Bounded Mocanu Variation Properties of Certain Subclass of Meromorphic Functions Involving a Family of Linear Operator

Ali Muhammad
Department of Basic Sciences University of Engineering and Technology Peshawar, Pakistan
e-mail: ali7887@gmail.com

Abstract. In this paper, we introduce a new subclass of meromorphic functions defined in the punctured unit disc. We derive inclusion relationships, radius problem and some other interesting properties of this class are investigated.

1. Introduction

Let $M$ denote the class of functions $f(z)$ of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

which are analytic in the punctured open unit disc $E^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = E \setminus \{0\}$.

If $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f \ast g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k = (g \ast f)(z) \quad (z \in E).$$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in $E$ with $p(0) = 1$ and

$$\int_{0}^{2\pi} |\frac{\Re p(z) - \rho}{1 - \rho}| \, d\theta \leq k\pi, z = re^{i\theta},$$

Received April 13, 2011; accepted January 28, 2014.
2010 Mathematics Subject Classification: 30C45, 30C50.
Key words and phrases: Meromorphic functions, Generalized hypergeometric functions, Functions with positive real part, Hadamard product (or convolution), Linear operators.
where \( k \geq 2 \) and \( 0 \leq \rho < 1 \). This class was introduced by Padmanbhan et. al. in [13]. We note that \( P_k(0) = P_k \), see Pinchuk [14], \( P_2(\rho) = P(\rho) \), the class of analytic functions with positive real part greater than \( \rho \) and \( P_2(0) = P \), the class of functions with positive real part. From (1.4) we can easily deduce that \( p(z) \in P_k(\rho) \) if, and only if, there exists \( p_1(z) \), \( p_2(z) \in P(\rho) \) such that for \( z \in E \),
\[
(1.5) \quad p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z).
\]

In recent years, several families of integral operators and differential operators were introduced using Hadamard product (or convolution). For example, we choose to mention the Ruscheweyh derivative [15], the Carlson-Shaffer operator [1], the Dzoik-Srivastava operator [4], the Noor integral operator [12] also see, [3, 6, 7, 11]. Motivated by the work of N. E. Cho and K. I. Noor [2, 9], we introduce a family of integral operators defined on the space of meromorphic functions in the class \( \mathcal{M} \), see [16]. By using these integral operators, we define a new subclass of meromorphic functions and investigate various inclusion relationships, radius problem and some other properties for the meromorphic function classes introduced here.

For a complex parameters \( \alpha_1, \ldots, \alpha_q \) and \( \beta_1, \ldots, \beta_s \) \( (\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0 = \{0, -1, -2, \ldots\}; \ j = 1, \ldots, s) \), we now define the function \( \phi(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \) by
\[
\phi(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \frac{1}{z} \left[ \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k+1} \ldots (\alpha_q)_{k+1}}{(\beta_1)_{k+1} \ldots (\beta_s)_{k+1}} \left( \frac{z}{k+1} \right)^k \right],
\]
where \( (v)_k \) is the Pochhammer symbol (or shifted factorial) defined in (terms of the Gamma function) by
\[
(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} \frac{1}{v(v+1)\ldots(v+k-1)} & \text{if } k = 0 \text{ and } v \in \mathbb{C} \setminus \{0\} \\ v(v+1)\ldots(v+k-1) & \text{if } k \in \mathbb{N} \text{ and } v \in \mathbb{C}. \end{cases}
\]

Now we introduce the following operator
\[
I_{\nu}^p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s): \mathcal{M} \longrightarrow \mathcal{M}
\]
as follows:

Let \( F_{\mu,p}(z) = \frac{1}{z} \sum_{k=0}^{\infty} \left( \frac{z}{k+1} \right)^p, p \in \mathbb{N}_0, \mu \neq 0 \) and let \( F_{\mu,1}^{-1}(z) \) be defined such that
\[
F_{\mu,p}(z) * F_{\mu,1}^{-1}(z) = \phi(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z).
\]

Then
\[
I_{\mu}^p(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_s) f(z) = F_{\mu,1}^{-1}(z) * f(z).
\]

From (1.6) it can be easily seen
\[
(1.7) \quad I_{\mu}^p(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_s) f(z) = \frac{1}{z} \sum_{k=0}^{\infty} \left( k \right)^p \phi(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z).
\]
For conveniences, we shall henceforth denote
\[ I^p_\mu(\alpha_1, ..., \alpha_q, \beta_1, ..., \beta_s) f(z) = I^p_\mu(\alpha_1, \beta_1) f(z). \]

For the choices of the parameters \( p = 0, q = 2, s = 1 \), the operator \( I^p_\mu(\alpha_1, \beta_1) f(z) \) is reduced to an operator by N. E. Cho and K. I. Noor [2] and K. I. Noor [9] and when \( p = 0, q = 2, s = 1, \alpha_1 = \lambda, \alpha_2 = 1, \beta_1 = (n + 1) \), the operator \( I^p_\mu(\alpha_1, \beta_1) f(z) \) is reduced to an operator recently introduced by S.-M. Yuan et. al. in [17].

It can be easily verified from the above definition of the operator \( I^p_\mu(\alpha_1, \beta_1) f(z) \) that
\[
(1.9) \quad z (I^{p+1}_\mu(\alpha_1, \beta_1) f(z))' = \mu I^p_\mu(\alpha_1, \beta_1) f(z) - (\mu + 1) I^{p+1}_\mu(\alpha_1, \beta_1) f(z),
\]
and
\[
(1.10) \quad z (I^p_\mu(\alpha_1, \beta_1) f(z))' = \alpha_1 I^p_\mu(\alpha_1 + 1, \beta_1) f(z) - (\alpha_1 + 1) I^p_\mu(\alpha_1, \beta_1) f(z).
\]

By using the operator \( I^p_\mu(\alpha_1, \beta_1) f(z) \), we now introduce the following subclass of meromorphic functions:

**Definition 1.3.** Let \( \lambda \in \mathbb{C} \) with \( \Re \lambda > 0 \), \( f \in \mathcal{M} \), \( p \in \mathbb{N}_0, 0 \leq \rho < 1, \alpha = \mu > 0 \) and \( k \geq 2 \). Then \( f \in B^\lambda_{k,p}(\alpha_1, \beta_1, \alpha, \rho) \), if and only if

\[
\left\{ \left( 1 - \lambda \right) \left( \frac{I^p_\mu(\alpha_1, \beta_1) f(z)}{I^p_\mu(\alpha_1, \beta_1) g(z)} \right)^\alpha + \lambda \left( \frac{I^p_\mu(\alpha_1 + 1, \beta_1) f(z)}{I^p_\mu(\alpha_1 + 1, \beta_1) g(z)} \right) \left( \frac{I^p_\mu(\alpha_1, \beta_1) f(z)}{I^p_\mu(\alpha_1, \beta_1) g(z)} \right)^{\alpha - 1} \right\} \in P_k(\rho),
\]

where \( g \in \mathcal{M} \) satisfies the condition:
\[
(1.11) \quad \left( \frac{I^p_\mu(\alpha_1 + 1, \beta_1) g(z)}{I^p_\mu(\alpha_1, \beta_1) g(z)} \right) \in P(\eta), \quad z \in E, \quad \text{with} \quad 0 \leq \eta < 1.
\]

Unless otherwise mentioned, we assume through this paper that \( p \in \mathbb{N}_0, 0 \leq \rho < 1, \alpha = \mu > 0 \).

### 2. Preliminary Results

In order to establish our main results, we need the following Lemma which is properly known as the Miller-Mocanu Lemma.

**Lemma 2.1** [8]. Let \( u = u_1 + iu_2, v = v_1 + iv_2 \) and \( \Psi(u, v) \) be a complex valued function satisfying the conditions:
(i) \( \Psi(u, v) \) is continuous in a domain \( D \subset \mathbb{C}^2 \),
(ii) \( \Re \Psi(1, 0) > 0 \),
(iii) \( \Im \Psi(iu_2, v_1) \leq 0 \), whenever \( (iu_2, v_1) \in D \) and \( v_1 \leq -\frac{1}{2} \left( 1 + u_2^2 \right) \).

If \( h(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is a function analytic in \( E \) such that \( (h(z), z h'(z)) \in D \),
and \( \Re \Psi(h(z), zh'(z)) > 0 \) for \( z \in E \), then \( \Re h(z) > 0 \) in \( E \).

3. Main Results

**Theorem 3.1.** Let \( \lambda \in \mathbb{C} \setminus \{0\} \) with \( \Re \lambda > 0 \) and \( f \in B^s_{K, \mu}(\alpha_1, \beta_1, \alpha, \rho) \). Then

\[
\left( \frac{I^p_\mu(\alpha_1, \beta_1)f(z)}{I^p_\mu(\alpha_1, \beta_1)g(z)} \right) ^\alpha \in P_\delta(\gamma),
\]

where

\[
\gamma = \frac{2 \mu \alpha_1 \rho + \lambda \delta}{2 \mu \alpha_1 + \lambda \delta},
\]

\( I^p_\mu(\alpha_1, \beta_1) \)

and \( g \in M \) satisfies the condition \( (1.11) \)

\[
\delta = \frac{\Re h_0(z)}{|h_0(z)|^2}, \quad h_0(z) = \left( \frac{I^p_\mu(\alpha_1 + 1, \beta_1)g(z)}{I^p_\mu(\alpha_1, \beta_1)g(z)} \right).
\]

**Proof.** Set

\[
(3.2) \quad \left( \frac{I^p_\mu(\alpha_1, \beta_1)f(z)}{I^p_\mu(\alpha_1, \beta_1)g(z)} \right) ^\alpha = (1 - \gamma)h(z) + \gamma,
\]

\( h(0) = 1 \), and \( h(z) \) is analytic in \( E \) and we can write

\[
(3.3) \quad h(z) = \left( \frac{k}{4} + \frac{1}{2} \right)h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right)h_2(z).
\]

Differentiating \( (3.2) \) with respect to \( z \) and using the identity \( (1.10) \), we have

\[
(3.4) \quad \left\{ (1 - \lambda) \left( \frac{I^p_\mu(\alpha_1, \beta_1)f(z)}{I^p_\mu(\alpha_1, \beta_1)g(z)} \right) ^\alpha + \lambda \left( \frac{I^p_\mu(\alpha_1 + 1, \beta_1)f(z)}{I^p_\mu(\alpha_1, \beta_1)g(z)} \right) \left( \frac{I^p_\mu(\alpha_1, \beta_1)f(z)}{I^p_\mu(\alpha_1 + 1, \beta_1)g(z)} \right) ^{\alpha - 1} \right\}
\]

\[
= \left( \frac{k}{4} + \frac{1}{2} \right) \left\{ (1 - \gamma)h_1(z) + \gamma - \rho + \frac{\lambda(1 - \gamma)z h_1'(z)}{\alpha \alpha_1 h_0(z)} \right\}
\]

\[
- \left( \frac{k}{4} - \frac{1}{2} \right) \left\{ (1 - \gamma)h_2(z) + \gamma - \rho + \frac{\lambda(1 - \gamma)z h_2'(z)}{\alpha \alpha_1 h_0(z)} \right\}.
\]

Now we form the functional \( \Psi(u, v) \) by choosing \( u = h_1(z) = u_1 + iv_2 \) and \( v = zh_1'(z) = v_1 + iv_2 \). Thus

\[
\Psi(u, v) = \left\{ (1 - \gamma)u + \gamma - \rho + \frac{\lambda(1 - \gamma)v}{\alpha \alpha_1 h_0(z)} \right\}.
\]

The first two conditions of Lemma 2.1 are clearly satisfied. We verify the condition (iii) as follows:

\[
\Psi(iv_2, v_1) = \gamma - \rho + \frac{\lambda(1 - \gamma)v_1 \Re h_0(z)}{\alpha \alpha_1 |h_0(z)|^2} = \gamma - \rho + \frac{\lambda(1 - \gamma)v_1 \delta}{\alpha \alpha_1},
\]
Let \( \frac{\Re h_2(z)}{|h_0(z)|} \).

Now, for \( v_1 \leq -\frac{1}{2}(1 + u_2^2) \), we have

\[
\Re \Psi(\rho u_2, v_1) \leq \gamma - \rho - \frac{1}{2} \frac{\lambda(1 - \gamma)(1 + u_2^2)\delta}{\alpha \alpha_1}.
\]

This implies that \( \lambda \frac{\Re(\rho u_2, v_1)}{\alpha \alpha_1} \leq 2\alpha \alpha_1(\gamma - \rho) - \lambda \delta(1 - \gamma) - \lambda \delta(1 - \gamma)u_2^2 = A + B u_2^2 \).

Now, for \( \Re \Psi(\rho u_2, v_1) \leq 0 \) if \( A \leq 0 \) and this gives us \( \gamma \) as defined by (3.1). We now applying Lemma 2.1 to conclude that \( h_i \in P \) for \( z \in E \) and thus \( h \in P_k \) which gives us the required result.

We note that \( \gamma = \rho \) when \( \eta = 0 \).

**Theorem 3.2.** For \( \lambda \geq 1 \), let \( f \in B_{k,\rho}^\lambda(\alpha_1, \beta_1, 1, \rho) \). Then

\[
\left( \frac{I_{\rho}^\lambda(\alpha_1 + 1, \beta_1)f(z)}{I_{\rho}^\lambda(\alpha_1 + 1, \beta_1)g(z)} \right) \in P_k(\rho), \text{ for } z \in E.
\]

**Proof.** We can write, for \( \lambda \geq 1 \),

\[
\lambda \left( \frac{I_{\rho}^\lambda(\alpha_1 + 1, \beta_1)f(z)}{I_{\rho}^\lambda(\alpha_1 + 1, \beta_1)g(z)} \right) = (1 - \lambda) \left( \frac{I_{\rho}^\lambda(\alpha_1, \beta_1)f(z)}{I_{\rho}^\lambda(\alpha_1, \beta_1)g(z)} \right) + \lambda \left( \frac{I_{\rho}^\lambda(\alpha_1, \beta_1)f(z)}{I_{\rho}^\lambda(\alpha_1, \beta_1)g(z)} \right).
\]

This implies that

\[
\left( \frac{I_{\rho}^\lambda(\alpha_1 + 1, \beta_1)f(z)}{I_{\rho}^\lambda(\alpha_1 + 1, \beta_1)g(z)} \right) = \frac{1}{\lambda} \left( (1 - \lambda) \left( \frac{I_{\rho}^\lambda(\alpha_1, \beta_1)f(z)}{I_{\rho}^\lambda(\alpha_1, \beta_1)g(z)} \right) + \lambda \left( \frac{I_{\rho}^\lambda(\alpha_1, \beta_1)f(z)}{I_{\rho}^\lambda(\alpha_1, \beta_1)g(z)} \right) \right)
\]

\[
= \frac{1}{\lambda} H_1(z) + (1 - \frac{1}{\lambda}) H_2(z).
\]

Since \( H_1(z), H_2(z) \in P_k(\rho) \), by Theorem 3.1, Definition 3.1 and \( P_k(\rho) \) is a convex set, see [10], we obtain the required result.

**Theorem 3.3.** Let \( \lambda \in \mathbb{C} \setminus \{0\} \) with \( \Re \lambda > 0 \). If \( f \in \mathcal{M} \) satisfies the following condition:
\[
\left\{(1 - \lambda) \left( z\left( I^p(\alpha_1, \beta_1)f(z)\right) \right)^{\alpha} + \lambda \left( z\left( I^p(\alpha_1 + 1, \beta_1)f(z)\right) \right) \left( z\left( I^p(\alpha_1, \beta_1)f(z)\right) \right)^{\alpha-1}\right\} \in P_k(\rho), \text{ for } \alpha > 0\left( z \in E^*\right), \text{ then }
\]
\[
\left( z\left( I^p(\alpha_1, \beta_1)f(z)\right) \right)^{\alpha} \in P_\lambda(\sigma),
\]
where
\[
\sigma = \rho + (1 - \rho)(2\sigma_1 - 1) \text{ with } \sigma_1 = \int_0^1 \left( 1 + t^{\frac{3\lambda}{\lambda_1}} \right) dt.
\]
The value of \(\sigma\) is best possible and cannot be improved.

**Proof.** We set
\[
\left( z\left( I^p(\alpha_1, \beta_1)f(z)\right) \right)^{\alpha} = h(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z),
\]
where \(h(0) = 1\) and \(h\) is analytic in \(E\). Then by a simple computation together with (1.10), we have
\[
\left\{(1 - \lambda) \left( z\left( I^p(\alpha_1, \beta_1)f(z)\right) \right)^{\alpha} + \lambda \left( z\left( I^p(\alpha_1 + 1, \beta_1)f(z)\right) \right) \left( z\left( I^p(\alpha_1, \beta_1)f(z)\right) \right)^{\alpha-1}\right\}
\]
\[
= \left\{h(z) + \frac{\lambda z h'(z)}{\mu \alpha_1}\right\} \in P_k(\rho), \text{ z } \in E.
\]
Using Lemma 2.2, we note that \(h_1(z) \in P(\sigma), \text{ }
\]
\[
\sigma = \rho + (1 - \rho)(2\sigma_1 - 1),
\]
(3.5)
\[
\sigma_1 = \int_0^1 \left( 1 + t^{\frac{3\lambda}{\lambda_1}} \right) dt,
\]
and consequently \(h(z) \in P_k(\sigma)\) and this gives the required result. \(\square\)

We note that \(\sigma_1\) given by (3.5) can be expressed in terms of hypergeometric function as
\[
\sigma_1 = \int_0^1 \left( 1 + t^{\frac{3\lambda}{\lambda_1}} \right) dt
\]
\[
= \frac{\mu \alpha_1}{\lambda_1} \int_0^1 u^{\frac{\alpha_1}{\lambda_1} - 1} (1 + u)^{-1} du, \quad (\lambda_1 = \Re \lambda > 0)
\]
\[
= 2 F_1(1, \frac{\mu \alpha_1}{\lambda_1}; 1 + \frac{\mu \alpha_1}{\lambda_1}; -1)
\]
\[
= 2 F_1(1, 1; 1 + \frac{\mu \alpha_1}{\lambda_1}; \frac{1}{2})
\]
\(\square\)
Consider the operator defined by

\begin{equation}
F_c = \left( c \int_0^z t^c ( f(t) ) dt \right) \quad (c > 0; z \in E^*). \tag{3.6}
\end{equation}

It is clear that the function $F_c \in M$ and

\begin{equation}
z((I^p_\mu(\alpha_1, \beta_1) F_c(f))(z)) = c(I^p_\mu(\alpha_1, \beta_1) f(z)) - (c + 1)(I^p_\mu(\alpha_1, \beta_1) F_c(f))(z). \tag{3.7}
\end{equation}

**Theorem 3.4.** Let $\lambda \in \mathbb{C}\setminus\{0\}$ with $\Re \lambda > 0$. If $f \in M$ satisfies the following condition:

\begin{equation}
(1 - \lambda) z ((I^p_\mu(\alpha_1, \beta_1) F_c(f))(z)) + \lambda z ((I^p_\mu(\alpha_1, \beta_1) f(z))] \in P_k(\rho), \text{ for } z \in E^* \tag{3.8}
\end{equation}

then the function defined by

\begin{equation}
(z(I^p_\mu(\alpha_1, \beta_1) F_c(f))(z)) \in P_k(\rho_1), \tag{3.9}
\end{equation}

where

\[ \rho_1 = \rho + (1 - \rho)(2\sigma_2 - 1) \quad \text{with} \quad \sigma_2 = \int_0^1 (1 + t^{\Re \lambda}) dt. \]

The value of $\rho_1$ is best possible and cannot be improved.

**Proof.** Set

\begin{equation}
(z(I^p_\mu(\alpha_1, \beta_1) F_c(f))(z)) = h(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z). \tag{3.10}
\end{equation}

Then $h(z)$ is analytic in $E$ with $h(0) = 1$.

Differentiating equation (3.10) with respect $z$ and using (3.7) in the resulting equation, we have

\[ \{ (1 - \lambda) z (I^p_\mu(\alpha_1, \beta_1) F_c(f))(z) \} + \lambda z ((I^p_\mu(\alpha_1, \beta_1) f(z))] \in P_k(\rho), \quad z \in E. \]

Using Lemma 2.2, we note that $h_1(z) \in P(\rho_1)$,

\[ \rho_1 = \rho + (1 - \rho)(2\sigma_2 - 1), \]

\[ \sigma_2 = \int_0^1 (1 + t^{\Re \lambda}) dt, \tag{3.11} \]

and consequently $h(z) \in P_k(\rho_1)$ and this gives the required result. □

In term of hypergeometric function $\sigma_2$ can be written as

\[ \sigma_2 = _2 F_1(1, 1; \frac{c}{\Re \lambda} + 1; \frac{1}{2}) \]
Theorem 3.5. For $0 \leq \lambda_2 < \lambda_1$,  

$$B_{\lambda_1, \mu}^{\alpha_1, \beta_1, \alpha, \rho} \subset B_{\lambda_1, \mu}^{\alpha_2, \beta_2, \alpha, \rho}.$$  

If $\lambda_2 = 0$, then the proof is immediate from Theorem 3.1. Let $\lambda_2 > 0$ and $f \in B_{\lambda_1, \mu}^{\alpha_1, \beta_1, \alpha, \rho}$. Then there exist two functions $H_1, H_2 \in P_\lambda(\rho)$ such that  

$$\left\{ (1 - \lambda_1) \left( \frac{(P_\mu f(z))}{(P_\mu(\alpha, \beta_1)g(z))} \right)^\alpha + \lambda_1 \left( \frac{(P_\mu f(\alpha_1 + 1, \beta_1)\rho(z))}{(P_\mu(\alpha_1 + 1, \beta_1)g(z))} \right) \left( \frac{(P_\mu f(\alpha_1, \beta_1))}{(P_\mu(\alpha_1, \beta_1)g(z))} \right)^{\alpha-1} \right\} = H_1(z),$$  

and  

$$\left( \frac{(P_\mu f(\alpha_1, \beta_1))}{(P_\mu(\alpha_1, \beta_1)g(z))} \right)^{\alpha} = H_2(z).$$  

Then  

$$(1 - \lambda_2) \left( \frac{(P_\mu f(\alpha_1, \beta_1))}{(P_\mu(\alpha_1, \beta_1)g(z))} \right)^\alpha + \lambda_2 \left( \frac{(P_\mu f(\alpha_1 + 1, \beta_1)\rho(z))}{(P_\mu(\alpha_1 + 1, \beta_1)g(z))} \right) \left( \frac{(P_\mu f(\alpha_1, \beta_1))}{(P_\mu(\alpha_1, \beta_1)g(z))} \right)^{\alpha-1}$$  

$$= \frac{\lambda_2}{\lambda_1} H_1(z) + (1 - \frac{\lambda_2}{\lambda_1}) H_2(z),$$  

and since $P_\lambda(\rho)$ is a convex set, see [10], it follows that the right hand side of (3.12) belongs to $P_\lambda(\rho)$ and this completes the proof. \qed

We next take the converse case of Theorem 3.1 as follows:

Theorem 3.6. Let $(\frac{(P_\mu f(\alpha_1, \beta_1))}{(P_\mu(\alpha_1, \beta_1)g(z))})^\alpha \in P_\lambda(\rho)$ with $(\frac{(P_\mu f(\alpha_1 + 1, \beta_1)\rho(z))}{(P_\mu(\alpha_1 + 1, \beta_1)g(z))}) \in P(\eta)$,  

for $z \in E$. Then $f \in B_{\lambda_1, \mu}^{\alpha_1, \beta_1, \alpha, \rho}$ for $|z| < r$, where $r$ is given by  

$$r = \frac{\mu \alpha_1}{\lambda + |\eta|} + \sqrt{\eta \mu (\alpha_1)^2 + |\lambda|^2 + 2 \lambda |(1 - \eta) \mu \alpha_1}. \tag{3.13}$$

Proof. Let  

$$\left( \frac{(P_\mu f(\alpha_1, \beta_1))}{(P_\mu(\alpha_1, \beta_1)g(z))} \right)^\alpha = H,$$  

$$\left( \frac{(P_\mu f(\alpha_1 + 1, \beta_1)\rho(z))}{(P_\mu(\alpha_1 + 1, \beta_1)g(z))} \right) = H_0,$$  

then $H \in P_\lambda(\rho), H_0 \in P(\eta)$.

Proceeding as in Theorem 3.1, for $\mu > 0, k \geq 2, \lambda \in \mathbb{C} \setminus \{0\}, 0 \leq \rho, \eta < 1$, and  

$$H = (1 - \rho) h + \rho,$$  

$$H_0 = (1 - \eta) h_0 + \eta,$$  

with $h \in P_k, h_0 \in P$,
we have
\[ \frac{1}{1 - \rho} \left\{ (1 - \lambda) \left( \frac{\left( \frac{d^p}{dz^p} \left( a_0 + \beta_1 \right) f(z) \right)}{\left( \frac{d^p}{dz^p} \left( a_0 + \beta_1 \right) g(z) \right)} \right)^{\alpha} + \lambda \left( \frac{\left( \frac{d^p}{dz^p} \left( a_0 + \beta_1 + 1 \right) f(z) \right)}{\left( \frac{d^p}{dz^p} \left( a_0 + \beta_1 + 1 \right) g(z) \right)} \right) \left( \frac{\left( \frac{d^p}{dz^p} \left( a_0 + \beta_1 \right) f(z) \right)}{\left( \frac{d^p}{dz^p} \left( a_0 + \beta_1 \right) g(z) \right)} \right)^{\alpha - 1 - \rho} \right\} \]
\[ = \left\{ h(z) + \frac{\lambda}{\mu \alpha_1} \frac{zh'(z)}{(1 - \eta)h_0(z) + \eta} \right\} \]
\[ = \left( \frac{k}{4} + \frac{1}{2} \right) \left( h_1(z) + \frac{\lambda}{\mu \alpha_1} \frac{zh'(z)}{(1 - \eta)h_0(z) + \eta} \right) - \left( \frac{k}{4} - \frac{1}{2} \right) \left( h_2(z) + \frac{\lambda}{\mu \alpha_1} \frac{zh'(z)}{(1 - \eta)h_0(z) + \eta} \right). \]

Using well known estimates, see [5], for \( h_i \in P \),
\[ |zh'_i(z)| \leq \frac{2r |Rh_i(z)|}{1 - r^2}, \]
\[ \frac{1 - r}{1 + r} \leq |h_i(z)| \leq \frac{1 + r}{1 - r}, \]
we have
\[ \Re \left\{ h_i(z) + \frac{\lambda}{\mu \alpha_1} \frac{zh'(z)}{(1 - \eta)h_0(z) + \eta} \right\} \geq \Re h_i(z) \left[ 1 - \frac{2 |\lambda| r}{\mu \alpha_1} \frac{1}{1 - r^2} \left( \frac{1 + r}{1 - (1 - 2\eta)r} \right) \right] \]
\[ \geq \Re h_i(z) \left[ 1 - \frac{2 |\lambda| r}{\mu \alpha_1} \frac{1}{1 - r} \left( \frac{1 + r}{1 - (1 - 2\eta)r} \right) \right] \]
\[ \geq \Re h_i(z) \left[ \frac{\mu \alpha_1 [(1 - r - (1 - 2\eta)r) + (1 - 2\eta)r^2] - 2 |\lambda| r}{\mu \alpha_1 (1 - r)(1 - (1 - 2\eta)r)} \right] \]
\[ \geq \Re h_i(z) \left[ \frac{\mu \alpha_1 (1 - 2\eta)r^2 - 2(1 - \eta)\mu \alpha_1 + |\lambda| r + \mu \alpha_1 \alpha_1 (1 - r)1 - (1 - (1 - 2\eta)r)}{\mu} \right]. \]

Right hand side of (3.14) is positive for \( |z| < r \), where \( r \) is given by (3.13). \( \square \)

References