Rings Whose Simple Singular Modules are $PS$-Injective

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Abstract. Let $R$ be a ring. A right $R$-module $M$ is $PS$-injective if every $R$-homomorphism $f : aR \to M$ for every principally small right ideal $aR$ can be extended to $R \to M$. We investigate, in this paper, rings whose simple singular modules are $PS$-injective. New characterizations of semiprimitive rings and semisimple Artinian rings are given.

1. Introduction

Throughout this paper, $R$ is an associative ring with identity and all modules are unitary. The Jacobson radical of $R$ is denoted by $J(R)$ and the right singular ideal is denoted by $Z(R_R)$. For $a \in R$, $l(a)$ (resp. $r(a)$) denote the left (resp. right) annihilator of $a$ in $R$. For the usual notations we refer the reader to [3], [7] and [10].

A right ideal $I$ of $R$ is called small if for every proper right ideal $K$ of $R$, $K + I \neq R$. A right $R$-module $M$ is right $PS$-injective if every $R$-homomorphism $f : aR \to M$ for every principally small right ideal $aR$ can be extended to $R \to M$ (see [13]). The ring $R$ is said to be right $PS$-injective if $R_R$ is right $PS$-injective. This concept was introduced as a non-trivial generalization of right small injective rings and right $P$-injective rings. Given a right $R$-module $M$, we set $Z(M) = \{x \in M | xI = 0$ for some essential right ideal $I$ of $R\}$. The module $M$ is called singular module provided $Z(M) = M$. In what follows, we say that $R$ satisfies (P) if every simple singular right $R$-module is $PS$-injective. Recall that:

(1) A ring $R$ is semiprimitive if $J(R) = 0$.

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Main Results

A right ideal of \( R \) is reduced if it contains no nonzero nilpotent elements.

A ring \( R \) is called an MERT if every essential maximal right ideal of \( R \) is an ideal.

\( R \) is a left (right) Kasch ring if every maximal left (right) ideal is a left (right) annihilator of \( R \).

Motivated by the well known result of Kaplansky (i.e., A commutative ring \( R \) is von Neumann regular if and only if every simple \( R \)-module is injective (or \( P \)-injective, \( GP \)-injective) (see [1], [2], [4-6], [9], [11], [12], [14], [15]). It was proven that:

1. \( R \) is strongly regular if and only if \( R \) is a left duo ring whose simple singular left \( R \)-modules are \( YJ \)-injective if and only if \( R \) is a weakly right duo ring whose simple singular right \( R \)-modules are \( YJ \)-injective (see [2]);
2. A ring \( R \) is strongly regular if and only if \( R \) is a left duo ring whose simple singular left \( R \)-modules are \( YJ \)-injective;
3. A ring \( R \) is strongly regular if and only if \( R \) is a left duo ring whose simple singular left \( R \)-modules are \( YJ \)-injective;
4. A ring \( R \) is strongly regular if and only if \( R \) is a weakly right duo ring whose simple singular right \( R \)-modules are \( GP \)-injective (see [6]).

The aim of present paper is to investigate rings whose simple singular right \( R \)-modules are \( PS \)-injective. We prove that a \( NI \) ring satisfying (P) are right nonsingular. Semiprimitive rings, nonsingular rings and semisimple Artinian rings are characterized in terms of \( PS \)-injectivity.

2. Main Results

We start with the following lemmas needed frequently in the sequel.

**Lemma 2.1.** Let \( R \) satisfy (P). Then for any \( a \in J(R) \), there exists a right ideal \( L \) of \( R \) such that \( (RaR + r(a)) \oplus L = R \).

**Proof.** For the right ideal \( RaR + r(a) \) of \( R \), there exists a right ideal \( L \) of \( R \) such that \( (RaR + r(a)) \oplus L \) is an essential right ideal of \( R \). Suppose \( (RaR + r(a)) \oplus L \neq R \). Then it must be contained in a maximal right ideal \( M \), whence \( M \) is essential.

Define \( f : aR \to R/M \) by \( f(ax) = x + M \) for \( x \in R \). It is easy to check that \( f \) is well-defined. Since \( R \) satisfies (P), \( R/M \) is \( PS \)-injective. Thus there exists \( b \in R \) such that \( 1 + M = f(a) = (b + M)a = ba + M \), and hence \( 1 - ba \in M \). Note that \( 1 - ba \) is invertible, contradicting with the maximality of \( M \). Thus, \( (RaR + r(a)) \oplus L = R \).

**Lemma 2.2.** Let \( R \) satisfy (P). Then \( J(R) \cap Z(R_R) = 0 \).

**Proof.** Take any \( 0 \neq b \in J(R) \cap Z(R_R) \). By Lemma 2.1, there exists a right ideal \( L \) of \( R \) such that \( (RbR + r(b)) \oplus L = R \). Since \( b \in Z(R_R) \), \( r(b) \) is an essential right ideal of \( R \). Now \( r(b) \cap L = 0 \), so \( L = 0 \). This proves that \( RbR + r(b) = R \), and hence \( r(b) = R \) because \( RbR \) is a small ideal of \( R \). This implies \( b = 0 \), a required contradiction.

Recall that a ring \( R \) is a \( NI \) ring [8] if the set of nilpotent elements \( N(R) \) in \( R \) is an ideal. A ring \( R \) is a \( NI \) ring if and only if the nilradical \( Nil^*(R) = N(R) \).
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Obviously, 2-primal rings (i.e., \( P(R) = N(R) \), where \( P(R) \) is the prime radical of \( R \)) are \( NI \) rings.

**Proposition 2.3.** If \( R \) is a \( NI \) ring and satisfies (P), then \( R \) is right nonsingular.

**Proof.** Suppose that \( Z(R_R) \neq 0 \). Then \( Z(R_R) \) contains nonzero nilpotent elements. To see this, let \( 0 \neq x \in Z(R_R) \), so \( r(x) \) is an essential right ideal of \( R \). Thus \( r(x) \cap xR \neq 0 \), and hence there exists \( r \in R \) such that \( xr \neq 0 \) and \( x^2r = 0 \). So we have \( (xr)^2 = 0 \), whence \( xrx = 0 \). It implies \( (xr)^2 = 0 \), and hence \( xrx = 0 \), a contradiction.

Now take \( 0 \neq b \in Z(R_R) \) with \( b^2 = 0 \), so \( b \in J(R) \) since \( R \) is a \( NI \) ring. Then \( b \in J(R) \cap Z(R_R) = 0 \) by Lemma 2.2. This is a contradiction. \( \Box \)

It is known that a ring \( R \) is semiprimitive if and only if every right simple \( R \)-module is \( PS \)-injective (cf. [13, Proposition 2.18]). But a ring satisfying (P) need not be semiprimitive by the following example.

**Example 2.4.** Let \( R = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix} \), where \( F \) is a field. Then \( 0 \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in J(R) \). Note that \( T = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix} \) is the unique essential maximal right ideal of \( R \). It is easy to show that every simple singular right \( R \)-module is \( PS \)-injective.

Now we consider when a ring \( R \) satisfying (P) is semiprimitive.

**Proposition 2.5.** If \( R \) satisfies (P) and every complement right ideal is an ideal, then \( R \) is semiprimitive.

**Proof.** We first prove that \( J(R) \) contains no nonzero nilpotent elements. Let \( a \in J(R) \) with \( a^2 = 0 \). So there exists a right ideal \( L \) of \( R \) such that \( r(a) \oplus L \) is right essential. By hypothesis, \( L \) is an ideal. Then \( aL \subseteq L \cap r(a) = 0 \), so \( L \subseteq r(a) \), and hence \( r(a) \) is an essential right ideal of \( R \). Then \( a \in Z(R_R) \). So \( a \in J(R) \cap Z(R_R) = 0 \) by Lemma 2.2.

Now let \( b \in J(R) \). By Lemma 2.1, there exists a right ideal \( L \) of \( R \) such that \( (RbR + r(b)) \oplus L = R \). Thus \( RbR + r(b) = eR \) with \( e^2 = e \in R \). So \( b^2 = beb = b^2ab \) for some \( a \in R \), and hence \( b^2(1 - ab) = 0 \), which implies \( b^2 = 0 \) because \( 1 - ab \) is invertible. Thus \( b = 0 \) by the preceding result. \( \Box \)

A ring is called a right duo ring if every right ideal is an ideal.

**Corollary 2.6.** If \( R \) is a right duo ring and satisfies (P), then it is semiprimitive.

Recall that a ring \( R \) is right weakly continuous \([10]\) if \( R \) is semiregular and \( J(R) = Z(R_R) \). Examples of this rings include mininjective semiregular rings \( R \) in which \( soc(R_R) \leq ess \ R_R \), \( P \)-injective semiregular rings, right continuous rings, and the endomorphism rings of free continuous right modules.

**Proposition 2.7.** Let \( R \) be a right weakly continuous ring. If \( R \) satisfies (P), then it is semiprimitive.
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Proof. Note that $J(R) = Z(R_R)$ since $R$ is right weakly continuous. Thus, the result follows by Lemma 2.2. □

A ring $R$ is called idempotent reflexive if $eRa = 0$ implies $aRe = 0$ for any $a$ and $e^2 = e \in R$. Abelian rings and semiprime rings are idempotent reflexive. Now we have the following results.

**Theorem 2.8.** The following are equivalent for a ring $R$.

1. $R$ is semiprimitive.
2. $R$ is a semiprime ring satisfying $(P)$.
3. $R$ is an idempotent reflexive ring satisfying $(P)$.
4. $R$ is a right $PS$-injective ring satisfying $(P)$.

Proof. (1)⇒(2), (2)⇒(3) and (1)⇒(4) are trivial. (3)⇒(1). For any $a \in J(R)$, by lemma 2.1, there exists a right ideal $L$ of $R$ such that $(RaR + r(a)) \oplus L = R$. Let $L = eR$, where $e^2 = e \in R$. Then $eRaR = LReR \subseteq RaR \cap L = 0$, and hence $eRa = 0$. Thus, $aRe = 0$ since $R$ is an idempotent reflexive ring. So $L \subseteq ReR \subseteq r(a)$. This implies $L = 0$. Then we have $RaR + r(a) = R$, and hence $r(a) = R$ since $RaR$ is a small ideal of $R$. Therefore, $a = 0$. (4)⇒(1). By [13, Theorem 2.6], $J(R) \subseteq Z(R_R)$ since $R$ is right $PS$-injective. Then $J(R) = J(R) \cap Z(R_R) = 0$ by Lemma 2.2. □

**Remark 2.9.** A left $PS$-injective ring satisfying $(P)$ need not be semiprimitive. For example, let $R = \left( \begin{array}{cc} K & 0 \\ K & A \end{array} \right)$, where $K = \mathbb{Z}_2$ and

$$A = \{(a_1, a_2, \ldots, a_n, a, a, \ldots) \mid a, a_1, a_2, \ldots \in K, n \in \mathbb{N}\}.$$ 

If $k \in K$ and $(a_1, a_2, \ldots, a_n, a, a, \ldots) \in A$, let $k \cdot (a_1, a_2, \ldots, a_n, a, a, \ldots) = ka$. Then $R = \left( \begin{array}{cc} K & 0 \\ K & \mathbb{Z}_2^{(N)} \end{array} \right)$ is the unique maximal essential right ideal of $R$, where

$$\mathbb{Z}_2^{(N)} = \{(a_1, a_2, \ldots, a_n, 0, 0, \ldots) \mid a, a_1, a_2, \ldots \in \mathbb{Z}_2, n \in \mathbb{N}\}.$$ 

Analogous to the proof of [12, Example 2.13], we can show that $R$ is a MERT, left $PS$-injective ring and satisfies $(P)$. But it is not semiprimitive because $J(R) = \left( \begin{array}{cc} 0 & 0 \\ \mathbb{Z}_2 & 0 \end{array} \right) \neq 0$.

**Lemma 2.10.** ([2, Lemma 3.8]) A ring $R$ is semisimple Artinian if and only if $R$ has no an essential maximal left(right) ideal.

**Theorem 2.11.** The following are equivalent for a ring $R$.

1. $R$ is a semisimple Artinian ring.
2. $R$ is a right Kasch ring satisfying $(P)$. 

Proof. (1) \(\Rightarrow\) (2) is clear.

(2) \(\Rightarrow\) (1). Suppose that \(M \neq 0\) is an essential maximal right ideal of \(R\). Since \(R\) is a right Kasch ring, \(M = r(a)\) for some \(0 \neq a \in R\). Then \(a \in Z(R_R)\). Note

that \(aR \cong R/M\) is simple, and hence \(aR \subseteq \text{soc}(R_R)\). Thus \((aR)^2 \subseteq aR\text{soc}(R_R) = a\text{soc}(R_R) \subseteq aM = 0\) since \(\text{soc}(R_R)\) is the intersection of all essential right ideals of \(R\), whence \(aR \subseteq J(R)\). Then \(aR \subseteq Z(R_R) \cap J(R) = 0\) by Lemma 2.2, and hence \(a = 0\), a contradiction. Therefore, \(R\) has no an essential maximal right ideal, and hence (1) follows by Lemma 2.10.

\[ \Box \]

**Proposition 2.12.** If \(R\) is an MERT, left Kasch, NI ring such that essential maximal right ideals are PS-injective, then it is semisimple Artinian.

Proof. Assume that \(R\) is not semisimple Artinian, then it has an essential maximal right ideal \(M\) by lemma 2.10. Thus \(M\) is an ideal since \(R\) is MERT. Let \(M_1\) be a maximal left ideal of \(R\) containing \(M\). Then, in view of [10, Proposition 1.44], \(M_1 = l(u)\) for some \(0 \neq u \in R\) since \(R\) is a left Kasch ring. Now \(M\) is an essential right ideal of \(R\), and hence \(M \cap uR \neq 0\). Thus, there exists \(r \in R\) such that \(ur \neq 0\) and \(ur \in M\), whence \(uru = 0\) because \(M \subseteq M_1\), which yields \((ur)^2 = 0\), and hence \(ur \in J(R)\) since \(R\) is a NI ring. Then the inclusion map \(urR \rightarrow M\) extends to \(R \rightarrow M\) by the PS-injectivity of \(M\). Therefore \(ur = c \cdot (ur)\) for some \(c \in M \subseteq M_1\), whence \(1 - c \in l(ur)\). But \(l(ur) = l(u) = M_1\). Thus \(1 - c \in M_1\), which yields \(1 \in M_1\), a contradiction. Then the result follows.

\[ \Box \]

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