The Spectral Radii of Graphs with Prescribed Degree Sequence

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Abstract. In this paper, we first present the properties of the graph which maximize the spectral radius among all graphs with prescribed degree sequence. Using these results, we provide a somewhat simpler method to determine the unicyclic graph with maximum spectral radius among all unicyclic graphs with a given degree sequence. Moreover, we determine the bicyclic graph which has maximum spectral radius among all bicyclic graphs with a given degree sequence.

1. Introduction

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Its order is $|V(G)|$, denoted by $n$, and its size is $|E(G)|$, denoted by $m$. For $v \in V(G)$, let $N_G(v)$ (or $N(v)$ for short) be the set of all neighbors of $v$ in $G$ and let $d(v) = |N(v)|$ be the degree of $v$. We use $G - e$ and $G + e$ to denote the graphs obtained by deleting the edge $e$ from $G$ and by adding the edge $e$ to $G$, respectively. For any nonempty subset $W$ of $V(G)$, the subgraph of $G$ induced by $W$ is denoted by $G[W]$. The distance of $u$ and $v$ (in $G$) is the length of the shortest path between $u$ and $v$, denoted by $d(u, v)$. For all other notions and definitions, not given here, see, for example, [1], or [4] (for graph spectra). For the basic notions and terminology on the spectral graph theory the readers are referred to [4].

Let $A(G)$ be the adjacency matrix of $G$. Its eigenvalues are called the eigenvalues
(or the spectrum) of \( G \). They are real because \( A(G) \) is symmetric. The eigenvalues of \( A(G) \) are usually denoted by \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). The largest eigenvalue \( \lambda_1 \) is also called the spectral radius of \( G \), denoted by \( \rho \). The following description of the spectral radius \( \rho \) of \( G \), is well known (see, for example, [5, p.49]):

\[
\rho = \sup_{\|x\|=1} x^T A(G) x \quad (x \in \mathbb{R}^n)
\]

We note here that the maximum is attained in (1.1) if and only if \( x \) is an eigenvector (for the largest eigenvalue) of \( A(G) \). When \( G \) is connected, then \( A(G) \) is simple and there is a unique positive unit eigenvector \( x = (x_v, \; v \in V(G)) \), where \( x_v \) is also called the \( \rho \)-weight of the vertex \( v \) (with respect to \( x \)). We refer to such an eigenvector as the Perron vector of \( G \). Then we have the following set of equations, known in general as eigenvalue equations:

\[
\rho x_v = \sum_{u \in N(v)} x_u \quad \text{for} \; v \in V(G).
\]

A nonincreasing sequence \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) of nonnegative integers is called a degree sequence (or graphic) if there exists a graph \( G \) of order \( n \) for which \( d_0, d_1, \ldots, d_{n-1} \) are the degrees of its vertices.

Given a degree sequence \( \pi = (d_0, d_1, \ldots, d_{n-1}) \), let \( S_n^\pi \) be the set of all connected graphs of order \( n \) with this degree sequence. For any \( G \in S_n^\pi \), we have \( \sum_{i=0}^{n-1} d_i = 2m \), and \( k = m - n + 1 \) is the number of independent cycles. Usually, such a graph \( G \) can be referred to as a \( k \)-cyclic graph (for example, a tree is a connected acyclic graph (so \( k = 0 \)), a unicyclic graph is a connected graph containing exactly one cycle (so \( k = 1 \)) and a bicyclic graph is a connected graph containing two independent cycles (so \( k = 2 \)). For any \( k \)-cyclic graph \( G \), the Perron-core of \( G \) is the set of vertices \( \{v_0, v_1, \ldots, v_{t-1}\} \) (\( t \leq n \)) having the largest degree such that the graph constructed on such vertices is a \( k \)-cyclic graph. The remaining vertices of \( G \) form the Perron-periphery. Clearly the number of elements in the Perron-core depends on \( k \), the number of independent cycles. So the vertices in the Perron-periphery lie on some hanging trees attached to the vertices of the Perron-core.

The Brualdi-Solheid problem (BSP for short) put forward the determination of graphs maximizing the spectral radius in a given set \( S \) of graphs. The BSP for \( S = S_n^\pi \) has not been solved in general. The BSP for \( S_n^\pi \) if restricted on trees has been solved in [2]. Recently, Belardo et al. [3] solved the BSP for \( S_n^\pi \) if restricted on unicyclic graphs, and make the following general conjecture.

**Conjecture 1.1.** Let \( G^\pi_{\text{max}} \) be the graph which has maximum spectral radius among all graphs in \( S_n^\pi \). Then \( G^\pi_{\text{max}} \) is the unique graph consisting of a \( k \)-cyclic Perron-core and the vertices of the Perron-periphery are inserted in spiral like disposition (for a formal definition of spiral like disposition see [2]) with respect to the Perron-core.
The rest of this paper is organized as follows. In section 2, we present some useful lemmas. In section 3, we first introduce some properties of the graphs which maximize the spectral radii among all graphs in $\mathcal{G}_n^u$. Then using these results, we give a somewhat simpler method to determine the unicyclic graph which has maximum spectral radius among all unicyclic graphs with prescribed degree sequence. Moreover, we determine the bicyclic graph which has maximum spectral radius among all bicyclic graphs with prescribed degree sequence, which confirms Conjecture 1.1 with $k = 2$.

2. Preliminaries

Generally, it is natural to expect that $\rho$ changes when $G$ is perturbed, and we can ask whether $\rho$ increases or decreases in such situations. The following two results are the part of standard folklore of graph perturbations. Their proofs appear in several literatures (see, for example, [5]). The first one is about the perturbation known as the (simultaneous) rotations (see Lemma 2.1), the second one is about the local switching (see Lemma 2.2). Recall that the local switching preserve the degree sequence and they play the crucial role in the next section.

Lemma 2.1. ([5]) Let $u$ and $v$ be two vertices of a connected graph $G$ (of order $n$) and let $N(u) \setminus N(v) = \{v_1, v_2, \ldots, v_s\}$ ($s \geq 1$). Let $G'$ be the graph obtained from $G$ by deleting the edges $uv_1(1 \leq i \leq s)$, and then adding the edges $vu_i (1 \leq i \leq s)$. If $x_v \geq x_u$, then $\rho(G') > \rho(G)$.

Lemma 2.2. ([5]) Let $G$ (of order $n$) be a connected graph with $u_1v_1, u_2v_2 \in E(G)$ and $u_1u_2, v_1v_2 \notin E(G)$. Let $G'$ be the graph obtained from $G$ by the local switching, that consists of the deletion of edges $u_1v_1$ and $u_2v_2$, followed by the addition of edges $u_1u_2$ and $v_1v_2$ (see Fig. 1). If $(x_{u_1} - x_{v_2})(x_{u_2} - x_{v_1}) \geq 0$, then $\rho(G') \geq \rho(G)$, and the equality holds if and only if $x_{u_1} = x_{v_2}$ and $x_{u_2} = x_{v_1}$.

To state the next result (due to Hoffman and Smith), we need more definitions. An internal path in a graph, denoted by $v_1v_2 \cdots v_r$ is a path joining vertices $v_1$ and $v_r$ which are both of degree greater than two (not necessarily distinct), while all other vertices (i.e., $v_2, \ldots, v_{r-1}$) are of degree equal to 2. We denote by $C_n$ and $W_n$ the cycle and the double-snake (the tree of order $n$ having two vertices of degree three which are at distance $n - 5$).

Lemma 2.3. ([8]) Let $G'$ be the graph obtained from a graph $G$, which is neither $C_n$ nor $W_n$, by inserting a vertex of degree two in an edge $e$. Then we have

(1) if $e$ does not lie on an internal path, then $\rho(G') > \rho(G)$;

(2) if $e$ lie on an internal path, then $\rho(G') < \rho(G)$.

If $G = C_n$ (resp. $W_n$) and $G' = C_{n+1}$ (resp. $W_{n+1}$), then $\rho(G') = \rho(G) = 2$.

Lemma 2.4. ([8]) Let $G(k, l)$ be a graph obtained from a connected graph $G$ by adding at a fixed vertex $v$ two hanging paths whose lengths are $k$ and $l$ ($k \geq l \geq 1$).
\[ \rho(G(k, l)) > \rho(G(k + 1, l - 1)). \]

3. Graphs with Maximum Spectral Radii

We introduce an ordering of the vertices \( v_0, v_1, \ldots, v_{n-1} \) of a graph \( G \in \mathcal{G}_\pi^\pi \) by means of breadth-first-search. Select a vertex \( v_0 \in V(G) \) and start with vertex \( v_0 \) in layer 0 as root; all neighbors of \( v_0 \) belong to layer 1. Now we continue by recursion to construct all other layers, i.e., all neighbors of vertices in layer \( i \), which are not in layers \( i \) or \( i - 1 \), build up layer \( i + 1 \). Note that all vertices in layer \( i \) have distance \( i \) from root \( v_0 \). We call this distance the height \( h(v) = d(v, v_0) \) of a vertex \( v \).

Note that one can draw these layers on circles, respectively. Thus such an ordering is also called spiral-like ordering.

For the description of graphs which have maximum spectral radii, we need the following notion.

**Definition 3.1.** Let \( G = (V, E) \) be a graph with root \( v_0 \). An ordering \( \prec \) of the vertices is called a breadth-first-search ordering (BFS-ordering for short) if the following hold for all vertices \( v_i, v_j \in V \) (\( i \neq j \)):

1. \( v_i \prec v_j \) implies \( h(v_i) \leq h(v_j) \)
2. \( v_i \prec v_j \) implies \( d(v_i) \geq d(v_j) \)
3. Let \( v_i v_j \in E, v_i v_k \notin E, v_j v_l \notin E \) with \( h(v_i) = h(v_j) = h(v_k) - 1 = h(v_l) - 1 \). If \( v_i \prec v_l \), then \( v_j \prec v_k \).

We call a connected graph which has a BFS-ordering for its vertices a BFS-graph.

Let \( G_{\pi}^\pi_{\max} \) be the graph which has maximum spectral radius among all graphs in \( \mathcal{G}_\pi^\pi \). Let \( \mathbf{x} = (x_{v_0}, x_{v_1}, \ldots, x_{v_{n-1}}) \) \((x_{v_0} \geq x_{v_1} \geq \cdots \geq x_{v_{n-1}})\) be the Perron vector of \( G_{\pi}^\pi_{\max} \).

The following result due to Biyukoğlu and Leydold [2], which provide a structural characterization for \( G_{\pi}^\pi_{\max} \).

**Lemma 3.2.** ([2]) There exists an ordering \( \prec \) of \( V(G_{\pi}^\pi_{\max}) \) which is consistent with its Perron vector \( \mathbf{x} \) in such a way that \( x_{v_i} \geq x_{v_j} \) implies that \( v_i \prec v_j \). Moreover, such an ordering \( \prec \) of \( V(G_{\pi}^\pi_{\max}) \) satisfies the conditions (1) and (2) in Definition 3.1.

In fact the ordering \( \prec \) of \( V(G_{\pi}^\pi_{\max}) \) in Lemma 3.2 also satisfies the condition (3) in Definition 3.1. Otherwise, by Lemma 3.2, there exists an ordering \( \prec \) of \( V(G_{\pi}^\pi_{\max}) \) such that \( v_0 \prec v_1 \prec \cdots \prec v_{n-1} \) (i.e., \( x_{v_0} \geq x_{v_1} \geq \cdots \geq x_{v_{n-1}} \)) implies that \( h(v_0) \leq h(v_1) \leq \cdots \leq h(v_{n-1}) \) and \( d(v_0) \geq d(v_1) \geq \cdots \geq d(v_{n-1}) \). Furthermore, if \( v_i v_j \in E, v_i v_k \notin E, v_j v_l \notin E \) with \( h(v_i) = h(v_j) = h(v_k) - 1 = h(v_l) - 1 \). Suppose that \( v_i \prec v_l \) and \( v_j \succ v_k \), i.e., \( x_{v_i} \geq x_{v_j} \) and \( x_{v_j} < x_{v_k} \). Let \( G' = G_{\pi}^\pi_{\max} - \{v_i v_j, v_i v_k\} + \{v_i v_k, v_j v_l\} \).
Clearly, \( G' \in G^n_\pi \). Hence, Lemma 2.2 implies that \( \rho(G_{\max}) < \rho(G') \). This is a contradiction.

Hence, we summary the above result as follows.

**Theorem 3.3.** \( G_{\max}^n \) is a BFS-graph. Moreover, \( G_{\max}^n \) has a BFS-ordering of its vertices \( v_0 < v_1 < \cdots < v_{n-1} \), which consists with the Perron vector \( x \) in such a way that \( x_{v_0} \geq x_{v_1} \geq \cdots \geq x_{v_{n-1}} \).

Let \( x = (x_{v_0}, x_{v_1}, \ldots, x_{v_{n-1}}) \) \( (x_{v_0} \geq x_{v_1} \geq \cdots \geq x_{v_{n-1}}) \) be the Perron vector of \( G_{\max}^n \). In fact, Theorem implies that there exists a well-ordering \( V(G_{\max}^n) = \{v_0, v_1, \ldots, v_{n-1}\} \) of \( G_{\max}^n \) with root \( v_0 \) such that

\[
v_0 < v_1 < \cdots < v_{n-1} \quad \text{(i.e., } x_{v_0} \geq x_{v_1} \geq \cdots \geq x_{v_{n-1}} \text{)}
\]

implies that

\[
h(v_0) \leq h(v_1) \leq \cdots \leq h(v_{n-1}) \text{ and } d(v_0) \geq d(v_1) \geq \cdots \geq d(v_{n-1}).
\]

Let \( V_i = \{v \in V(G_{\max}^n), h(v) = i\} \) for \( i = 0, 1, \ldots, h(v_{n-1}) = p \). Hence we may relabeled the vertices of \( G_{\max}^n \) in such a way that \( V_i = \{v_{i,1}, \ldots, v_{i,s_i}\} \) with \( x_{v_{i,1}} \geq x_{v_{i,2}} \geq \cdots \geq x_{v_{i,s_i}} \) and \( x_{v_{i,j}} \geq x_{v_{i+1,k}} \) for \( i = 0, 1, \ldots, p-1 \) and \( 1 \leq j \leq s_i \), \( 1 \leq k \leq s_{i+1} \). (Following, if a graph is a BFS-graph, we may keep this labeling and notation). Clearly, \( V(G_{\max}^n) = V_1 \cup V_2 \cup \cdots \cup V_p \), \( |V_1| = s_1 = d_0 \) and \( |V_i| = s_i \) for \( 2 \leq i \leq p \). Following, we give an example to explain this concept.

**Example 3.4.** In Figure 2 the unique graph maximizing the spectral radius among all uncyclic graphs with degree sequence \( (5^{(1)}, 4^{(2)}, 3^{(1)}, 2^{(3)}, 1^{(5)}) \) is depicted. The exponent in the degree sequence denote the number of vertices in the graph having such a degree. On the left there is a original graph, and on the right there is its relabeled graph.

First, we show the following result.

**Lemma 3.5.** If \( G_{\max}^n \) is not a regular graph, then \( x_{v_{0,1}} > x_{v_{1,1}} \) and \( x_{v_{1,1}} > \min_{v_{2,1} \in N(v_{1,1})} \{x_{v_{2,1}}\} \).

**Proof.** Recall that for any graph \( G \) with maximum degree \( \Delta \), we have \( \rho(G) \leq \Delta \), and the equality holds if and only if \( G \) is regular (see [4]). Thus \( d_0 = d(v_{0,1}) > \rho(G_{\max}^n) \), since \( G_{\max}^n \) is not a regular graph. Hence, from (1.2), we have

\[
d_0x_{v_{0,1}} > \rho(G_{\max}^n)x_{v_{0,1}} = \sum_{i=1}^{s_1} x_{v_{1,i}} \geq d_0x_{v_{1,1}}.
\]

Therefore, \( x_{v_{0,1}} > x_{v_{1,1}} \).

Let \( x_{v_{2,t}} = \min_{v_{2,1} \in N(v_{1,1})} \{x_{v_{2,1}}\} \). Suppose that \( x_{v_{1,1}} = x_{v_{2,t}} \). Then \( x_{v_{1,1}} = \cdots = x_{v_{1,s_1}} = x_{v_{2,t}} \). From (1.2), we have...
It confirms Conjecture 1.1 with $k$ maximum spectral radius among all bicyclic graphs with prescribed degree sequence. Using above results, let $G$ be a unicyclic graph with prescribed degree sequence. We have

$$\rho(G_{\text{max}}^\pi)x_{v_0,1} = \sum_{i=1}^s x_{v_1,i} = d(v_1,1)x_{v_1,1}$$

and

$$\rho(G_{\text{max}}^\pi)x_{v_1,1} = x_{v_0,1} + (d(v_{1,1}) - 1)x_{v_1,1},$$

i.e., \( (\rho(G_{\text{max}}^\pi) - d(v_{1,1}) + 1)x_{v_1,1} = x_{v_0,1} \).

Combining (3.1) and (3.2), we have

$$\rho(G_{\text{max}}^\pi)(\rho(G_{\text{max}}^\pi) - d(v_{1,1}) + 1) = d(v_{0,1}).$$

On the other hand, we have

$$\rho(G_{\text{max}}^\pi)x_{v_2,t} = \sum_{u \in v_2, \in E(G_{\text{max}}^\pi)} x_u \leq d(v_{2,t})x_{v_1,1},$$

i.e., \( \rho(G_{\text{max}}^\pi) \leq d(v_{2,t}) \leq d(v_{1,1}) \).

Note that $G$ is non-regular, hence $\rho(G_{\text{max}}^\pi) < d(v_{0,1}) = d_0$. Then in view of (3.3), we have $d(v_{0,1}) \leq \rho(G_{\text{max}}^\pi) < d(v_{0,1})$, a contradiction.

We define a partial ordering on degree sequences as follows: for two degree sequences $\pi = (d_0, d_1, \ldots, d_{n-1})$ and $\pi' = (d'_0, d'_1, \ldots, d'_{n-1})$, we write $\pi \preceq \pi'$ if and only if $\sum_{i=0}^n d_i = \sum_{i=0}^{n-1} d'_i$ and $\sum_{i=0}^k d_i \leq \sum_{i=0}^k d'_i$ for all $k = 0, \ldots, n-1$. Recall that the degree sequences are non-increasing. Such an ordering is also called a majorization. Biyukoglu and Leydold [2] proved that

Lemma 3.6. ([2]) Let $\pi$ and $\pi'$ be two distinct degree sequences with $\pi \preceq \pi'$. Let $G_{\text{max}}^\pi$ and $G_{\text{max}}^\pi'$ be two graphs with maximum spectral radii among all graphs in the sets $S_n^\pi$ and $S_n^\pi'$, respectively. Then $\rho(G_{\text{max}}^\pi) < \rho(G_{\text{max}}^\pi')$.

Belardo et al. [3] characterize the unicyclic graph with maximum spectral radius among all unicyclic graphs with prescribed degree sequence. Following, we will provide a somewhat simpler method to determine the unicyclic graph which has maximum spectral radius among all unicyclic graphs with prescribed degree sequence. Moreover, we also determine the bicyclic graph which has maximum spectral radius among all bicyclic graphs with prescribed degree sequence. It confirms Conjecture 1.1 with $k = 2$.

### 3.1. Unicyclic graphs

Given a non-increasing sequence $\pi = (d_0, d_1, \ldots, d_{n-1})$ with $\sum_{i=0}^{n-1} d_i = 2n$, if there exists a unicyclic graph having $\pi$ as its degree sequence, then $\pi$ is called unicyclic graphic. Let $U_n^\pi$ be the set of all unicyclic graphs of order $n$ with degree sequence $\pi$. Let $\pi = (d_0, d_1, \ldots, d_{n-1})$ be unicyclic graphic with $n \geq 3$. We construct a special unicyclic graph $U_n^\pi$ as follows: If $d_0 = 2$, then let $U_n^\pi = C_n$. If $d_0 \geq 3$ and $d_1 = 2$, then let $U_n^\pi$ be the unicyclic graph obtained by attaching $d_0 - 2$
paths of almost equal lengths to one vertex of $C_3$. If $d_1 \geq 3$, then let $d(v_i) = d_i$ for $0 \leq i \leq n - 1$. Then let $U^*_n$ be the unicyclic graph consisting of $C_3 = v_0v_1v_2v_0$ and the remaining vertices (i.e., $v_3, \ldots, v_{n-1}$) appear in spiral like disposition with respect to $C_3$ starting from $v_3$ that is adjacent to $v_0$.

**Lemma 3.7.** Let $\pi = (d_0, d_1, \ldots, d_{n-1})$ be unicyclic graphic with $d_1 = 2$. Then $U^*_n$ is the only unicyclic graph with maximum spectral radius among all graphs in $\mathfrak{U}^*_n$.

**Proof.** Let $G$ be a unicyclic graph with maximum spectral radius among all unicyclic graphs with degree sequence $\pi = (d_0, d_1, \ldots, d_{n-1})$. If $d_0 = 2$, then $G$ must be the cycle $C_n$ and the assertion holds. Now assume that $d_0 \geq 3$. Since $d_1 = 2$, $G$ must be the graph obtained from a cycle $C_k$ and $d_0 - 2$ paths $P_{n_i}$ by identifying one vertex of $C_k$ and one end vertex of each $P_{n_i}$ for $i = 1, 2, \ldots, d_0 - 2$. Clearly, $n = k + \sum_{i=1}^{d_0-2} n_i$. Lemma 2.3 implies that $k = 3$. Otherwise, we may contract an edge of $C_k$ (which decreases the cycle length), and subdivide an edge of one path $P_{n_i}$. Clearly, the resulting graph $G' \in \mathfrak{U}^*_n$. By Lemma 2.3, we have $\rho(G) < \rho(G')$. This is a contradiction. Moreover, Lemma 2.4 implies that $|n_i - n_j| \leq 1$ for all $1 \leq i, j \leq d_0 - 2$. Therefore, $G \cong U^*_n$. \hfill \Box

**Lemma 3.8.** Let $\pi = (d_0, d_1, \ldots, d_{n-1})$ be unicyclic graphic with $d_1 \geq 3$. Then $U^*_n$ is the only unicyclic graph with maximum spectral radius among all graphs in $\mathfrak{U}^*_n$.

**Proof.** Let $G$ be a unicyclic graph with maximum spectral radius among all graphs in $\mathfrak{U}^*_n$. Then Theorem 3.3 implies that $G$ must be a BFS-graph. We keep the labeling of its vertices as defined below Theorem 3.3. It suffices to show that $v_{1,1}v_{1,2} \in E(G)$.

Since $G$ is a unicyclic graph, there exists only one cycle $C$ in $G$. Let $v_{r,q}$ be the smallest height among vertices in $V(C)$, i.e., $h(v_{r,q}) = r \leq h(u)$ for every $u \in V(C)$.

Suppose that $v_{1,1}v_{1,2} \notin E(G)$. Now we consider the following three cases:

**Case 1.** $v_{r,q} = v_0$.

Since $G$ is a unicyclic graph, $|E(G[V_1])| \leq 1$.

(i) $|E(G[V_1])| = 1$.

Suppose that $v_{1,1}v_{1,i} \in E(G)$ ($i \geq 3$). Since $G$ is a unicyclic graph and $d(v_{1,2}) \geq d(v_{1,i}) \geq 2$, there exists a vertex $v_{2,t} \in V_2$ such that $v_{1,2}v_{2,t} \in E(G)$ but $v_{1,1}v_{2,t} \notin E(G)$. Let

$$G' = G - \{v_{1,1}v_{1,i}, v_{1,2}v_{2,t}\} + \{v_{1,1}v_{1,2}, v_{1,1}v_{2,t}\}.$$  

Then $G' \in \mathfrak{U}^*_n$. Moreover, Lemma 3.5 implies that $x_{v_{1,1}} > \min_{v_{2,t} \in N(v_{1,1})} \{x_{v_{2,t}}\} \geq x_{v_{2,t}}$. And $x_{v_{1,2}} > x_{v_{1,i}}$. Therefore, by Lemma 2.2 we have $\rho(G') < \rho(G)$. This is a contradiction.

Suppose that $v_{1,i}v_{1,j} \in E(G)$ ($j > i \geq 2$). Since $G$ is a unicyclic graph and $d(v_{1,1}) = d_1 \geq 3$, there exists a vertex $v_{2,t} \in V_2$ such that $v_{1,1}v_{2,t} \in E(G)$ and
$x_{v_{2,i}} = \min_{v_{2,j} \in N(v_{1,i})} \{x_{v_{2,j}}\}$. Clearly, $v_{1,1}v_{1,t} \notin E(G)$ and $v_{1,i}v_{2,t} \notin E(G)$. Let

$$G' = G - \{v_{1,1}v_{2,t}, v_{1,i}v_{1,j}\} + \{v_{1,1}v_{1,i}, v_{1,j}v_{2,t}\}.$$ 

Then $G' \in \mathcal{U}_n$. Moreover, we claim that $x_{v_{1,1}} > x_{v_{1,j}}$ or $x_{v_{1,1}} > x_{v_{2,t}}$ (otherwise, $x_{v_{1,1}} = x_{v_{1,j}} = x_{v_{1,t}} = x_{v_{2,t}}$, by Lemma 3.5, it is impossible). Therefore, Lemma 2.2 implies that $\rho(G) < \rho(G')$. This is a contradiction too.

(ii) $|E(G[V_1])| = 0$.

In this case, there exist two vertices $v_{1,i} \in V_1$ ($i \geq 2$) and $v_{2,j} \in V_2$ such that $v_{1,i}v_{2,j} \in E(C)$. Since $G$ is a unicyclic graph and $d(v_{1,i}) = d_1 \geq 3$, there exists a vertex $v_{2,t} \in V_2$ such that $v_{1,i}v_{2,t} \in E(G)$ but $v_{1,1}v_{2,t} \notin E(G)$. Clearly, $v_{1,i}v_{2,t} \notin E(G)$. Let

$$G' = G - \{v_{1,1}v_{2,t}, v_{1,i}v_{2,j}\} + \{v_{1,1}v_{1,i}, v_{1,j}v_{2,t}\}.$$ 

Then $G' \in \mathcal{U}_n$. Moreover, by Lemma 3.5, we have $x_{v_{1,1}} > \min_{v_{2,t} \in N(v_{1,i})} \{x_{v_{2,t}}\} > x_{v_{2,t}}$. Recall that $x_{v_{1,1}} > x_{v_{2,t}}$. Therefore, Lemma 2.2 implies that $\rho(G) < \rho(G')$. This is a contradiction.

**Case 2.** $v_{r,q} = v_{1,1}$.

There exists a vertex $v_{2,t} \in V_2$ such that $v_{1,1}v_{2,t} \in E(C)$. Since $G$ is a unicyclic graph and $d(v_{1,i}) = d_1 \geq 3$, there exists a vertex $v_{2,t} \in V_2$ such that $v_{1,1}v_{2,t} \in E(G)$ but $v_{1,2}v_{2,t} \notin E(G)$. Clearly, $v_{1,2}v_{2,t} \notin E(G)$. Let

$$G' = G - \{v_{1,1}v_{2,t}, v_{1,2}v_{2,t}\} + \{v_{1,1}v_{1,2}, v_{2,t}v_{2,t}\}.$$ 

Then $G' \in \mathcal{U}_n$. Similarly, we have $x_{v_{1,1}} > x_{v_{2,t}}$ and $x_{v_{1,2}} \geq x_{v_{2,t}}$. Hence, Lemma 2.2 implies that $\rho(G) < \rho(G')$. This is a contradiction.

**Case 3.** $v_{r,q} \neq v_{0,1}, v_{1,1}$.

There exists $v_{r+1,t} \in V_{r+1}$ such that $v_{r,q}v_{r+1,t} \in E(C)$. Since $G$ is a unicyclic graph, from the choice of $v_{r,q}$, it is clear that there is no edges in $G[V_r]$ for $0 \leq i \leq r$. Hence $|V_r| \geq d_0 \geq 3$. Since $d(v_{r+1,t}) \geq 2$, there must exist two vertices $v_{r,i} \in V_r$ and $v_{r+1,j} \in V_{r+1}$ such that $v_{r,i}v_{r+1,j} \in E(G)$ but $v_{r,i}v_{r+1,j} \notin E(G)$. Let

$$G^* = G - \{v_{r,q}v_{r+1,t}, v_{r,i}v_{r+1,j}\} + \{v_{r,i}v_{r,q}, v_{r+1,j}v_{r+1,t}\}.$$ 

Then $G^* \in \mathcal{U}_n$. Since $x_{v_{r,q}} \geq x_{v_{r+1,t}}$ and $x_{v_{r,i}} \geq x_{v_{r+1,j}}$, Lemma 2.2 implies that $\rho(G) \leq \rho(G^*)$. Clearly, the smallest height of the cycle in $G^*$ is less than $r$. By repeating the argument of Case 3 or Cases 1 and 2, it is easy to see that $G$ is not a unicyclic graph with maximum spectral radius among all graphs in $\mathcal{U}_n^\pi$. This is a contradiction.

From the above discussions, the proof is completed. 

Combining Lemmas 3.7 and 3.8, we have the following result.
The Spectral Radii of Graphs with Prescribed Degree Sequence

**Theorem 3.9.** Let \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) be unicyclic graphic. Then \( U^*_n \) is the only unicyclic graph having maximum spectral radius among all graphs in \( \mathcal{U}^*_n \).

Let \( \mathcal{U}_n \) be the set of all unicyclic graphs of order \( n \), \( \mathcal{U}_{n,k} \) be the set of all unicyclic graphs of order \( n \) with \( k \) pendant vertices and \( \mathcal{U}^*_n \) be the set of all unicyclic graphs of order \( n \) with \( k \) pendant vertices and degree sequence \( \pi = (d_0, d_1, \ldots, d_{n-1}) \). Thus \( d_{n-k-1} > 1 \) and \( d_{n-k} = \cdots = d_{n-1} = 1 \). Let \( U^*_{n,k} \) be the unicyclic graph with maximum spectral radius among all graphs in \( \mathcal{U}^*_{n,k} \), where \( \pi' = (k + 2, 2, \ldots, 2, 1, \ldots, 1) \).

By Lemma 3.7, it is obvious that \( U^*_n \) is the unicyclic graph obtained from a triangle and \( k \) paths of almost equal lengths by identifying one vertex of the triangle and one end of each path of the \( k \) paths. It is easy to see that \( \pi \preceq \pi' \) for each \( \pi \), where \( \pi \) is the degree sequence of unicyclic graph of order \( n \) with \( k \) pendant vertices. So that by Lemma 3.6, the following result is obvious.

**Theorem 3.10.** ([6]) Let \( G \in \mathcal{U}_{n,k} \). Then \( \rho(G) \leq \rho(U^*_n) \), and the equality holds if and only if \( G \cong U^*_n \), where \( \pi' = (k + 2, 2, \ldots, 2, 1, \ldots, 1) \).

Since \( U^*_n \) is the only unicyclic graph with degree sequence \( \pi^* = (n - 1, 2, 2, 1, \ldots, 1) \) and for each unicyclic graphic degree sequence \( \pi \), we have \( \pi \preceq \pi^* \). So that Lemma 3.6 implies that

**Theorem 3.11.** ([6]) Let \( G \in \mathcal{U}_n \). Then \( \rho(G) \leq \rho(U^*_n) \), and the equality holds if and only if \( G \cong U^*_n \).

### 3.2. Bicyclic graphs

To state the main results in this subsection, we need to define the following two kinds of bicyclic graphs.

Let \( B(l, s, k) \) be the bicyclic graph obtained from two cycles \( C_l \) and \( C_k \), by joining a path of length \( s - 1 \) between them, where \( l \geq k \geq 3 \) and \( s \geq 1 \) (see Fig. 4).

Let \( P(p, l, q) \) \((1 \leq l \leq \min(p, q)) \) be the bicyclic graph obtained from the cycle \( C_{p+q}: v_1v_2 \cdots v_pv_{p+1} \cdots v_{p+q}v_1 \) by connecting vertices \( v_1 \) and \( v_{p+1} \) with a new path \( v_1u_1 \cdots u_{l-1}v_{p+1} \) of length \( l \) (see Fig. 4).

Similarly, for a given non-increasing sequence \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) with \( \sum_{i=1}^{n-1} d_i = 2(n + 1) \), if there exists a bicyclic graph having \( \pi \) as its degree sequence, then \( \pi \) is called bicyclic graphic. Let \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) be bicyclic graphic with \( n \geq 4 \). We construct a special bicyclic graph \( B^*_n \) as follows: if \( d_0 = 4 \) and \( d_1 = 2 \), then let \( B^*_n = B(n - 2, 1, 3) \). If \( d_0 \geq 5 \) and \( d_1 = 2 \), then let \( B^*_n \) be the bicyclic graph obtained from \( B(3, 1, 3) \) by attaching \( d_0 - 4 \) paths of almost equal lengths to the vertex of degree 4. If \( d_0 = d_1 = 3 \) and \( d_2 = 2 \), then let \( B^*_n = P(n - 2, 1, 2) \) (or \( P(2, 1, n - 2) \)). If \( d_0 = d_1 = d_2 = 3 \), then let \( d(v_i) = d_i \) for \( 0 \leq i \leq n - 1 \). Then let \( B^*_n \) be the bicyclic graph consisting of \( P(2, 1, 2) \) (shown in Fig. 5.) and the remaining vertices (i.e., \( v_4, \ldots, v_{n-1} \)) appear in spiral like disposition with respect to \( P(2, 1, 2) \) starting from \( v_3 \) that is adjacent to \( v_2 \). If \( d_0 \geq 4 \) and \( d_1 \geq 3 \),
let \( d(v_i) = d_i \) for \( 0 \leq i \leq n - 1 \). Then let \( B_2^* \) be the bicyclic graph consisting of \( P(2, 1, 2) \) (shown in Fig. 5) and the remaining vertices (i.e., \( v_4, \ldots, v_{n-1} \)) appear in spiral like disposition with respect to \( P(2, 1, 2) \) starting from \( v_4 \) that is adjacent to \( v_0 \).

**Lemma 3.12.** Let \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) be bicyclic graphic with \( d_1 = 2 \). Then \( B_2^* \) is the only bicyclic graph with maximum spectral radius among all graphs in \( B_n^\pi \).

**Proof.** Let \( G \) be a bicyclic graph with maximum spectral radius among all graphs in \( B_n^\pi \). Theorem 3.3 implies that \( G \) is a BFS-graph. Suppose that \( d_0 \geq 5 \). Similar to the proof of Lemma 3.7, the assertion holds. Now assume that \( d_0 = 4 \). Then \( G \) must be \( B(l, 1, k) \) \((l + k = n + 1)\). It suffices to prove that \( k = 3 \). Moreover, since \( G \) is a BFS-graph, keeping the labeling as mentioned above, we only need to show that \( |E(G[V_i])| \geq 1 \). Otherwise, there exists \( v_{2,t} \in V_2 \) such that \( v_{1,1}v_{2,t} \in E(G) \), and there exists an edge \( v_{1,i}v_{2,t} \) \((2 \leq i \leq 4)\) such that \( v_{1,i}v_{2,t} \) and \( v_{1,1}v_{2,t} \) lie on the different cycles since \( d(v_{0,1}) = d_0 = 4 \). Now, let

\[
G' = G - \{v_{1,1}v_{2,t}, v_{1,i}v_{2,t}\} + \{v_{1,1}v_{1,i}, v_{2,t}v_{2,j}\}.
\]

Then \( G' \in B_n^\pi \). From Lemma 3.5, we know that \( x_{v_{1,1}} > x_{v_{2,2}} \geq x_{v_{2,1}} \) and \( x_{v_{2,2}} \geq x_{v_{2,1}} \). Therefore, Lemma 2.2 implies that \( \rho(G) < \rho(G') \). This is a contradiction. \( \square \)

To deal with the case \( d_1 \geq 3 \), we need consider the following two subcases:

**Subcase A:** \( d_0 = 3 \). Since \( d_0 \geq d_1 \geq \cdots \geq d_{n-1} \), \( d_0 = d_1 = 3 \) and \( d_i \leq 3 \) \((3 \leq i \leq n - 1)\).

**Subcase B:** \( d_0 \geq 4 \).

Such two cases will be solved in Lemmas 3.13 and 3.14, respectively.

**Lemma 3.13.** Let \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) be bicyclic graphic with \( d_0 = d_1 = 3 \). Then \( B_2^* \) is the only bicyclic graph with maximum spectral radius among all graphs in \( B_n^\pi \).

**Proof.** Let \( G \) be a bicyclic graph with maximum spectral radius among all graphs in \( B_n^\pi \). Theorem 3.3 implies that \( G \) is a BFS-graph.

Suppose that \( d_2 = 2 \). Then \( G \) must be either \( B(l, 2, k) \) or \( P(p, 1, q) \). If \( G \cong B(l, 2, k) \), then let

\[
G' = G - \{v_{1,1}v_{2,1}, v_{1,2}v_{2,3}\} + \{v_{1,1}v_{1,2}, v_{2,1}v_{2,3}\}.
\]

Clearly, \( G' \in B_n^\pi \) and \( \rho(G) < \rho(G') \). Thus, \( G \cong P(p, 1, q) \). Similarly, we can show that \( p = n - 2 \) or \( q = n - 2 \). So that the assertion holds.

Suppose that \( d_2 = 3 \). By using the same argument as the proof of Theorem 3.14 below, the assertion holds. \( \square \)

**Theorem 3.14.** Let \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) be bicyclic graphic with \( d_0 \geq 4 \) and \( d_1 \geq 3 \). Then \( B_2^* \) is the only bicyclic graph with maximum spectral radius among
implies that only need to consider the following two subcases:

Let \( v_{r,q} \) be the smallest height among vertices in \( V(C_1) \) or \( V(C_2) \), i.e., \( h(v_{r,q}) = r \leq h(u) \) for any \( u \in V(C_1 \cup C_2) \) (if there exist \( v_{r,q_1} \in V(C_1) \) and \( v_{r,q_2} \in V(C_2) \), then \( q = \min\{q_1, q_2\} \)). Now we consider the following three cases:

**Case 1.** \( v_{r,q} = v_{0,1} \).

First, we claim that \( |E(G[V_1])| \geq 1 \).

Suppose not, i.e., \( |E(G[V_1])| = 0 \). Since \( v_{0,1} \in V(C_1 \cup C_2) \), there exist \( v_{1,i} \in V_1 \) (\( i \geq 2 \)) and \( v_{2,j} \in V_2 \) such that \( (v_{1,i}v_{2,j} \in E(C_1 \cup C_2) \).

Suppose that there exists \( v_{2,t} \in V_2 \) such that \( v_{1,1}v_{2,t} \in E(G) \) but \( v_{1,1}v_{2,t} \notin E(C_1 \cup C_2) \).

Let

\[
G' = G - \{v_{1,1}v_{2,t}, v_{1,i}v_{2,t}, v_{1,1}v_{2,t} \}.
\]

Then \( G' \in \mathcal{B}_n^\pi \). Moreover, recall that \( x_{v_{1,i}} > x_{v_{2,j}} \) and \( x_{v_{1,i}} \geq x_{v_{2,t}} \). Lemma 2.2 implies that \( \rho(G) < \rho(G') \). This is a contradiction.

Suppose that for each \( v_{2,k} \in N(v_{1,1}) \), \( v_{1,1}v_{2,k} \in E(C_1 \cup C_2) \). Since \( d(v_{1,1}) = d_1 \geq 3 \) and \( G \) is a bicyclic graph, there exist at least one edge \( v_{1,1}v_{2,t} (v_{2,t} \in N(v_{1,1})) \) such that \( v_{1,1}v_{2,t} \) and \( v_{1,i}v_{2,t} \) are in the different independent cycles. Let

\[
G' = G - \{v_{1,1}v_{2,t}, v_{1,i}v_{2,t}, v_{1,1}v_{2,t} \}.
\]

Similarly, we have \( G' \in \mathcal{B}_n^\pi \) and \( \rho(G) < \rho(G') \). This is a contradiction too.

On the other hand, since \( G \) is a bicyclic graph, \( |E(G[V_1])| \leq 2 \). Therefore, we only need to consider the following two subcases:

(a) \( |E(G[V_1])| = 2 \). It suffices to prove that \( v_{1,1}v_{1,2}, v_{1,1}v_{1,3} \in E(G) \).

Suppose that \( v_{1,1}v_{1,i}, v_{1,1}v_{1,k} \in E(G) \) (\( j > i \geq 2, k > l \geq 2 \)). Since \( d(v_{1,1}) = d_1 \geq 3 \), there exists \( v_{2,t} \in V_2 \) such that \( x_{v_{2,t}} = \min_{v_{2,t} \in N(v_{1,1})} \{x_{v_{2,t}}\} \) and \( v_{1,1}v_{2,t} \in E(G) \).

Let

\[
G' = G - \{v_{1,1}v_{2,t}, v_{1,i}v_{1,j} \} + \{v_{1,1}v_{1,i}, v_{1,j}v_{2,t} \}.
\]

Similarly, we have \( G' \in \mathcal{B}_n^\pi \) and \( \rho(G) < \rho(G') \). This is a contradiction.

Suppose that \( v_{1,1}v_{1,i}, v_{1,1}v_{1,k} \in E(G) \) (\( k > l \geq 2 \)). Then there exists \( v_{2,t} \in V_2 \) such that \( x_{v_{2,t}} = \min_{v_{2,t} \in N(v_{1,1})} \{x_{v_{2,t}}\} \), \( v_{1,1}v_{2,t} \in E(G) \) and \( v_{1,k}v_{2,t} \notin E(G) \) since \( d(v_{1,1}) = d_1 \geq 3 \) and \( G \) is a bicyclic graph. Let

\[
G' = G - \{v_{1,1}v_{2,t}, v_{1,i}v_{1,k} \} + \{v_{1,1}v_{1,i}, v_{1,k}v_{2,t} \}.
\]
Then $G' \in \mathcal{B}^n_\eta$. Moreover, we claim that $x_{v_{1,1}} > x_{v_{1,k}}$ or $x_{v_{1,i}} > x_{v_{2,t}}$. (Otherwise, $x_{v_{1,1}} = x_{v_{1,j}} = x_{v_{1,k}} = x_{v_{2,t}}$. By Lemma 3.5, it is impossible). Hence Lemma 2.2 implies that $\rho(G) < \rho(G')$. This is a contradiction.

Suppose that $v_{1,1}v_{1,i}, v_{1,1}v_{1,j} \in E(G)$ for $2 \leq i < j \leq s_1$. We claim that $i = 2$ and $j = 3$.

Suppose that, i.e., $i \neq 2$ or $j \neq 3$. If $i \neq 2$, since $d(v_{1,2}) \geq d(v_{1,1}) \geq 2$ and $G$ is a bicyclic graph, there exists $v_{2,t} \in V_2$ such that $v_{1,2}v_{2,t} \in E(G)$ and $v_{1,1}v_{2,t} \notin E(G)$. Let
\[ G' = G - \{v_{1,1}v_{1,i}, v_{1,2}v_{2,t}\} + \{v_{1,1}v_{1,2}, v_{1,i}v_{2,t}\}. \]
Then $G' \in \mathcal{B}^n_\eta$. Moreover, recall that $x_{v_{1,1}} > x_{v_{2,t}}$ and $x_{v_{1,2}} \geq x_{v_{1,i}}$. Lemma 2.2 implies that $\rho(G) < \rho(G')$. This is a contradiction.

Similarly, if $j \neq 3$, since $d(v_{1,3}) \geq d(v_{1,1}) \geq 2$, there exists $v_{2,t} \in V_2$ such that $v_{1,3}v_{2,t} \in E(G)$ and $v_{1,1}v_{2,t} \notin E(G)$. Let
\[ G' = G - \{v_{1,1}v_{1,j}, v_{1,3}v_{2,t}\} + \{v_{1,1}v_{1,3}, v_{1,j}v_{2,t}\}. \]
Then $G' \in \mathcal{B}^n_\eta$ and $\rho(G) < \rho(G')$. This is a contradiction too.

(b) $|E(G[V_1])| = 1$.

First, similar to the proof of Case 1 (i) in Lemma 3.8, we claim that $v_{1,1}v_{1,2} \in E(G)$.

Hence, without loss of generality, we may assume that $C_1 = v_{0,1}v_{1,1}v_{1,2}v_{0,1}$. Let $v_{k,i}$ be the smallest height among vertices in $V(C_2)$, i.e., $h(v_{k,i}) = k \leq h(v)$ for any $v \in V(C_2)$. Then there are four cases:

(1) $v_{k,i} = v_{0,1}$.
Since $C_1$ and $C_2$ are two independent cycles in $G$, there exist $v_{1,s} \in V_1 (i \geq 3)$ and $v_{2,t} \in V_2$ such that $v_{1,i}v_{2,j} \in E(G)$. On the other hand, since $d_0 = |V_1| \geq 4$, $d(v_{1,1}) = d_1 \geq 3$ and $d(v_{1,i}) \geq 2$ for $2 \leq i \leq |V_1|$, there exist $v_{1,s} \in V_1 (s \neq i)$ and $v_{2,t} \in V_2$ such that $v_{1,s}v_{2,t} \in E(G)$ but $v_{1,i}v_{2,t} \notin E(G)$. Let
\[ G_1 = G - \{v_{1,i}v_{2,j}, v_{1,s}v_{2,t}\} + \{v_{1,i}v_{1,s}, v_{2,t}v_{2,j}\}. \]
Then $G_1 \in \mathcal{B}^n_\eta$. If $s = 1$, then similarly we claim that $x_{v_{1,s}} > x_{v_{2,j}}$ or $x_{v_{1,i}} > x_{v_{2,t}}$. Hence Lemma 2.2 implies that $\rho(G) < \rho(G_1)$ yielding a contradiction. If $s \neq 1$, since $x_{v_{1,i}} \geq x_{v_{2,t}}$ and $x_{v_{1,i}} \geq x_{v_{2,t}}$, Lemma 2.2 implies that $\rho(G) \leq \rho(G_1)$. Moreover, since $v_{1,1}v_{1,2}, v_{1,i}v_{1,s} \in E(G_1)$, using the same argument as in Subcase (a) (on $G_1$), we can obtain a contradiction too.

(2) $v_{k,i} = v_{0,1}$.
There exists $v_{2,t} \in V_2$ such that $v_{1,1}v_{2,t} \in E(G)$. Since $d(v_{0,1}) = d_0 \geq 4$ and $d(v_{1,1}) \geq d(v_{1,t}) \geq 2$, there exist $v_{1,i} (i \geq 3)$ and $v_{2,t}$ such that $v_{1,i}v_{2,t} \in E(G)$ but $v_{1,1}v_{2,t} \notin E(G)$. Let
\[ G' = G - \{v_{1,1}v_{2,t}, v_{1,i}v_{2,t}\} + \{v_{1,1}v_{1,i}, v_{2,t}v_{2,t}\}. \]
Then $G' \in \mathcal{B}^n_\eta$. Moreover, recall that $x_{v_{1,1}} > x_{v_{2,t}}$ and $x_{v_{1,i}} \geq x_{v_{2,t}}$. Lemma 2.2 implies that $\rho(G) < \rho(G')$. This is a contradiction.
(3) \( v_{k,1} = v_{1,2} \).

Similarly, there exist \( v_{1,i} (i \geq 3) \), \( v_{2,j} \) and \( v_{2,t} \) such that \( v_{1,2}v_{2,t} \in E(C_2) \), \( v_{1,j}v_{2,j} \in E(G) \) but \( v_{1,i}v_{2,j} \notin E(C_2) \). Let

\[
G_1 = G - \{v_{1,2}v_{2,t}, v_{1,i}v_{2,j}\} + \{v_{1,2}v_{1,i}, v_{2,t}v_{2,j}\}.
\]

Clearly, \( G_1 \in \mathcal{B}_n \) and \( \rho(G) \leq \rho(G_1) \). Moreover, since \( v_{1,1}v_{1,2}, v_{1,2}v_{1,1} \in E(G_1) \), using the same argument as in Subcase (a) (on \( G_1 \)), we can obtain a contradiction too.

(4) \( v_{k,1} \notin \{v_{0,1}, v_{1,1}, v_{1,2}\} \).

There exists \( v_{k+1,t} \in V_{k+1} \) such that \( v_{k,t}v_{k+1,t} \in E(C_2) \). Note that \( |E(G[V_1])| = 1 \) and \( |E(G[V_i])| = 0 \) for \( 2 \leq i \leq k \). Thus \( |V_k| \geq d_0 - 1 \geq 3 \). Then there exist \( v_{k,i} \) \((i \neq l)\) and \( v_{k+1,j} \) \((j \neq t)\) such that \( v_{k,i}v_{k+1,j} \in E(G) \) but \( v_{k,i}v_{k+1,j} \notin E(C_2) \). Let

\[
G^* = G - \{v_{k,t}v_{k+1,t}, v_{k,i}v_{k+1,j}\} + \{v_{k,t}v_{k}, v_{k+1,t}v_{k+1,j}\}.
\]

Then \( G^* \in \mathcal{B}_n \). Moreover, since \( x_{v_{k,i}} \geq x_{v_{k+1,j}} \) and \( x_{v_{k,t}} \geq x_{v_{k+1,t}} \), Lemma 2.2 implies that \( \rho(G) \leq \rho(G^*) \). Clearly, the smallest height of \( C_2 \) in \( G^* \) is less than \( k \). Furthermore, by repeating the argument of (4), we will get a case referred to (1), (2) or (3). Thus \( G \) is not a bicyclic graph with maximum spectral radius among all graphs in \( \mathcal{B}_n \). This is a contradiction.

**Case 2.** \( v_{r,q} = v_{1,1} \).

There exists \( v_{2,t} \in V_2 \) such that \( v_{1,1}v_{2,t} \in E(C_1 \cup C_2) \). Since \( d_0 \geq 4 \) and \( E(G[V_1]) = 0 \), there exist \( v_{1,i} \in V_1 (i \geq 2) \) and \( v_{2,j} \in V_2 \) such that \( v_{1,i}v_{2,j} \in E(G) \) but \( v_{1,i}v_{2,j} \notin E(C_1 \cup C_2) \). Let

\[
G' = G - \{v_{1,1}v_{2,t}, v_{1,i}v_{2,j}\} + \{v_{1,1}v_{1,i}, v_{2,t}v_{2,j}\}.
\]

Then \( G' \in \mathcal{B}_n \). Moreover, recall that \( x_{v_{1,1}} > x_{v_{2,t}} \) and \( x_{v_{1,i}} \geq x_{v_{2,j}} \). Lemma 2.2 implies that \( \rho(G) < \rho(G') \). This is a contradiction.

**Case 3.** \( v_{r,q} \neq v_{0,1}, v_{1,1} \).

There exists \( v_{r+1,t} \in V_{r+1} \) such that \( v_{r,q}v_{r+1,t} \in E(C_1 \cup C_2) \). Recall that there is no edge in \( G[V_r] \) for \( 0 < i \leq r \). Thus \( |V_r| \geq d_0 \geq 4 \). Then there exist \( v_{r,i} \in V_r (i \neq q) \) and \( v_{r+1,j} \in V_{r+1} (j \neq t) \) such that \( v_{r,i}v_{r+1,j} \in E(G) \) but \( v_{r,i}v_{r+1,j} \notin E(C_1 \cup C_2) \). Let

\[
G^* = G - \{v_{r,q}v_{r+1,t}, v_{r,i}v_{r+1,j}\} + \{v_{r,q}v_{r,i}, v_{r+1,t}v_{r+1,j}\}.
\]

Then \( G^* \in \mathcal{B}_n \). Moreover, since \( x_{v_{r,q}} \geq x_{v_{r+1,j}} \) and \( x_{v_{r,i}} \geq x_{v_{r+1,t}} \), Lemma 2.2 implies that \( \rho(G) \leq \rho(G^*) \). Clearly, the smallest height of the cycles in \( G^* \) is less than \( r \). Furthermore, by repeating the argument of Case 3, or Cases 1 and 2. It is easy to know that \( G \) is not a bicyclic graph with maximum spectral radius among all graphs in \( \mathcal{B}_n \). This is a contradiction.

From the above discussions, the proof is completed. \( \Box \)
Combining Lemmas 3.12, 3.13 and 3.14, we have the following result.

**Theorem 3.15.** Let \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) be bicyclic graphic. Then \( B_\pi^* \) is the only bicyclic graph which has maximum spectral radius among all graphs in \( \mathcal{B}_n^\pi \).

Let \( \mathcal{B}_n \) be the set of all bicyclic graphs of order \( n \), \( \mathcal{B}_{n,k} \) be the set of all bicyclic graphs of order \( n \) with \( k \) pendant vertices and \( \mathcal{B}^*_{n,k} \) be the set of all bicyclic graphs of order \( n \) with \( k \) pendant vertices and degree sequence \( \pi = (d_0, d_1, \ldots, d_{n-1}) \). Thus \( d_{n-k-1} > 1 \) and \( d_{n-k} = \cdots = d_{n-1} = 1 \). If \( 1 \leq k \leq n-5 \), then it is easy to see that for each degree sequence \( \pi \) of bicyclic graph with \( k \) pendant vertices, we have \( \pi \succeq \pi' \), where \( \pi' = (k + 4, 2, \ldots, 2, 1, \ldots, 1) \). Moreover, Lemma 3.12 implies that \( B_{\pi'}^\ast \) is the bicyclic graph with maximum spectral radius among all graphs in \( \mathcal{B}_n^\pi \). If \( k = n-4 \), then there is only one bicyclic graph \( B^\ast \) with degree sequence \( \pi^\ast = (n-1, 3, 2, 2, 1, \ldots, 1) \). Hence, by Theorem 3.6, we have the following result.

**Theorem 3.16.** Let \( G \in \mathcal{B}_{n,k} \). Then \( \rho(G) \leq \rho(B_{\pi'}^\ast) \) for \( 1 \leq k \leq n-5 \), and the equality holds if and only if \( G \cong B_{\pi'}^\ast \), where \( \pi' = (k + 4, 2, \ldots, 2, 1, \ldots, 1) \); and

\[ \rho(G) = \rho(B^\ast) \quad \text{for} \quad k = n-4. \]

When \( 1 \leq k \leq n-6 \). It is easy to check that \( \pi' = (k + 4, 2, \ldots, 2, 1, \ldots, 1) \) \succeq \( \pi'' = (k + 5, 2, \ldots, 2, 1, \ldots, 1) \). Then Theorem 3.6 implies that \( \rho(B_{\pi'}^\ast) < \rho(B_{\pi''}^\ast) \). Hence \( \rho(B_{\pi'}^\ast) \) is an increasing function for \( 1 \leq k \leq n-5 \). Moreover, when \( k = n-5 \) there is only one bicyclic graph \( B^\ast \) with degree sequence \( (n-1, 2, 2, 2, 1, \ldots, 1) \). He et al. [7] proved that \( \rho(B^\ast) < \rho(B^\ast) \). Then by Theorem 3.16, we have the following theorem.

**Theorem 3.17.** ([7]) Let \( G \in \mathcal{B}_n \). Then \( \rho(G) \leq \rho(B^\ast) \), and the equality holds if and only if \( G \cong B^\ast \). Moreover, if \( G \in \mathcal{B}_n \) and \( G \neq B^\ast \), then \( \rho(G) \leq \rho(B^\ast) \), and the equality holds if and only if \( G \cong B^\ast \).

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References


Figure 1: $G$ and $G'$ in Lemma 2.2

Figure 2: Original graph and its relabeled graph, where $V_1 = \{v_0\}$, $V_2 = \{v_1, v_2, v_3, v_4, v_5\}$ and $V_3 = \{v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$

Figure 3: Spiral like disposition with respect to the cycle $C$ (On the left) staring from $v_3$ that is adjacent to $v_0$
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Figure 4: Bicyclic graphs $B(l, s, k)$ and $P(p, l, q)$.

Figure 5: Bicyclic graph $P(2, 1, 2)$. 