The Order of Normal Form Generalized Hypersubstitutions of Type $\tau = (2)$

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Abstract. In 2000, K. Denecke and K. Mahdavi showed that there are many idempotent elements in $\text{Hyp}_{\phi}(V)$ the set of normal form hypersubstitutions of type $\tau = (2)$ which are not idempotent elements in $\text{Hyp}(2)$ the set of all hypersubstitutions of type $\tau = (2)$. They considered in which varieties, idempotent elements of $\text{Hyp}(2)$ are idempotent elements of $\text{Hyp}_{\phi}(V)$. In this paper, we study the similar problems on the set of all generalized hypersubstitutions of type $\tau = (2)$ and the set of all normal form generalized hypersubstitutions of type $\tau = (2)$ and determine the order of normal form generalized hypersubstitutions of type $\tau = (2)$.

1. Introduction

The order of generalized hypersubstitutions of type $\tau = (2)$ was studied by W. Puninagool and S. Leeratanavalee [6]. In this paper, we used the order of generalized hypersubstitutions of type $\tau = (2)$ as a tool to characterize the order of normal form generalized hypersubstitutions of type $\tau = (2)$.

A generalized hypersubstitution of type $\tau = (n_i)_{i \in I}$ is a mapping $\sigma$ which maps each $n_i$-ary operation symbol to the set $W_{\tau}(X)$ of all terms of type $\tau$ built up by operation symbols from $\{f_i | i \in I\}$ where $f_i$ is $n_i$-ary and variables from a countably infinite alphabet $X := \{x_1, x_2, x_3, \ldots\}$ which dose not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type $\tau$ by $\text{Hyp}_G(\tau)$. To
define a binary operation on $Hyp_G(\tau)$, we define at first the concept of generalized superposition of terms $S^m: W_\tau(X)^{m+1} \rightarrow W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, ..., t_m) := t_j$.
(ii) If $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, ..., t_m) := x_j$.
(iii) If $t = f_i(s_1, ..., s_{n_i})$, then 
\[ S^m(t, t_1, ..., t_m) := f_i(S^m(s_1, t_1, ..., t_m), ..., S^m(s_{n_i}, t_1, ..., t_m)). \]

We extend a generalized hypersubstitution $\sigma$ to a mapping $\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ inductively defined as follows:

(i) $\hat{\sigma}[x] := x \in X$,
(ii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] := S^m(\sigma[f_i], \hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}]),$ for any $n_i$-ary operation symbol $f_i$, supposed that $\hat{\sigma}[t_j], 1 \leq j \leq n_i$ are already defined.

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \hat{\sigma}_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let $\sigma_{id}$ be the hypersubstitution which maps each $n_i$-ary operation symbol $f_i$ to the term $f_i(x_1, ..., x_{n_i})$.

In [3], S. Leeratanavalee and K. Denecke proved that: For arbitrary terms $t, t_1, ..., t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have

(i) $S^n(\sigma[t], \sigma[t_1], ..., \sigma[t_n]) = \hat{\sigma}[S^n(t, t_1, ..., t_n)],$
(ii) $(\hat{\sigma}_1 \circ \hat{\sigma}_2) = \hat{\sigma}_1 \circ \hat{\sigma}_2.$

It turns out that $Hyp_G(\tau) = (Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid where $\sigma_{id}$ is the identity element and the set of all hypersubstitutions of type $\tau$ forms a submonoid of $Hyp_G(\tau)$.

For more details on generalized hypersubstitutions see [3]. In this paper, we consider the type $\tau = (2)$ with the binary operation symbol, say $f$. Let $W_1$ denote the set of all words using only the letter $x_1$, and dually for $W_{x_2}$. For $s \in W_2(X)$, we denote:

\[ \sigma_s := \text{the generalized hypersubstitution which maps the binary operation } f \text{ to the term } s, \]
\[ \text{leftmost}(s) := \text{the first variable (from the left) that occurs in } s, \]
\[ \text{rightmost}(s) := \text{the last variable (from the right) that occurs in } s, \]
\[ W^G_1(\{x_1\}) := \{s \in W_2(X)|x_1 \in \text{var}(s), x_2 \notin \text{var}(s)\}, \]
\[ W^G_2(\{x_2\}) := \{s \in W_2(X)|x_2 \in \text{var}(s), x_1 \notin \text{var}(s)\}, \]
\[ W^G := \{t \in W_2(X)|t \notin X, x_1, x_2 \in \text{var}(t)\}, \]
\[ G := \{\sigma_s \in Hyp_G(2)|s \in W_2(X)\}/X, x_1, x_2 \notin \text{var}(s)\}, \]
Let \( P_G(2) := \{ \sigma_s \leq HYP_G(2) | i \in \mathbb{N}, x_1 \in X \} \),
\[ E^G_{x_1} := \{ \sigma_{f(x_1, x)} \leq HYP_G(2) | s \in W^G(2)(X), x_2 \notin \text{var}(s) \}, \]
\[ E^G_{x_2} := \{ \sigma_{f(x, x_2)} \leq HYP_G(2) | s \in W^G(2)(X), x_1 \notin \text{var}(s) \}, \]
\[ T_1 := \{ \sigma_s \leq HYP_G(2) | s \in W^G(2)(\{x_1\}) \text{ and leftmost}(s) = x_m \text{ where } m > 2, \]
\[ T_2 := \{ \sigma_s \leq HYP_G(2) | s \in W^G(2)(\{x_2\}) \text{ and rightmost}(s) = x_m \text{ where } m > 2. \]

In [6], W. Puninagool and S. Leeratanavalee proved that the following statements hold.

(i) Let \( \sigma_1 \) be a generalized hypersubstitution of type \( \tau = (2) \). Then \( \sigma_1 \) is idempotent if and only if \( \hat{\sigma}[t] \equiv t \).

(ii) \( P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{ \sigma_{id} \} \) is the set of all idempotent elements in \( HYP_G(2) \).

(iii) \( T_1 \cup T_2 \cup \{ \sigma_{f(x_2, x_1)} \} \) is the set of all elements has order 2 in \( HYP_G(2) \).

(iv) If \( \sigma \in HYP_G(2) \setminus (P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{ \sigma_{id} \} \cup T_1 \cup T_2 \cup \{ \sigma_{f(x_2, x_1)} \}) \), then \( \sigma^n \neq \sigma^{n+1} \) for all \( n \in \mathbb{N} \) with \( n \geq 1 \) (i.e. \( \sigma \) has infinite order).

(v) If \( \sigma \in HYP_G(2) \setminus (P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{ \sigma_{id} \} \cup T_1 \cup T_2 \cup \{ \sigma_{f(x_2, x_1)} \}) \), then the length of the word \( (\sigma \circ_h \sigma)(f) \) is greater than the length of \( \sigma(t) \).

2. Normal Form Generalized Hypersubstitutions

The concept of normal form hypersubstitutions was introduced by J. Plonka in 1994 [5]. In [4], S. Leeratanavalee and K. Denecke generalized the concept of normal form hypersubstitutions to normal form generalized hypersubstitutions. We recall first the definition of \( V \)-generalized equivalent.

**Definition 2.1.** Let \( V \) be a variety of type \( \tau \). Two generalized hypersubstitutions \( \sigma_1 \) and \( \sigma_2 \) of type \( \tau \) are called \( V \)-generalized equivalent if \( \sigma_1(f_i) \approx \sigma_2(f_i) \) are identities in \( V \) for all \( i \in I \). In this case we write \( \sigma_1 \sim_{VG} \sigma_2 \).

Clearly, the relation \( \sim_{VG} \) is an equivalence relation on \( HYP_G(\tau) \) and has the following properties:

**Proposition 2.2.** ([4]) Let \( V \) be a variety of type \( \tau \) and let \( \sigma_1, \sigma_2 \in HYP_G(\tau) \). Then the following are equivalent.

(i) \( \sigma_1 \sim_{VG} \sigma_2 \).

(ii) For all \( t \in W_\tau(X) \) the equation \( \hat{\sigma_1}[t] \approx \hat{\sigma_2}[t] \) is an identity in \( V \).

In general, the relation \( \sim_{VG} \) is not a congruence relation on \( HYP_G(\tau) \). Let \( V \) be a variety of type \( \tau \) and \( IdV \) be the set of identities satisfied in the variety \( V \). If \( s \approx t \) is an identity and for any \( \sigma \in HYP_G(\tau) \), \( \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV \) then \( s \approx t \) is called a strong hyperidentity. A variety \( V \) is called strongly solid if every identity in \( V \) is satisfied as a strong hyperidentity. For a strongly solid variety \( V \) the relation \( \sim_{VG} \) is a congruence relation on \( HYP_G(\tau) \) and the factor monoid \( HYP_G(\tau)/\sim_{VG} \) exists.

In the arbitrary case we form also \( HYP_G(2)/\sim_{VG} \) and consider a choice function
where \( m, k, n > 1 \).

Let \( \text{Proposition 2.3.} \)
\[
\mathcal{W} := \varphi(\text{Hyp}(2) / \sim_{VG})
\]
by \( \sigma \circ \text{id} := \varphi(\sigma \circ \text{id}) \). This mapping is well-defined, but in general not associative.

For example, we consider the variety \( V = \text{Mod} \{ xy^z \approx x(yz), xuy \approx xuyv, x^3 \approx x \}. \) Let \( f \) be our binary operation symbol and \( x_1x_2 \) abbreviates \( f(x_1, x_2) \). So we can construct the set \( W_{(2)}(X)/\text{IdV} \). These are some elements in \( W_{(2)}(X)/\text{IdV} : \)

\( [x_1]_{\text{IdV}}, [x_2]_{\text{IdV}}, [x_m]_{\text{IdV}}, [x_1x_m]_{\text{IdV}}, [x_2x_m]_{\text{IdV}}, [x_1x_2]_{\text{IdV}}, [x_1x_m]_{\text{IdV}}, [x_mx_1]_{\text{IdV}}, [x_2x_1]_{\text{IdV}}, [x_2x_m]_{\text{IdV}}, [x_2x_2]_{\text{IdV}}, [x_2x_1]_{\text{IdV}}, [x_2x_m]_{\text{IdV}}, [x_mx_1]_{\text{IdV}}, [x_mx_2]_{\text{IdV}}, [x_mx_2x_m]_{\text{IdV}}, [x_mx_1x_m]_{\text{IdV}}, [x_mx_2x_m]_{\text{IdV}}, [x_mx_1x_2]_{\text{IdV}}, [x_mx_2x_2]_{\text{IdV}}, [x_mx_1x_2x_m]_{\text{IdV}}, [x_mx_2x_2x_m]_{\text{IdV}}, [x_mx_1x_2x_2]_{\text{IdV}}, [x_mx_2x_2x_2]_{\text{IdV}}, \)

where \( m, k, n > 2 \).

So we get some corresponding elements in \( \text{Hyp}_{GN_v}(V) : \)

\( \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_m}, \sigma_{x_1x_m}, \sigma_{x_2x_m}, \sigma_{x_1x_2}, \sigma_{x_1x_1}, \sigma_{x_2x_1}, \sigma_{x_2x_2}, \sigma_{x_1x_2x_m}, \sigma_{x_2x_2x_m}, \sigma_{x_1x_1x_2}, \sigma_{x_1x_2x_2}, \sigma_{x_2x_2x_2x_m}, \)

where \( m, k, n > 2 \).

We call this structure a groupoid of normal form generalized hypersubstitutions. Next, we consider, how to characterize the idempotent elements of \( \text{Hyp}_{GN_v}(V) \) where \( V \) is a variety of semigroups.

**Proposition 2.3.** Let \( V \) be a variety of semigroups and let

\( \varphi : \text{Hyp}(2) / \sim_{VG} \rightarrow \text{Hyp}(2) \)

be a choice function. Then

(i) \( \sigma \in \text{Hyp}_{GN_v}(V) \) is an idempotent element iff \( \sigma \circ \sigma \sim_{VG} \sigma \).

(ii) \( \sigma \circ \sigma \sim_{VG} \sigma \circ \sigma \) if \( \sigma \circ \sigma \in \text{Hyp}_{GN_v}(V) \).

Thus \( \varphi : \text{Hyp}(2) / \sim_{VG} \rightarrow \text{Hyp}(2) \) of all normal form generalized hypersubstitutions with respect to \( \sim_{VG} \) and \( \varphi \).
Proof. (i) If $\sigma$ is an idempotent of $Hyp_{GN}(V)$, then $\sigma \circ_{GN} \sigma = \sigma \sim_{V} (\sigma \circ_{G} \sigma)$. Conversely, we assume that $\sigma \sim_{V} (\sigma \circ_{G} \sigma)$. Because of $\sigma \in Hyp_{GN}(V)$, so $\sigma \circ_{GN} \sigma = \sigma$.

(ii) Since $(\sigma_{x_{1}} \circ_{GN} \sigma_{x_{2}}) \sim_{V} (\sigma_{x_{1}} \circ_{G} \sigma_{x_{2}}) = \sigma_{x_{1}x_{2}} \in Hyp_{GN}(V)$. Thus $\sigma_{x_{1}} \circ_{GN} \sigma_{x_{2}} = \sigma_{x_{1}x_{2}}$.

We consider $\sigma_{x_{1}x_{2}} \circ_{GN} \sigma_{x_{1}x_{2}} = \sigma_{x_{1}x_{2}}^{2}$.

We get $\sigma_{x_{1}x_{2}}$ is idempotents in $Hyp_{GN}(V)$ which is not idempotents in $Hyp_{GN}(2)$.

All idempotent elements of $Hyp_{GN}(V)$ are $\{\sigma_{x_{1}}, \sigma_{x_{2}}, \sigma_{x_{1}x_{2}}, \sigma_{x_{1}x_{2}}, \sigma_{x_{1}x_{2}}, \sigma_{x_{1}x_{2}}, \sigma_{x_{1}x_{2}}, \sigma_{x_{1}x_{2}}\}$.

3. Idempotents in $Hyp_{GN}(V)$

In general, if $\sigma$ is an idempotent of $Hyp_{G}(2)$ and $\sigma \in Hyp_{GN}(V)$, then it is also an idempotent of $Hyp_{GN}(V)$ for any variety $V$ of semigroups and any choice function $\varphi$. But if $\sigma$ is an idempotent in $Hyp_{GN}(V)$, then it is not necessarily be idempotent in $Hyp_{G}(2)$. As an example, let $V = Mod\{(xy)z \approx x(yz), xyw \approx yxw, x^{3} \approx x\}$. We consider

$\sigma_{x_{1}x_{2}} \circ_{GN} \sigma_{x_{1}x_{2}} = \sigma_{x_{1}x_{2}}^{2}$.

Theorem 3.1. For a variety $V$ of semigroups the following are equivalent:

(i) $Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$.

(ii) $\{\sigma | \sigma \in Hyp_{GN}(V) \text{ and } \sigma \circ_{GN} \sigma = \sigma \} = \{\sigma | \sigma \in Hyp_{G}(2) \text{ and } \sigma \circ_{G} \sigma = \sigma \} \cap Hyp_{GN}(V)$ for each choice function $\varphi$. 


Proof. Let $x_i \in W_{(2)}(X)$ where $i > 2$ and $l(t)$ denote the length of $t$ where $t \in W_{(2)}(X)$. Let $\varphi$ be an arbitrary choice function.

(i) $\Rightarrow$ (ii)

Let $\text{Mod}\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$. It is clear that $\{\sigma | \sigma \in HypG(2) \}$ and $\sigma \circ_G \sigma = \sigma \cap HypG_N(\sigma) \subseteq \{\sigma | \sigma \in HypG_N(\sigma) \}$ and $\sigma \circ_G \sigma = \sigma$ for each choice function $\varphi$.

Conversely, let $\sigma_w \subseteq \{\sigma | \sigma \in HypG_N(\sigma) \}$ and $\sigma_w$ is not idempotent in $HypG(2)$. Since $xy \approx x_i, x_i, x_{i+1} \approx x_{i+1}$ and $tx_i \approx tx_2$ (where $s \in W_{(2)}(\{x_1\})$), $t \in W_{(2)}(\{x_2\})$ are identity in $V$, so we choose $xy, x_i, ts_i, ttx_2$ are representatives of its classes in $HypG_N(\sigma)$. Then $xy, x_i, ts_i, ttx_2 \notin HypG_N(\sigma)$. Since $\sigma_w$ is not idempotent in $HypG(2)$ and $\sigma_w \neq \sigma y, \sigma x_i, \sigma z_i$, so $\sigma_w$ has infinite order.

Since $\sigma_w$ has infinite order, so $l(\sigma_w) \neq l(\sigma_w \circ_G \sigma_w)$. We get $\sigma_w \approx (\sigma_w \circ_G \sigma_w) \notin \text{IdMod}\{(xy)z \approx x(yz), xy \approx yx\}$. But $\sigma_w$ is idempotent on $HypG_N(\sigma)$, so $l(\sigma_w) = l(\sigma_w \circ_G \sigma_w) = l(\varphi(\sigma_w \circ_G \sigma_w))$, i.e., $\sigma_w \approx (\sigma_w \circ_G \sigma_w) \in IdV$, a contradiction. So $IdV \subseteq \text{IdMod}\{(xy)z \approx x(yz), xy \approx yx\}$.

(ii) $\Rightarrow$ (i)

Let $\{\sigma | \sigma \in HypG_N(\sigma) \}$ and $\sigma \circ_G \sigma = \sigma \cap HypG_N(\sigma) \subseteq \{\sigma | \sigma \in HypG(2) \}$ and $\sigma \circ_G \sigma = \sigma$ for each choice function $\varphi$.

Assume that $\text{Mod}\{(xy)z \approx x(yz), xy \approx yx\} \not\subseteq V$. Then there exists $x^k \approx x^n \in IdV$ with $1 \leq k \leq n \in \mathbb{N}$. Next, we will construct an idempotent element of $HypG_N(\sigma)$ which is not in $P_G(2) \cup E_G(2) \cup E_G \cup G \cup \{\sigma_{id}\}$. We consider into six cases:

Case 1 : We set $m = n - k$ and $w = f(f(x_1, x_1), u)$ where $u \in W_{x_i}$. Clearly, $\sigma_w \notin P_G(2) \cup E_G(2) \cup E_G \cup G \cup \{\sigma_{id}\}$. It is easy to see that the length of $\sigma_w$ is $3km$ and the length of $(\sigma_w \circ_G \sigma_w)$ is $(3km)^2$. In fact, from $x^k \approx x^n$ in $IdV$ it follows that $x^a \approx x^{a+b+m}$ in $IdV$ for all $a \geq k$ and $b \geq 1$ where $a, b \in \mathbb{N}$. Then we have $x_{3km} \approx x_{3km+(9k^2m-3k)m} = x_{3km+9k^2m^2-3km} = x_{3km+9k^2m^2} = x_{(3km)^2}$. Hence $\sigma_w(f) \approx x_{(3km)^2} \approx x_{(3km)^2} \approx (\sigma_w \circ_G \sigma_w)(f)$.

Case 2 : We set $m = n - k$ and $w = f(f(f(x_1, x_1), ..., x_1), x_1, x_1)$. Clearly, $\sigma_w \notin P_G(2) \cup E_G(2) \cup E_G \cup G \cup \{\sigma_{id}\}$. It is easy to see that the length of $\sigma_w$ is $km + 1$ and the length of $(\sigma_w \circ_G \sigma_w)$ is $(km)^2 + 1$. In fact, from $x^k \approx x^n$ in $IdV$ it follows that $x^a \approx x^{a+b+m}$ in $IdV$ for all $a \geq k$ and $b \geq 1$ where $a, b \in \mathbb{N}$. Then we have $x_{km} \approx x_{km+(k^2m-k)m} = x_{km+k^2m^2-km} = x_{km+k^2m^2} = x_{(km)^2}$. Hence $\sigma_w(f) \approx x_{(km)^2} \approx x_{(km)^2} \approx (\sigma_w \circ_G \sigma_w)(f)$.

Case 3 : From $x^k \approx x_n$ in $IdV$ implies $x^i \approx x^i \in IdV$. We set $m = n - k$, $t = r-s$ and $w = f(f(x_1, x_1), u)$ where $u \in W_{(2)}(\{x_1\})$. Clearly, $\sigma_w \notin P_G(2) \cup E_G(2) \cup E_G \cup G \cup \{\sigma_{id}\}$. It is easy to see that the length of $\sigma_w$ is $(2km)^2 + st(2km + 1)$. In fact, from $x^t \approx x^t$ it follows that $x^t \approx x^{s+b+m}$ in $IdV$ for all $a \geq k$, $b \geq 1$ and $c \geq s$ and $d \geq 1$ where $a, b, c, d \in \mathbb{N}$. Then we have $x_{2km} \approx x_{2km+(4k^2m-2km)} = x_{2km+4k^2m^2-2km} = x_{4k^2m^2} = x_{4k^2m^2}$. Hence $\sigma_w(f) \approx x_{2km} \approx x_{2km} \approx (\sigma_w \circ_G \sigma_w)(f)$.
4. Elements of Infinite Order

In this section, we will characterize the set of all elements in $\text{Hyp}_{\mathcal{G}_N}(V)$ which have infinite order where $V = \text{Mod}\{(xy)z \approx x(yz), xy \approx yx\}$. Let $O(\sigma)$ denote the order of the generalized hypersubstitution $\sigma \in \text{Hyp}_{\mathcal{G}_N}(V)$.

**Theorem 4.1.** Let $V$ be a variety of semigroups and $(\sigma)_{\mathcal{G}_N}$ be the cyclic subsemigroup generated by $\sigma$. Then following are equivalent:

(i) $\text{Mod}\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$.

(ii) $\{\sigma|\sigma \in \text{Hyp}_{\mathcal{G}_N}(V) \text{ and the order of } \sigma \text{ is infinite} \} = \text{Hyp}_{\mathcal{G}_N}(V) \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)$ where:

$A_1 = P_2 G(2) \cup E_2^G \cup E_2^G \cup G \cup \{\sigma_{id}\} \cup \{\sigma_{x_2,x_1}\}$

$A_2 = \{\sigma|\sigma \in \text{Hyp}_{\mathcal{G}_N}(V) \cap (T_1 \cup \{\sigma_v \mid v \in W_2^G(\{x_1\}) \text{ where leftmost}(v) = x_1\}) \sigma_{x_1,v} \sigma_{f_1,x_1} \text{ where } s \in W_2^G(\{x_1\}) \text{ and } (\sigma)_{\mathcal{G}_N} \cap \{\sigma_{x_1,v} \mid u \in W_2(X)\} \neq \emptyset\}$

$A_3 = \{\sigma|\sigma \in \text{Hyp}_{\mathcal{G}_N}(V) \cap (T_2 \cup \{\sigma_v \mid v \in W_2^G(\{x_2\}) \text{ where rightmost}(v) = x_2\}) \sigma_{x_2,v} \sigma_{f_2,x_2} \text{ where } s \in W_2^G(\{x_2\}) \text{ and } (\sigma)_{\mathcal{G}_N} \cap \{\sigma_{x_2,v} \mid u \in W_2(X)\} \neq \emptyset\}$.

**Proof.** Let $x_i \in W_2(X)$ where $i > 2$ and $l(t)$ denote the length of $t$ where $t \in W_2(X)$. Let $\varphi$ be an arbitrary choice function.

(i)$\Rightarrow$(ii)

Let $\text{Mod}\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$. We will show that $\{\sigma|\sigma \in \text{Hyp}_{\mathcal{G}_N}(V) \text{ and the order of } \sigma \text{ is infinite} \} = \text{Hyp}_{\mathcal{G}_N}(V) \setminus (A_1 \cup A_2 \cup A_3)$. Let $\sigma_w$ has infinite order on $\text{Hyp}_{\mathcal{G}_N}(V)$. Since $A_1$ is set of all idempotent on $\text{Hyp}_{\mathcal{G}_N}(V)$, i.e., all elements of $A_1$ has order $1$. So $\sigma_w \notin A_1$. Assume that $\sigma_w \in A_2 \cup A_3$, then there exists a word $u \in W_2(X)$ ($s \in W_2(X)$) such that $\sigma_{x_1,u} \in (\sigma)_{\mathcal{G}_N}$ $(\sigma_{x_2} \in (\sigma)_{\mathcal{G}_N})$. We get $\sigma_{x_1,u} = \sigma_{x_2,u}$ for each $m \in N (\sigma_{x_2} = \sigma_{x_2}^\infty$ for each $n \in N)$ and $O(\sigma_{x_1,u}) = 1$ $(O(\sigma_{x_2}) = 1$, so $O(\sigma^\infty) = 1 \ (O(\sigma^\infty) = 1)$, i.e., $\sigma^\infty$ is idempotent on $\text{Hyp}_{\mathcal{G}_N}(V)$, contradicts to $O(\sigma^\infty) = \infty$. Thus $\sigma_w \notin (A_1 \cup A_2 \cup A_3)$. Hence $\sigma_w \in \text{Hyp}_{\mathcal{G}_N}(V) \setminus (A_1 \cup A_2 \cup A_3)$.
(ii)$\Rightarrow$(i)

Let $\{\sigma|\sigma \in H_{\text{yp}G_{\mathcal{N}}_{c}}(V)\}$ and the order of $\sigma$ is infinite $= H_{\text{yp}G_{\mathcal{N}}_{c}}(V) \setminus (A_{1} \cup A_{2} \cup A_{3} \cup A_{4})$.

Assume that $\text{Mod} \{\{xy\}z \approx x(yz), xy \approx yx\} \nsubseteq V$. Then there exists $x^{k} \approx x^{n} \in IdV$ with $1 \leq k \leq n \in \mathbb{N}$. We consider into two cases:

Case 1: We set $m = n - k$ and $w = f(f(...f(x_{1}, x_{2}), ..., x_{2}), x_{2})$. Clearly, $\sigma_{w} \notin (A_{1} \cup A_{2} \cup A_{3})$. It is easy to see that the length of $\sigma_{w}$ is $km + 1$ and the length of $(\sigma_{w} \circ G \sigma_{w})$ is $(km)^{2} + 1$. In fact, from $x^{k} \approx x^{n} \in IdV$ it follows that $x^{k} \approx x^{n+bm} \in IdV$ for all $a \geq k$ and $b \geq 1$ where $a, b \in \mathbb{N}$. Then we have $x^{km} \approx x^{km+(k^{2}m-k)m} = x^{km+k^{2}m^{2}-km} = x^{(km)^{2}}$. Hence $\sigma_{w}(f) \approx x_{1}x_{2}^{km} \approx x_{1}x_{2}^{(km)^{2}} \approx (\sigma_{w} \circ G \sigma_{w})(f)$.

Case 2: We set $m = n - k$ and $w = f(f(...f(x_{1}, f(x_{2}, x_{1})), ..., x_{1}), x_{1})$. Clearly, $\sigma_{w} \notin (A_{1} \cup A_{2} \cup A_{3})$. It is easy to see that the length of $\sigma_{w}$ is $km$ and the length of $(\sigma_{w} \circ G \sigma_{w})$ is $km(km + 2)$. In fact, from $x^{k} \approx x^{n} \in IdV$ it follows that $x^{k} \approx x^{a+bm} \in IdV$ for all $a \geq k$ and $b \geq 1$ where $a, b \in \mathbb{N}$. Then we have $x^{km} \approx x^{km+(k^{2}m-k)m} = x^{km+k^{2}m^{2}+km} = x^{(km)^{2}+2km} = x^{km(km+2)}$. Hence $\sigma_{w}(f) \approx x_{1}x_{2}x_{m} \approx x_{1}x_{2}x_{m}^{km(km+2)} \approx (\sigma_{w} \circ G \sigma_{w})(f)$.

From all cases, we have $(\sigma_{w} \circ G \sigma_{w}) \sim_{V_{G}} \sigma_{w}$. And from (ii), $(\sigma_{w} \circ G \sigma_{w}) \sim_{V_{G}} \sigma_{w}$. So $(\sigma_{w} \circ G \sigma_{w}) \sim_{V_{G}} \sigma_{w} \circ G \sigma_{w} \sim_{V_{G}} \sigma_{w}$ it follows $\sigma_{w} \circ G \sigma_{w} = \sigma_{w}$. Therefore $\sigma_{w}$ is idempotent on $H_{\text{yp}G_{\mathcal{N}}_{c}}(V)$, a contradiction.

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\textbf{Acknowledgements.} The corresponding author was supported by Chiang Mai University, Chiang Mai 50200, Thailand.

\section*{References}


