The Line $n$-sigraph of a Symmetric $n$-sigraph-V

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Abstract. An $n$-tuple $(a_1, a_2, ..., a_n)$ is symmetric, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric $n$-tuples. A symmetric $n$-sigraph (symmetric $n$-marked graph) is an ordered pair $\mathcal{S}_n = (G, \sigma)$ $(\mathcal{S}_n = (G, \mu))$, where $G = (V, E)$ is a graph called the underlying graph of $\mathcal{S}_n$ and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function. The restricted super line graph of index $r$ of a graph $G$, denoted by $RL_r(G)$. The vertices of $RL_r(G)$ are the $r$-subsets of $E(G)$ and two vertices $P = \{p_1, p_2, ..., p_r\}$ and $Q = \{q_1, q_2, ..., q_r\}$ are adjacent if there exists exactly one pair of edges, say $p_i$ and $q_j$, where $1 \leq i, j \leq r$, that are adjacent edges in $G$. Analogously, one can define the restricted super line symmetric $n$-sigraph of index $r$ of a symmetric $n$-sigraph $\mathcal{S}_n = (G, \sigma)$ as a symmetric $n$-sigraph $RL_r(\mathcal{S}_n) = (RL_r(G), \sigma')$, where $RL_r(G)$ is the underlying graph of $RL_r(\mathcal{S}_n)$, where for any edge $PQ$ in $RL_r(S_n)$, $\sigma'(PQ) = \sigma(P) \sigma(Q)$. It is shown that for any symmetric $n$-sigraph $\mathcal{S}_n$, its $RL_r(\mathcal{S}_n)$ is $i$-balanced and we offer a structural characterization of super line symmetric $n$-sigraphs of index $r$. Further, we characterize symmetric $n$-sigraphs $\mathcal{S}_n$ for which $RL_r(\mathcal{S}_n) \sim L_r(\mathcal{S}_n)$ and $RL_r(S_n) \cong L_r(\mathcal{S}_n)$, where $\sim$ and $\cong$ denotes switching equivalence and isomorphism and $RL_r(S_n)$ and $L_r(S_n)$ are denotes the restricted super line symmetric $n$-sigraph of index $r$ and super line symmetric $n$-sigraph of index $r$ of $\mathcal{S}_n$ respectively.

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1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [2]. We consider only finite, simple graphs free from self-loops.

Let \( n \geq 1 \) be an integer. An \( n \)-tuple \((a_1, a_2, \ldots, a_n)\) is symmetric, if \( a_k = a_{n-k+1}, 1 \leq k \leq n \). Let \( H_n = \{(a_1, a_2, \ldots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\} \) be the set of all symmetric \( n \)-tuples. Note that \( H_n \) is a group under coordinate wise multiplication, and the order of \( H_n \) is \( 2^m \), where \( m = \lceil \frac{n}{2} \rceil \).

A symmetric \( n \)-sigraph (symmetric \( n \)-marked graph) is an ordered pair \( S_n = (G, \sigma) \) (\( S_n = (G, \mu) \)), where \( G = (V, E) \) is a graph called the underlying graph of \( S_n \) and \( \sigma : E \rightarrow H_n \) (\( \mu : V \rightarrow H_n \)) is a function.

In this paper by an \( n \)-tuple/\( n \)-sigraph/\( n \)-marked graph we always mean a symmetric \( n \)-tuple/symmetric \( n \)-sigraph/symmetric \( n \)-marked graph.

An \( n \)-tuple \((a_1, a_2, \ldots, a_n)\) is the identity \( n \)-tuple, if \( a_k = + \), for \( 1 \leq k \leq n \), otherwise it is a non-identity \( n \)-tuple. In an \( n \)-sigraph \( S_n = (G, \sigma) \) an edge labelled with the identity \( n \)-tuple is called an identity edge, otherwise it is a non-identity edge.

Further, in an \( n \)-sigraph \( S_n = (G, \sigma) \), for any \( A \subseteq E(G) \) the \( n \)-tuple \( \sigma(A) \) is the product of the \( n \)-tuples on the edges of \( A \).

In [17], the authors defined two notions of balance in \( n \)-sigraph \( S_n = (G, \sigma) \) as follows (See also R. Rangarajan and P.S.K.Reddy [6]):

**Definition 1.1.** Let \( S_n = (G, \sigma) \) be an \( n \)-sigraph. Then,

(i) \( S_n \) is identity balanced (or \( i \)-balanced), if product of \( n \)-tuples on each cycle of \( S_n \) is the identity \( n \)-tuple, and

(ii) \( S_n \) is balanced, if every cycle in \( S_n \) contains an even number of non-identity edges.

**Note:** An \( i \)-balanced \( n \)-sigraph need not be balanced and conversely.

The following characterization of \( i \)-balanced \( n \)-sigraphs is obtained in [17].

**Proposition 1.1.** (E. Sampathkumar et al. [17])

An \( n \)-sigraph \( S_n = (G, \sigma) \) is \( i \)-balanced if, and only if, it is possible to assign \( n \)-tuples to its vertices such that the \( n \)-tuple of each edge uv is equal to the product of the \( n \)-tuples of u and v.

Let \( S_n = (G, \sigma) \) be an \( n \)-sigraph. Consider the \( n \)-marking \( \mu \) on vertices of \( S_n \) defined as follows: each vertex \( v \in V \), \( \mu(v) \) is the \( n \)-tuple which is the product of the \( n \)-tuples on the edges incident with \( v \). **Complement** of \( S_n \) is an \( n \)-sigraph
Given a graph \( G = (V, E) \), the \( r \)-restricted super line \( n \)-sigraph \( S_n = (G, \sigma) \) is defined as follows: (See also [3, 7, 8] & [9]-[16])

Let \( S_n = (G, \sigma) \) and \( S'_n = (G', \sigma') \), be two \( n \)-sigraphs. Then \( S_n \) and \( S'_n \) are said to be isomorphic, if there exists an isomorphism \( \phi : G \rightarrow G' \) such that if \( uv \) is an edge in \( S_n \) with label \((a_1, a_2, ..., a_n)\) then \( \phi(u)\phi(v) \) is an edge in \( S'_n \) with label \((a_1, a_2, ..., a_n)\).

Given an \( n \)-marking \( \mu \) of an \( n \)-sigraph \( S_n = (G, \sigma) \), switching \( S_n \) with respect to \( \mu \) is the operation of changing the \( n \)-tuple of every edge \( uv \) of \( S_n \) by \( \mu(u)\sigma(uv)\mu(v) \). The \( n \)-sigraph obtained in this way is denoted by \( S_\mu(S_n) \) and is called the \( \mu \)-switched \( n \)-sigraph or just switched \( n \)-sigraph.

Further, an \( n \)-sigraph \( S_n \) switches to \( n \)-sigraph \( S'_n \) (or that they are switching equivalent to each other), written as \( S_n \sim S'_n \), whenever there exists an \( n \)-marking of \( S_n \) such that \( S_\mu(S_n) \cong S'_n \).

Two \( n \)-sigraphs \( S_n = (G, \sigma) \) and \( S'_n = (G', \sigma') \) are said to be cycle isomorphic, if there exists an isomorphism \( \phi : G \rightarrow G' \) such that the \( n \)-tuple \( \sigma(C) \) of every cycle \( C \) in \( S_n \) equals to the \( n \)-tuple \( \sigma(\phi(C)) \) in \( S'_n \). We make use of the following known result (see [17]).

**Proposition 1.2.** ([E. Sampathkumar et al. [17]])

*Given a graph \( G \), any two \( n \)-sigraphs with \( G \) as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

In this paper, we introduced the notion called restricted super line \( n \)-sigraph of index \( r \) and we obtained some interesting results in the following sections. The restricted super line \( n \)-sigraph of index \( r \) is the generalization of line \( n \)-sigraph.

### 2. Restricted Super Line \( n \)-sigraph \( L_r(S_n) \)

In [4], K. Manjula introduced the concept of the \emph{restricted super line graph}, which generalizes the notion of line graph. For a given \( G \), its restricted super line graph \( \mathcal{RL}_r(G) \) of index \( r \) is the graph whose vertices are the \( r \)-subsets of \( E(G) \), and two vertices \( P = \{p_1, p_2, ..., p_r\} \) and \( Q = \{q_1, q_2, ..., q_r\} \) are adjacent if there exists exactly one pair of edges, say \( p_i \) and \( q_j \), where \( 1 \leq i, j \leq r \), that are adjacent edges in \( G \). In [1], the authors introduced the concept of the \emph{super line graph} as follows: For a given \( G \), its super line graph \( L_r(G) \) of index \( r \) is the graph whose vertices are the \( r \)-subsets of \( E(G) \), and two vertices \( P \) and \( Q \) are adjacent if there exist \( p \in P \) and \( q \in Q \) such that \( p \cap q \neq \emptyset \).

The \emph{restricted super line graph} \( L_r(S_n) \) is defined as follows: For a given \( n \)-sigraph \( S_n = (G, \sigma) \), its restricted super line graph \( L_r(S_n) \) is the graph whose vertices are the \( r \)-subsets of \( E(G) \), and two vertices \( P \) and \( Q \) are adjacent if there exist \( p \in P \) and \( q \in Q \) such that \( p \cap q \neq \emptyset \).
and $q \in Q$ such that $p$ and $q$ are adjacent edges in $G$. Clearly $RL_r(G)$ is a spanning subgraph of $L_r(G)$. From the definitions of $RL_r(G)$ and $L_r(G)$, it turns out that $RL_1(G)$ and $L_1(G)$ coincides with the line graph $L(G)$.

In this paper, we extend the notion of $RL_r(G)$ to realm of $n$-sigraphs as follows: The restricted super line $n$-sigraph of index $r$ of an $n$-sigraph $S_n = (G, \sigma)$ as an $n$-sigraph $RL_r(S_n) = (RL_r(G), \sigma')$, where $RL_r(G)$ is the underlying graph of $RL_r(S_n)$, where for any edge $PQ$ in $RL_r(S_n)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$.

Hence, we shall call a given $n$-sigraph $S_n$ is a restricted super line $n$-sigraph of index $r$ if it is isomorphic to the restricted super line $n$-sigraph of index $r$, $RL_r(S'_n)$ of some $n$-sigraph $S'_n$. In the following subsection, we shall present a characterization of restricted super line $n$-sigraph of index $r$.

The following result indicates the limitations of the notion $RL_r(S_n)$ as introduced above, since the entire class of $i$-unbalanced $n$-sigraphs is forbidden to be restricted super line $n$-sigraphs of index $r$.

**Proposition 2.1.** For any $n$-sigraph $S_n = (G, \sigma)$, its $RL_r(S_n)$ is $i$-balanced.

*Proof.* Let $\sigma'$ denote the $n$-tuple of $RL_r(S_n)$ and let the $n$-tuple $\sigma$ of $S_n$ be treated as an $n$-marking of the vertices of $RL_r(S_n)$. Then by definition of $RL_r(S_n)$ we see that $\sigma'(P, Q) = \sigma(P)\sigma(Q)$, for every edge $PQ$ of $RL_r(S_n)$ and hence, by Proposition 1.1, the result follows. \qed

For any positive integer $k$, the $k^{th}$ iterated restricted super line $n$-sigraph of index $r$, $RL_r(S_n)$ of $S_n$ is defined as follows:

$$RL_r^0(S_n) = S_n, \quad RL_r^k(S_n) = RL_r(RL_r^{k-1}(S_n))$$

**Corollary 2.2.** For any $n$-sigraph $S_n = (G, \sigma)$ and any positive integer $k$, $RL_r^k(S_n)$ is $i$-balanced.

In [16], the authors introduced the notion of the super line $n$-sigraph, which generalizes the notion of line $n$-sigraph [18]. The super line $n$-sigraph of index $r$ of an $n$-sigraph $S_n = (G, \sigma)$ as an $n$-sigraph $L_r(S_n) = (L_r(G), \sigma')$, where $L_r(G)$ is the underlying graph of $L_r(S_n)$, where for any edge $PQ$ in $L_r(S_n)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$. The above notion restricted super line $n$-sigraph is another generalization of line $n$-sigraphs.

**Proposition 2.3.** (P.S.K.Reddy et al. [16])

*For any $n$-sigraph $S_n = (G, \sigma)$, its $L_r(S_n)$ is $i$-balanced.*

In [4], the author characterized whose restricted super line graphs of index $r$ that are isomorphic to $L_r(G)$. 
Proposition 2.4. (K. Manjula [4])

For a graph $G = (V,E)$, $RL_r(G) \cong L_r(G)$ if, and only if, $G$ is either $K_{1,2} \cup nK_2$ or $nK_2$.

We now characterize $n$-sighraphs those $RL_r(S_n)$ are switching equivalent to their $L_r(S_n)$.

Proposition 2.5. For any $n$-sighraph $S_n = (G, \sigma)$, $RL_r(S_n) \sim L_r(S_n)$ if, and only if, $G$ is either $K_{1,2} \cup nK_2$ or $nK_2$.

Proof. Suppose $RL_r(S_n) \sim L_r(S_n)$. This implies, $RL_r(G) \cong L_r(G)$ and hence by Proposition 2.4, we see that the graph $G$ must be isomorphic to either $K_{1,2} \cup nK_2$ or $nK_2$.

Conversely, suppose that $G$ is either $K_{1,2} \cup nK_2$ or $nK_2$. Then $RL_r(G) \cong L_r(G)$ by Proposition 2.4. Now, if $S_n$ any $n$-sighraph on any of these graphs, by Proposition 2.1 and Proposition 2.3, $RL_r(S_n)$ and $L_r(S_n)$ are $i$-balanced and hence, the result follows from Proposition 1.2. \qed

We now characterize $n$-sighraphs those $RL_r(S_n)$ are isomorphic to their $L_r(S_n)$. The following result is a stronger form of the above result.

Proposition 2.6. For any $n$-sighraph $S_n = (G, \sigma)$, $RL_r(S_n) \cong L_r(S_n)$ if, and only if, $G$ is either $K_{1,2} \cup nK_2$ or $nK_2$.

Proof. Clearly $RL_r(S_n) \cong L_r(S_n)$, where $G$ is either $K_{1,2} \cup nK_2$ or $nK_2$. Consider the map $f : V(RL_r(G)) \to V(L_r(S))$ defined by $f(e_1e_2, e_2e_3) = (e_1'e_2', e_2'e_3')$ is an isomorphism. Let $\sigma$ be any $n$-tuple on $K_{1,2} \cup nK_2$ or $nK_2$. Let $e = (e_1e_2, e_2e_3)$ be an edge in $RL_r(G)$, where $G$ is $K_{1,2} \cup nK_2$ or $nK_2$. Then the $n$-tuple of the edge $e$ in $RL_r(G)$ is the $\sigma(e_1e_2)e(e_2e_3)$ which is the $n$-tuple of the edge $(e_1'e_2', e_2'e_3')$ in $L_r(G)$, where $G$ is $K_{1,2} \cup nK_2$ or $nK_2$. Hence the map $f$ is also an $n$-sighraph isomorphism between $RL_r(S_n)$ and $L_r(S_n)$. \qed

3. Characterization of Restricted Super Line $n$-sighraphs $RL_r(S_n)$

The following result characterize $n$-sighraphs which are restricted super line $n$-sighraphs of index $r$.

Proposition 3.1. An $n$-sighraph $S_n = (G, \sigma)$ is a restricted super line $n$-sighraph of index $r$ if and only if $S_n$ is $i$-balanced $n$-sighraph and its underlying graph $G$ is a restricted super line graph of index $r$.

Proof. Suppose that $S_n$ is $i$-balanced and $G$ is a $RL_r(G)$. Then there exists a graph $H$ such that $L_r(H) \cong G$. Since $S_n$ is $i$-balanced, by Proposition 1.1, there exists an $n$-marking $\mu$ of $G$ such that each edge $uv$ in $S_n$ satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the $n$-sighraph $S'_n = (H, \sigma')$, where for any edge $e$ in $H$, $\sigma'(e)$ is the
n-marking of the corresponding vertex in $G$. Then clearly, $\mathcal{R}_r(S'_n) \cong S_n$. Hence $S_n$ is a restricted super line $n$-sigraph of index $r$.

Conversely, suppose that $S_n = (G, \sigma)$ is a restricted super line $n$-sigraph of index $r$. Then there exists an $n$-sigraph $S'_n = (H, \sigma')$ such that $\mathcal{R}_r(S'_n) \cong S_n$. Hence $G$ is the $\mathcal{R}_r(G)$ of $H$ and by Proposition 2.1, $S_n$ is $i$-balanced.

If we take $r = 1$ in $\mathcal{R}_r(S_n)$, then this is the ordinary line $n$-sigraph. In [18], the authors obtained structural characterization of line $n$-sigraphs and clearly Proposition 3.1 is the generalization of line signed graphs.

**Proposition 3.2.** An $n$-sigraph $S_n = (G, \sigma)$ is a line $n$-sigraph if, and only if, $S_n$ is $i$-balanced $n$-sigraph and its underlying graph $G$ is a line graph.

4. Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a sigraph) in the more general context of graphs with multiple signs on their edges. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge.

For any $m \in H_n$, the $m$-complement of $a = (a_1, a_2, \ldots, a_n)$ is: $a^m = a_m$. For any $M \subseteq H_n$, and $m \in H_n$, the $m$-complement of $M$ is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the $m$-complement of an $n$-sigraph $S_n = (G, \sigma)$, written $(S_n)_m$, is the same graph but with each edge label $a = (a_1, a_2, \ldots, a_n)$ replaced by $a^m$.

For an $n$-sigraph $S_n = (G, \sigma)$, the $\mathcal{R}_r(S_n)$ is $i$-balanced (Proposition 2.1). We now examine, the condition under which $m$-complement of $\mathcal{R}_r(S_n)$ is $i$-balanced, where for any $m \in H_n$.

**Proposition 4.1.** Let $S_n = (G, \sigma)$ be an $n$-sigraph. Then, for any $m \in H_n$, if $\mathcal{R}_r(G)$ is bipartite then $(\mathcal{R}_r(S_n))^m$ is $i$-balanced.

Proof. Since, by Proposition 2.1, $\mathcal{R}_r(S_n)$ is $i$-balanced, for each $k$, $1 \leq k \leq n$, the number of $n$-tuples on any cycle $C$ in $\mathcal{R}_r(S_n)$ whose $k^{th}$ co-ordinate are $-i$ is even. Also, since $\mathcal{R}_r(G)$ is bipartite, all cycles have even length; thus, for each $k$, $1 \leq k \leq n$, the number of $n$-tuples on any cycle $C$ in $\mathcal{R}_r(S_n)$ whose $k^{th}$ co-ordinate are $+i$ is also even. This implies that the same thing is true in any $m$-complement, where for any $m \in H_n$. Hence $(\mathcal{R}_r(S_n))^m$ is $i$-balanced. □
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References