot||, \ldots, ||\cdot||)$, $c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot||, \ldots, ||\cdot||)$ and $l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot||, \ldots, ||\cdot||)$.

Theorem 2.1. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers, then the classes of sequences $c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot||, \ldots, ||\cdot||)$, $c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot||, \ldots, ||\cdot||)$ and $l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot||, \ldots, ||\cdot||)$ are linear spaces.

Proof. Let $x = (x_k)$, $y = (y_k) \in c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot||, \ldots, ||\cdot||)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers $\rho_1$ and $\rho_2$ such that
\[
\lim_{k \to \infty} u_k \left[ M_k \left( \left\| \frac{\Delta_m^n x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right] = 0, \text{ and }
\]
\[
\lim_{k \to \infty} u_k \left[ M_k \left( \left\| \frac{\Delta_m^n y_k}{\rho_2}, z_1, \ldots, z_{n-1} \right\| \right)^{p_k} \right] = 0.
\]
Let \( \rho_\alpha = \max(2|\alpha|, 2|\beta|) \). Since \( M = (M_k) \) is non-decreasing convex function and so by using inequality (1.1), we have

\[
\lim_{k \to \infty} u_k \left[ M_k \left( \left\| \frac{\Delta_n^k(ax_k + \beta y_k)}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \\
\leq \lim_{k \to \infty} u_k \left[ M_k \left( \left\| \frac{\Delta_m^k x_k}{\rho_1}, z_1, \cdots, z_{n-1} \right\| + \left\| \frac{\Delta_n^k y_k}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \\
\leq K \lim_{k \to \infty} \frac{1}{2^{p_k}} u_k \left[ M_k \left( \left\| \frac{\Delta_n^k x_k}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \\
+ K \lim_{k \to \infty} \frac{1}{2^{p_k}} u_k \left[ M_k \left( \left\| \frac{\Delta_n^k y_k}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \\
\leq K \lim_{k \to \infty} u_k \left[ M_k \left( \left\| \frac{\Delta_n^k x_k}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \\
+ K \lim_{k \to \infty} u_k \left[ M_k \left( \left\| \frac{\Delta_n^k y_k}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \\
= 0.
\]

So, \( ax + \beta y \in c_0(M, \Delta_m^k, p, u, ||\cdot||, \cdots, ||\cdot||) \). Hence \( c_0(M, \Delta_m^k, p, u, ||\cdot||, \cdots, ||\cdot||) \) is a linear space. Similarly, we can prove that \( c(M, \Delta_m^k, p, u, ||\cdot||, \cdots, ||\cdot||) \) and \( l_\infty(M, \Delta_m^k, p, u, ||\cdot||, \cdots, ||\cdot||) \) are linear spaces.

**Theorem 2.2.** Let \( M = (M_k) \) be a Musielak-Orlicz function, \( p = (p_k) \) be a bounded sequence of positive real numbers and \( u = (u_k) \) be a sequence of strictly positive real numbers. For \( Z = l_\infty, c \) and \( c_0 \), the spaces \( Z(M, \Delta_m^k, p, u, ||\cdot||, \cdots, ||\cdot||) \) are paranormed spaces, paranormed by

\[
g(x) = \sum_{k=1}^{m,n} ||x_k, z_1, \cdots, z_{n-1}|| + \inf \left\{ \frac{\sup_{\rho \in H} \rho}{\rho} : \sup_{k} u_k M_k \left( \left\| \frac{\Delta_m^k x_k}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \leq 1 \right\}
\]

where \( H = \max(1, \sup_k p_k) \).

**Proof.** Clearly \( g(-x) = g(x) \), \( g(0) = 0 \). Let \( (x_k) \) and \( (y_k) \) be any two sequences belong to any one of the space \( Z(M, \Delta_m^k, p, u, ||\cdot||, \cdots, ||\cdot||) \), for \( Z = c_0, c \) and \( l_\infty \). Then, we get \( \rho_1, \rho_2 > 0 \) such that

\[
\sup_{k} u_k M_k \left( \left\| \frac{\Delta_m^k x_k}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \leq 1
\]

and

\[
\sup_{k} u_k M_k \left( \left\| \frac{\Delta_m^k y_k}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \leq 1.
\]

Let \( \rho = \rho_1 + \rho_2 \). Then by convexity of \( M = (M_k) \), we have
\[
\sup_k u_k M_k \left( \left\| \frac{\Delta_m^n (x_k + y_k)}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)
\]
\[
\leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_k u_k M_k \left( \left\| \frac{\Delta_m^n x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right\| \right)
\]
\[
+ \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_k u_k M_k \left( \left\| \frac{\Delta_m^n y_k}{\rho_2}, z_1, \ldots, z_{n-1} \right\| \right)
\]
\[
\leq 1.
\]

Hence we have,
\[
g(x + y) = \sum_{k=1}^{mn} \left\| (x_k + y_k), z_1, \ldots, z_{n-1} \right\|
\]
\[
\leq \sum_{k=1}^{mn} \left\| x_k, z_1, \ldots, z_{n-1} \right\| \quad \text{+} \quad \inf \left\{ \rho_1^{\frac{p_k}{p}} : \sup_k u_k M_k \left( \left\| \frac{\Delta_m^n x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right\| \right) \leq 1 \right\}
\]
\[
\leq \sum_{k=1}^{mn} \left\| x_k, z_1, \ldots, z_{n-1} \right\| \quad \text{+} \quad \inf \left\{ \rho_1^{\frac{p_k}{p}} : \sup_k u_k M_k \left( \left\| \frac{\Delta_m^n x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right\| \right) \leq 1 \right\}
\]
\[
\leq \sum_{k=1}^{mn} \left\| y_k, z_1, \ldots, z_{n-1} \right\| \quad \text{+} \quad \inf \left\{ \rho_2^{\frac{p_k}{p}} : \sup_k u_k M_k \left( \left\| \frac{\Delta_m^n y_k}{\rho_2}, z_1, \ldots, z_{n-1} \right\| \right) \leq 1 \right\}.
\]

This implies that
\[
g(x + y) \leq g(x) + g(y).
\]

The continuity of the scalar multiplication follows from the following inequality
\[
g(\mu x) = \sum_{k=1}^{mn} \left\| \mu x_k, z_1, \ldots, z_{n-1} \right\| \quad \text{+} \quad \inf \left\{ \rho_1^{\frac{p_k}{p}} : \sup_k u_k M_k \left( \left\| \frac{\Delta_m^n \mu x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \leq 1 \right\}
\]
\[
= |\mu| \sum_{k=1}^{mn} \left\| x_k, z_1, \ldots, z_{n-1} \right\| \quad \text{+} \quad \inf \left\{ \rho_1^{\frac{p_k}{p}} : \sup_k u_k M_k \left( \left\| \frac{\Delta_m^n x_k}{\rho_1}, z_1, \ldots, z_{n-1} \right\| \right) \leq 1 \right\},
\]

where \( t = \frac{\rho}{|\mu|} \). Hence the space \( Z(M, \Delta_m^n, p, u, \|\cdot\|, \|\cdot\|) \), for \( Z = c_0, c \) and \( l_\infty \) is a paranormed space, paranormed by \( g \).

**Theorem 2.3.** Let \( \mathcal{M} = (M_k) \) be a Musielak-Orlicz function, \( p = (p_k) \) be a bounded sequence of positive real numbers and \( u = (u_k) \) be a sequence of strictly positive real numbers. For \( Z = l_\infty, c \) and \( c_0 \), the spaces \( Z(M, \Delta_m^n, p, u, \|\cdot\|, \|\cdot\|) \) are complete.
paranormed spaces, paranormed by

\[ g(x) = \sum_{k=1}^{mn} ||x_k, z_1, \ldots, z_{n-1}|| + \inf \left\{ \rho \frac{\mu_k}{\rho} : \sup_k u_k M_k \left( \left\| \frac{\Delta_m^n(x_k - x^j)}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \leq 1 \right\} , \]

where \( H = \max(1, \sup p_k) \).

**Proof.** We prove the result for the space \( l_\infty(M, \Delta_m^n, p, u, ||\cdot||, \cdots, ||\cdot||) \). Let \( (x^i) \) be any Cauchy sequence in \( l_\infty(M, \Delta_m^n, p, u, ||\cdot||, \cdots, ||\cdot||) \). Let \( x_0 > 0 \) be fixed and \( t > 0 \) be such that for a given \( 0 < \epsilon < 1, \frac{t}{x_0} > 0 \) and \( x_0 t \geq 1 \). Then there exists a positive integer \( n_0 \) such that \( g(x^i - x^j) \leq \frac{\epsilon}{x_0 t}, \) for all \( i, j \geq n_0 \). Using the definition of paranorm, we get

\[
(2.1) \sum_{k=1}^{mn} ||(x^i_k - x^j_k), z_1, \cdots, z_{n-1}|| + \inf \left\{ \rho \frac{\mu_k}{\rho} : \sup_k u_k M_k \left( \left\| \frac{\Delta_m^n(x^i_k - x^j_k)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \leq 1 \right\} < \epsilon, \quad \text{for all } i, j \geq n_0.
\]

Hence we have,

\[
\sum_{k=1}^{mn} ||(x^i_k - x^j_k), z_1, \cdots, z_{n-1}|| < \epsilon, \quad \text{for all } i, j \geq n_0.
\]

This implies that

\[
||(x^i_k - x^j_k), z_1, \cdots, z_{n-1}|| < \epsilon, \quad \text{for all } i, j \geq n_0 \text{ and } 1 \leq k \leq mn.
\]

Thus \( (x^i_k) \) is a Cauchy sequence for \( k = 1, 2, \ldots, mn \). Hence \( (x^i_k) \) is convergent for \( k = 1, 2, \ldots, mn \). Let

\[
(2.2) \lim_{i \to \infty} x^i_k = x_k, \quad \text{say for } k = 1, 2, \cdots, mn.
\]

Again from equation (2.1) we have,

\[
\inf \left\{ \rho \frac{\mu_k}{\rho} : \sup_k u_k M_k \left( \left\| \frac{\Delta_m^n(x^i_k - x^j_k)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \leq 1 \right\} < \epsilon, \quad \text{for all } i, j \geq n_0.
\]

Hence we get

\[
\sup_k u_k M_k \left( \left\| \frac{\Delta_m^n(x^i_k - x^j_k)}{g(x^i - x^j)}, z_1, \cdots, z_{n-1} \right\| \right) \leq 1, \quad \text{for all } i, j \geq n_0.
\]
It follows that $u_k M_k \left( \left\| \frac{\Delta^n(x^i_k - x^j_k)}{g(x^i - x^j)} \cdot z_1, \cdots, z_{n-1} \right\| \right) \leq 1$, for each $k \geq 1$ and for all $i, j \geq n_0$. For $t > 0$ with $u_k M_k \left( \frac{tx_0}{2} \right) \geq 1$, we have

$$u_k M_k \left( \left\| \frac{\Delta^n(x^i_k - x^j_k)}{g(x^i - x^j)} \cdot z_1, \cdots, z_{n-1} \right\| \right) \leq u_k M_k \left( \frac{tx_0}{2} \right).$$

This implies that

$$\left\| \Delta^n x^i_k - \Delta^n x^j_k, z_1, \cdots, z_{n-1} \right\| < \frac{tx_0}{2} \frac{\epsilon}{tx_0} = \frac{\epsilon}{2}.$$  

Hence $(\Delta^n x^i_k)$ is a Cauchy sequence for all $k \in \mathbb{N}$. This implies that $(\Delta^n x^i_k)$ is convergent for all $k \in \mathbb{N}$. Let $\lim_{i \to \infty} \Delta^n x^i_k = y_k$ for each $k \in \mathbb{N}$. Let $k = 1$, then we have

$$\lim_{i \to \infty} \Delta^n x^i_1 = \lim_{i \to \infty} \sum_{v=0}^{\infty} (-1)^v \left( \frac{n}{v} \right) x^i_{1 + mv} = y_1.$$  

We have by equation (2.2) and equation (2.3) $\lim_{i \to \infty} x^i_{mn+1} = x_{mn+1}$, exists. Proceeding in this way inductively, we have $\lim_{i \to \infty} x^i_k = x_k$ exists for each $k \in \mathbb{N}$. Now we have for all $i, j \geq n_0$,

$$\sum_{k=1}^{mn} \left\| (x^i_k - x^j_k), z_1, \cdots, z_{n-1} \right\|$$

$$+ \inf \left\{ \rho_{\mathbb{R}}^n : \sup_k u_k M_k \left( \left\| \frac{\Delta^n(x^i_k - x^j_k)}{\rho} - x^j_k, \cdots, z_{n-1} \right\| \right) \leq 1 \right\} < \epsilon.$$  

This implies that

$$\lim_{j \to \infty} \left\{ \sum_{k=1}^{mn} \left\| (x^i_k - x^j_k), z_1, \cdots, z_{n-1} \right\|$$

$$+ \inf \left\{ \rho_{\mathbb{R}}^n : \sup_k u_k M_k \left( \left\| \frac{\Delta^n(x^i_k - x^j_k)}{\rho} - x^j_k, \cdots, z_{n-1} \right\| \right) \leq 1 \right\} \right\} < \epsilon,$$

for all $i \geq n_0$. Using the continuity of $(M_k)$, we have

$$\sum_{k=1}^{mn} \left\| (x^i_k - x_k), z_1, \cdots, z_{n-1} \right\|$$

$$+ \inf \left\{ \rho_{\mathbb{R}}^n : \sup_k u_k M_k \left( \left\| \frac{\Delta^n x^i_k - \Delta^n x_k}{\rho} \right\|, z_1, \cdots, z_{n-1} \right\| \right) \leq 1 \right\} < \epsilon,$$

for all $i \geq n_0$. It follows that $(x^i - x) \in l_\infty(M, \Delta^n, p, u, ||\cdot||, ||\cdot||)$ and $l_\infty(M, \Delta^n, p, u, ||\cdot||, ||\cdot||)$ is a linear space, so
we have \( x = x^i - (x^i - x) \in l_\infty(M, \Delta_m^n, p, u, ||\cdot||, \cdot||) \). This completes the proof.

Similarly, we can prove that \( c(M, \Delta_m^n, p, u, ||\cdot||, \cdot||) \) and \( c_0(M, \Delta_m^n, p, u, ||\cdot||, \cdot||) \) are complete paranormed spaces in view of the above proof. 

\[ \square \]

**Theorem 2.4.** If \( 0 < p_k \leq q_k < \infty \) for each \( k \), then \( Z(M, \Delta_m^n, p, u, ||\cdot||, \cdot||) \subseteq Z(M, \Delta_m^n, q, u, ||\cdot||, \cdot||) \), for \( Z = c_0 \) and \( c \).

**Proof.** Let \( x = (x_k) \in c(M, \Delta_m^n, p, u, ||\cdot||, \cdot||) \). Then there exists some \( \rho > 0 \) and \( L \in X \) such that

\[
\lim_{k \to \infty} u_k \left( M_k \left( \left| \frac{\Delta_m x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \right)^{p_k} = 0.
\]

This implies that

\[
\lim_{k \to \infty} u_k M_k \left( \left| \frac{\Delta_m x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right| \right) < \epsilon, \quad (0 < \epsilon < 1)
\]

for sufficiently large \( k \). Hence we get

\[
\begin{align*}
\lim_{k \to \infty} u_k \left( M_k \left( \left| \frac{\Delta_m x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \right)^{q_k} & \leq \lim_{k \to \infty} u_k \left( M_k \left( \left| \frac{\Delta_m x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \right)^{p_k} \\
& = 0.
\end{align*}
\]

This implies that \( x = (x_k) \in c(M, \Delta_m^n, q, u, ||\cdot||, \cdot||) \). This completes the proof.

Similarly, we can prove for the case \( Z = c_0 \). \[ \square \]

**Theorem 2.5.** If \( M' = (M'_k) \) and \( M'' = (M''_k) \) be two Musielak-Orlicz functions. Then

(i) \( Z(M', \Delta_m^n, p, u, ||\cdot||, \cdot||) \subseteq Z(M'' \circ M', \Delta_m^n, p, u, ||\cdot||, \cdot||), \)

(ii) \( Z(M', \Delta_m^n, p, u, ||\cdot||, \cdot||) \cap Z(M'' \circ M', \Delta_m^n, p, u, ||\cdot||, \cdot||) \)

\[ \subseteq Z(M' + M'', \Delta_m^n, p, u, ||\cdot||, \cdot||), \]

for \( Z = l_\infty, c \) and \( c_0 \).

**Proof.** (i) We prove this part for \( Z = l_\infty \) and the rest of the cases will follow similarly. Let \( (x_k) \in l_\infty(M', \Delta_m^n, p, u, ||\cdot||, \cdot||) \), then there exists \( 0 < U < \infty \) such that

\[
u_k \left( M'_k \left( \left| \frac{\Delta_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \right)^{p_k} \leq U, \quad \text{for all} \ k \in \mathbb{N}.
\]

Let \( y_k = u_k M'_k \left( \left| \frac{\Delta_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \). Then \( y_k \leq U \frac{p_k}{\rho} \leq V \), say for all \( k \in \mathbb{N} \).

Hence we have

\[
\left( (M''_k \circ M'_k) \left( \left| \frac{\Delta_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right| \right) \right)^{p_k} = (M''_k(y_k))^{p_k} \leq (M''_k(V))^{p_k} < \infty,
\]
similarly. Let $u_k \left( \frac{\Delta_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) < \infty$. Thus $x = (x_k) \in l_\infty(M'' \circ M', \Delta_m, p, u, ||\cdot||)$. 

(ii) We prove the result for the case $Z = c$ and the rest of the cases will follow similarly. Let $x = (x_k) \in c(M', \Delta_m', p, u, ||\cdot||) \cap c(M'', \Delta_m, p, u, ||\cdot||)$, then there exist some $\rho_1, \rho_2 > 0$ and $L \in X$ such that

$$
\lim_{k \to \infty} u_k \left( M'_k \left( \frac{\Delta_m x_k - L}{\rho_1}, z_1, \cdots, z_{n-1} \right) \right) = 0
$$

and

$$
\lim_{k \to \infty} u_k \left( M''_k \left( \frac{\Delta_m x_k - L}{\rho_2}, z-1, \cdots, z_{n-1} \right) \right) = 0.
$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$
u_k \left( \left( M'_k + M''_k \right) \left( \frac{\Delta_m x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right) \leq K \left[ \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) u_k M'_k \left( \frac{\Delta_m x_k - L}{\rho_1}, z_1, \cdots, z_{n-1} \right) \right] + K \left[ \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) u_k M''_k \left( \frac{\Delta_m x_k - L}{\rho_2}, z_1, \cdots, z_{n-1} \right) \right].
$$

This implies that

$$
\lim_{k \to \infty} u_k \left( \left( M'_k + M''_k \right) \left( \frac{\Delta_m x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right) = 0.
$$

Thus $x = (x_k) \in c(M' + M'', \Delta_m', p, u, ||\cdot||)$. This completes the proof. \qed

**Theorem 2.6.** Let $M = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers, then $Z(M, \Delta_m^{-1}, p, u, ||\cdot||) \subset Z(M, \Delta_m, p, u, ||\cdot||)$, for $Z = l_\infty, c$ and $c_0$.

Proof. We prove the result for the case $Z = l_\infty$ and the rest of the cases will follow similarly. Let $x = (x_k) \in l_\infty(M, \Delta_m^{-1}, p, u, ||\cdot||)$. Then we can have $\rho > 0$ such that

$$
u_k \left( M_k \left( \frac{\Delta_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) \right) < \infty, \quad \text{for all } k \in \mathbb{N}.
$$

On considering 2p and using the convexity of $(M_k)$, we have

$$
u_k M_k \left( \frac{\Delta_m x_k}{2\rho}, z_1, \cdots, z_{n-1} \right) \leq \frac{1}{2} \nu_k M_k \left( \frac{\Delta_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right) + \frac{1}{2} \nu_k M_k \left( \frac{\Delta_m x_k + m}{\rho}, z_1, \cdots, z_{n-1} \right).
$$
Theorem 2.7. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then
\[
c_0(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||) \subset c(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||)
\subset l_\infty(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||).
\]

Proof. It is obvious that $c_0(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||) \subset c(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||)$. We shall prove that $c(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||) \subset l_\infty(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||)$. Let $x = (x_k) \in c(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||)$. Then there exists some $\rho > 0$ and $L \in X$ such that
\[
\lim_{k \to \infty} u_k \left( M_k \left( \left\| \frac{\Delta^*_m x_k - L}{\rho} \right\|, z_1, \cdots, z_{n-1} \right) \right)^{p_k} = 0.
\]
On taking $\rho = 2\rho_1$, we have
\[
u_k \left( M_k \left( \left\| \frac{\Delta^*_m x_k}{\rho} \right\|, z_1, \cdots, z_{n-1} \right) \right)^{p_k}
\leq K \left[ \frac{1}{2} u_k \left( M_k \left( \left\| \frac{\Delta^*_m x_k - L}{\rho_1} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k}
\quad + K \left[ \frac{1}{2} u_k \left( M_k \left( \left\| \frac{L}{\rho_1} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k}
\leq K \left( \frac{1}{2} \right)^{p_k} u_k \left( M_k \left( \left\| \frac{\Delta^*_m x_k - L}{\rho_1} \right\|, z_1, \cdots, z_{n-1} \right) \right)^{p_k}
\quad + K \left( \frac{1}{2} \right)^{p_k} \max \left( 1, u_k \left( M_k \left( \left\| \frac{L}{\rho_1} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right)^{H},
\]
where $H = \max(1, \sup p_k)$. Thus we get $x = (x_k) \in l_\infty(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||)$. Hence
\[
c_0(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||) \subset c(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||)
\subset l_\infty(\mathcal{M}, \Delta^*_m, p, u, ||\cdot, \cdot||).
\]
Theorem 2.8. The sequence space \( l_\infty(M, \Delta^n_m, p, u, ||\cdot||) \) is solid.

Proof. Let \( x = (x_k) \in l_\infty(M, \Delta^n_m, p, u, ||\cdot||) \), that is
\[
\lim_{k \to \infty} u_k \left( M_k \left( \left\| \frac{\Delta^n_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right)^{p_k} < \infty.
\]
Let \( (\alpha_k) \) be a sequence of scalars such that \( |\alpha_k| \leq 1 \) for all \( k \in \mathbb{N} \). Thus we have
\[
\lim_{k \to \infty} u_k \left[ M_k \left( \left\| \frac{\Delta^n_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \leq \lim_{k \to \infty} u_k \left[ M_k \left( \left\| \frac{\Delta^n_m x_k}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} < \infty.
\]
This shows that \((\alpha_k x_k) \in l_\infty(M, \Delta^n_m, p, u, ||\cdot||)\) for all sequences of scalars \((\alpha_k)\) with \( |\alpha_k| \leq 1 \) for all \( k \in \mathbb{N} \), whenever \((x_k) \in l_\infty(M, \Delta^n_m, p, u, ||\cdot||)\). Hence the space \( l_\infty(M, \Delta^n_m, p, u, ||\cdot||) \) is a solid sequence space. \( \square \)

Theorem 2.9. The sequence space \( l_\infty(M, \Delta^n_m, p, u, ||\cdot||) \) is monotone.

Proof. The proof of the theorem is obvious and so we omit it.

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References


