Poset Properties Determined by the Ideal - Based Zero-divisor Graph

KASI PORSELVI AND BALASUBRAMANIAN ELAVARASAN*
Department of Mathematics, School of Science and Humanities, Karunya University, Coimbatore - 641 114, Tamilnadu, India
e-mail: porselvi94@yahoo.co.in and belavarasan@gmail.com

ABSTRACT. In this paper, we study some properties of finite or infinite poset \( P \) determined by properties of the ideal based zero-divisor graph properties \( G_J(P) \), for an ideal \( J \) of \( P \).

1. Introduction

Throughout this paper, \((P, \leq)\) denotes a poset and the graph \( G \) denotes the ideal based zero-divisor graph of a poset \( P \) with respect to ideal \( I \) of \( P \). For \( M \subseteq P \), let \( L(M) := \{x \in P : x \leq m \text{ for all } m \in M\} \) denotes the lower cone of \( M \) in \( P \), and dually let \( U(M) := \{x \in P : m \leq x \text{ for all } m \in M\} \) be the upper cone of \( M \) in \( P \). For \( A, B \subseteq P \) we shall write \( L(A, B) \) instead of \( L(A \cup B) \) and dually for upper cones. If \( M = \{x_1, ..., x_n\} \) is finite, then we use the notation \( L(x_1, ..., x_n) \) instead of \( L(\{x_1, ..., x_n\}) \) (and dually). By an ideal we mean a non-empty subset \( I \subseteq P \) such that if \( b \in I \) and \( a \leq b \), then \( a \in I \). A proper order-ideal \( I \) of \( P \) is called prime if for any \( a, b \in P \), \( L(a, b) \subseteq I \) implies \( a \in I \) or \( b \in I \). In [2], Beck introduced the concept of a zero-divisor graph of a commutative ring with identity, but this work was mostly concerned with coloring of rings. Later D. F. Anderson and Livingston in [1] studied the subgraph \( \Gamma(R) \) of \( G(R) \) whose vertices are the nonzero zero-divisors of \( R \). In [10], Redmond has generalized the notion of the zero-divisor graph. For a given ideal \( I \) of a commutative ring \( R \), he defined an undirected graph \( \Gamma_I(R) \) with vertices \( \{x \in R \setminus I : xy \in I \text{ for some } y \in R \setminus I\} \), where distinct vertices \( x \) and \( y \) are adjacent if and only if \( xy \in I \). The zero-divisor graph of various algebraic structures has been studied by several authors [4],[5],[7] and[11].

In [8], Radomr Halas and Marek Jukl have introduced the concept of a graph structure of a posets, let \((P, \leq)\) be a poset with \( 0 \). Then the zero-divisor graph of \( P \), denoted by \( \Gamma(P) \), is an undirected graph whose vertices are just the elements of \( P \) with two distinct vertices \( x \) and \( y \) are joined by an edge if and only if \( L(x, y) = \{0\} \),

* Corresponding Author.
Received November 29, 2011; accepted May 24, 2013.
2010 Mathematics Subject Classification: 06D6.
Key words and phrases: Posets, ideals, prime ideals, graph, cycle and cut-set.
and proved some interesting results related with clique and chromatic number of this graph structure. In [6], we generalized the notion of zero-divisor graph of P. Let P be a poset and J be an ideal of P. Then the graph of P with respect to the ideal J, denoted by G_J(P), is the graph whose vertices are the set \{x \in P \mid J : L(x, y) \subseteq J \} for some y \in P \setminus J with distinct vertices x and y are adjacent if and only if L(x, y) \subseteq J. If J = \{0\}, then G_J(P) = G(P), and J is a prime ideal of P if and only if G_J(P) = \phi. And investigated the interplay between the poset properties of P and the graph-theoretics properties of G_J(P). Following [9], let I be an ideal of P. Then the extension of I by x \in P is meant the set \langle x, I \rangle = \{a \in P : L(a, x) \subseteq I\}. For any s \in V(G), N(s) denotes the set of all vertices adjacent to s and \overline{K(G)} denotes the core of G. In this paper the notations of graph theory are from [3], the notations of posets from [8].

2. Poset Properties Related to a Single Vertex

**Theorem 2.1.** Let G be the graph of a poset P. If there exist s, t \in V(G) such that N(s) \neq \phi, N(t) \neq \phi, then L(x, y) \subseteq (N(s) \cap N(t)) \cup I for x \in N(s), y \in N(t). In addition, if x is an end vertex, then I \cup \{s\} is an ideal of P.

**Proof.** Let t_1 \in L(x, y) \setminus I for x \in N(s) and y \in N(t). Then L(x, s) \subseteq I and L(y, t) \subseteq I. If t_1 \in \{x, y\}, then it is easy to see that t_1 \in N(s) \cap N(t). If t_1 = s, then s \in L(x, s) \subseteq I, a contradiction. So t_1 \neq s. Similar way, we can get t_1 \neq t. Now, L(t_1, s) \subseteq L(x, s) \subseteq I and L(t_1, t) \subseteq L(y, t) \subseteq I which imply t_1 \in N(s) \cap N(t). If x \in N(s) is an end vertex of G, then \langle x, I \rangle = I \cup \{s\} is an ideal of P.

**Corollary 2.2.** Let G be a graph of a poset P and y = s = t = x be a path in G. Then

(i) K(G) is non-empty and it contains at least \(|L(x, y) \setminus I|\) triangles.

(ii) If x and y are end vertices, then P has at least two ideals of the form I \cup \{s\}.

**Theorem 2.3.** Let P be a poset with corresponding graph G such that P = V(G) \cup \{I\}. For an element x \in P \setminus I, assume that V(G) = C_x \cup \{x\} \cup T(x) is a disjoint union of three subsets satisfying the following conditions:

(i) T(x) contains all end vertices adjacent to x.

(ii) There is no edge linking a vertex in T(x) with a vertex in C_x, whenever T(x) \neq \phi and C_x \neq \phi.

(iii) Either C_x \neq \phi or |V(G)| \geq 3 and x is adjacent to at least one end vertex. Then L(a, b) \subseteq C_x \cup \{x\} \cup I for all a, b \in C_x \cup \{x\} \cup I.

**Proof.** Let us assume that T(x) \neq \phi and let a, b \in C_x \cup \{x\} \cup I. If C_x = \phi, by assumption (iii), there exists an end vertex y adjacent with x which gives I \cup \{y\} is an ideal of P. So L(a, b) \subseteq I \cup \{y\}. If C_x \neq \phi, then there is at least one element z \in C_x such that z \neq x. Suppose L(x) \cap T(x) \neq \phi. Then there exists y \in L(x) \cap T(x) with such that L(z, y) \subseteq I, contradicting condition (ii). So L(x) \cap T(x) = \phi, i.e., L(x) \subseteq C_x \cup \{x\} \cup I. It remains to consider the case a, b \in C_x \cup \{x\} \cup I. Assume to the contrary that there is an element t \in L(a, b) such that t \notin C_x \cup \{x\} \cup I. If a
is not adjacent to $x$, then there exists $c \in C_x$ such that $L(a, c) \subseteq I$ which implies there is an edge $c - t$, where $c \in C_x, t \in T(x)$, contradicting condition (ii). If $a$ is adjacent to $x$, then by condition (i), $a$ is not an end vertex, then by condition (ii), there is an element $c(\neq a) \in C_x$ such that $a - c$. In this case also we have an edge $c - t$, contradicting condition (ii). So $L(a, b) \subseteq C_x \cup \{x\} \cup I$. 

For any vertex $x \in V(G)$, $T_x$ denotes the set of all end vertices adjacent to $x$ in $G$.

**Corollary 2.4.** Let $P$ be a poset with corresponding graph $G$ such that $V(G) = P\setminus I$. If $G$ is not a star graph, then for any $x \in V(G)$, we have $L(a, b) \subseteq P\setminus T_x$ for all $a, b \in P\setminus T_x$.

**Proof.** In Theorem 2.3, let $T(x) = T_x$. If $G$ is not a star graph, then $C_x \neq \emptyset$ and $P\setminus T(x) = C_x \cup \{x\} \cup I$. The result then follows from Theorem 2.3. 

**Theorem 2.5.** Let $G$ be the graph of a poset $P$ and assume that $G$ has a cycle. For any vertex $x$ in $G$ that is not an end vertex. If any two vertices in $L(a)$ are comparable ((i.e) $a \leq b$, for $a, b \in L(u)$), then $L(u, v) \subseteq T_x \cup I$ for all $u, v \in T_x \cup I$.

**Proof.** Suppose $L(u, v) \notin I$ for some $u, v \in T_x$. Then there exists $e \in L(u, v) \setminus I$ such that $e \neq x$. If $e$ is not an end vertex of $G$, by Theorem 3.4 of [6], it is in the core of $G$. Then there exists a vertex $d$ in the core such that $d \notin \{x, e\}$ and $x - c - d$. Since $L(d, u) \notin I$, there exists $e \in L(d, u) \setminus I$ such that $e \in I$ or $e \in I$ as $L(e, c) \subseteq I$, a contradiction. So $e$ is an end vertex of $G$.

Note that if we consider $x = \{a\}$ and $u = \{b, c\}$ in Example 2.8, then $\{b\}$ and $\{c\}$ are not comparable, but $L(\{b, c\}) \notin T_x \cup I$. Therefore, Theorem 2.5 is not valid in general. Hence, the condition comparable on the set $L(u)$ is not superficial in Theorem 2.5.

**Theorem 2.6.** Let $G$ be a graph of a poset $P$. If $G$ does not contain an infinite clique, then $P$ satisfies the a.c.c on $< x, I >$ for $x \in P$.

**Proof.** Suppose that $< x_1, I > \subseteq< x_2, I > \subseteq ... < x_i, I > ...$ be an increasing chain of ideals, for $x_i \in P$. If some $x_i \in I$, then the proof is trivial. So assume that $x_i \in P\setminus I$ for all $i$. For each $i \geq 2$, let $a_i < x_i, I > \setminus< x_{i-1}, I >$. Then $L(x_{n-1}, a_n) \notin I$ for $n = 2, 3, ...$. So there exists $y_n \in L(x_{n-1}, a_n) \setminus I$ such that $L(y_i, y_i) \subseteq I$ for any $i \neq j$. i.e., we have an infinite clique in $G$, a contradiction. So $P$ satisfies the a.c.c on $< x, I >$ for $x \in P$. 

**Example 2.7.** Let $G$ be a graph of a poset $P$. For any $x, y \in V(G)$ with $U(x, y) \cap V(G) \neq \emptyset$, then the edge $x - y$ is contained in a triangle.

**Proof.** Let $x, y \in V(G)$ with $x - y$ and $U(x, y) \cap V(G) \neq \emptyset$. Then there exists $t \in U(x, y) \cap V(G)$ such that $t \notin \{x, y\}$. Since $diam(G) \leq 3$, we have either $x - a - t$ or $x - a - b - t$ for some $a, b \in V(G)$. If $x - a - t$, then $x - a - y - x$. If $x - a - b - t$, then $x - b - t$ which implies $x - b - y - x$. 

We now show by an example that the Theorem 2.7 will fail if $U(x, y) \cap V(G) = \emptyset$ for any edge $x - y$ in $G$. 


Theorem 2.8. Let $P(X)$ be the power set of a set $X = \{a, b, c\}$. Then $P(X)$ is a poset under the set inclusion $\subseteq$. If $I = \{\phi\}$, then the graph $G$ is:

Here $U(\{a, b\}, \{c\}) \cap V(G) = \phi$ but the edge $\{a, b\} - \{c\}$ is not contained in a triangle.

The distance $d(v)$ of a vertex $v$ in a connected finite graph $G$ is the sum of the distances $v$ to each vertex of $G$. The median $M(G)$ of a graph $G$ is the subgraph induced by the set of vertices having minimum distance. Let $G$ be a connected graph, and $T \subseteq V(G)$. We say $T$ is a cut vertex set if $G \setminus T$ is disconnected. Also the cut vertex set $T$ is called a minimal cut vertex set for $G$ if no proper subset of $T$ is a cut vertex set. In addition, if $T = \{x\}$, then $x$ is called a cut vertex. \[\Box\]

Theorem 2.9. Let $G$ be a graph of a poset $P$. Then $V(M(G)) \cup I$ is an ideal of $P$. In addition, if $T$ is a minimal cut vertex set of $G$, then $T \cup I$ is an ideal of $P$.

Proof. Let $x \in V(M(G)) \setminus I$ and $y \in P$ with $y \leq x$. Suppose $y \notin I$. Then $y \in V(G)$ and $d(y, z) \leq d(x, z)$ for any $z \in V(G)$ which implies $d(y) = \sum_{z \in V(G)} d(y, z) \leq \sum_{z \in V(G)} d(x, z) = d(x)$. Since $x \in V(M(G))$, we have $d(y) = d(x)$, and hence $y \in V(M(G))$. Let $T$ be a minimal cut vertex set of $G$ and $x \in T, p \in P$ such that $p \leq x$. Then there exist two vertices $z, y$ of the graph $G$ such that $y - x - z$ is a path in $G$ and $y, z$ belong to two distinct connected components of $G \setminus T$, as $T \setminus \{x\}$ is not a cut vertex. Suppose $p \notin T \cup I$. Then there exists a path $y - p - z$ in $G \setminus T$, a contradiction. \[\Box\]

Corollary 2.10. Let $G$ be a graph of a poset $P$. If $x$ is cut-vertex of $G$, then $I \cup \{x\}$ is an ideal of $P$. For $y \in G \setminus \{x\}$, $x$ is adjacent to $y$ or $x \leq y$.

Corollary 2.11. Let $G$ be a graph of $G$, and let $x - y$ be a bridge $e$ of $G$ such that $G_1$ and $G_2$ are the two connected components of $G \setminus \{e\}$. Then the following conclusions hold:

(i) If $G_1$ and $G_2$ have at least two vertices, then $I \cup \{x\}$ and $I \cup \{y\}$ are ideals of $P$. Also, if $G_1$ or $G_2$ has only one vertex, then $I \cup \{x\}$ or $I \cup \{y\}$ is an ideal of $P$.

(ii) If $G_1$ and $G_2$ have exactly one vertex, then $I \cup \{x\}$ and $I \cup \{y\}$ are ideals of $P$, and hence $I \cup \{x, y\}$ is an ideal of $P$.

Proof. It follows from Corollary 2.10 and Theorem 2.1. \[\Box\]

The center $C(G)$ of a connected finite graph $G$ is the subgraph induced by the vertices of $G$ with eccentricity equal the radius of $G$. 

Theorem 2.12. Let $G$ be a graph of a poset $P$. For a finite poset, $V(C(G)) \cup I$ is an ideal of $P$.

**Proof.** Let $x \in V(C(G)) \cup I$ and $p \in P$ such that $p \leq x$. Suppose $p \notin I$. Then $p \in V(G)$ and $e(p) = \max\{d(u,p) : u \in V(G)\} \leq \max\{d(u,x) : u \in V(G)\} = e(x)$. Since $x \in V(C(G))$, we have $e(p) = e(x)$, hence $p \in V(C(G))$.

**Acknowledgments.** The authors express their sincere thanks to the referee for his/her valuable comments and suggestions which improve the paper a lot.

**References**


