Special Right Jacobson Radicals for Right Near-rings

RAVI SRINIVASA RAO*

Department of Mathematics, R. V. R. & J. C. College of Engineering, Chandramoulipuram, Choudavaram, Guntur-522019, Andhra Pradesh, India
e-mail: dr.rsrao@yahoo.com

KORRAPATI SIVA PRASAD
Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar-522510, Guntur (Dist.), Andhra Pradesh, India
e-mail: siva235prasad@yahoo.co.in

Abstract. In this paper three more right Jacobson-type radicals, $J_{\nu}^r$, are introduced for near-rings which generalize the Jacobson radical of rings, $\nu \in \{0, 1, 2\}$. It is proved that $J_{\nu}^r$ is a special radical in the class of all near-rings. Unlike the known right Jacobson semisimple near-rings, a $J_{\nu}^r$-semisimple near-ring $R$ with DCC on right ideals is a direct sum of minimal right ideals which are right $R$-groups of type-$g_{\nu}$, $\nu \in \{0, 1, 2\}$. Moreover, a finite right $g_2$-primitive near-ring $R$ with $eRe$ a non-ring is a near-ring of matrices over a near-field (which is isomorphic to $eRe$), where $e$ is a right $g_2$-primitive idempotent in $R$.

1. Introduction

Special radicals for near-rings are introduced in [1] by G. L. Booth and N. J. Groeneveld using equiprime near-rings. Among the known left Jacobson-type radicals, $J_3$, $J_{3(0)}$ are the only special radicals in the class of zero-symmetric near-rings and in the class of all near-rings respectively.

Srinivasa Rao and Siva Prasad [6, 7] introduced and studied $J_{\nu}^r$, the right Jacobson radical type-$\nu$, $\nu \in \{0, 1, 2\}$. In [9, 10] Srinivasa Rao and Siva Prasad along with T. Srinivas showed that $J_{\nu}^r$ is a Kurosh-Amitsur radical in the Fuchs variety $\mathcal{F}$ of all near-rings $R$ in which the constant part $R_c$ of $R$ is an ideal of $R$, $\nu \in \{0, 1, 2\}$.

But $J_{\nu}^r$ is not s-hereditary in the class of all zero-symmetric near-rings and hence it is not an ideal-hereditary radical in that class, $\nu \in \{0, 1, 2\}$.

Also in [5][11] Srinivasa Rao and Siva Prasad (along with T. Srinivas) intro-
duced and studied the right Jacobson type of radical $J_{\nu}^e$, $\nu \in \{1, 2\}$ ($J_{\nu}^e$) and showed that it is a Kurosh-Amitsur radical in the class of all near-rings and is an ideal hereditary Kurosh-Amitsur radical in the class of all zero-symmetric near-rings. Moreover, they are special radicals in the class of all near-rings.

In this paper we introduce three more right Jacobson radicals, $J_{\nu}^e$, $\nu \in \{0, 1, 2\}$. We show that they are special radicals in the class of all near-rings. So, in the class of all near-rings, they are Kurosh-Amitsur radicals, their semisimple classes are hereditary and radicals classes are c-hereditary. Unlike the known right Jacobson semisimple hereditary and radicals classes are c-hereditary. Unlike the known right Jacobson semisimple near-rings, a $J_{\nu}^e$-semisimple near-ring $R$ with DCC on right ideals is a direct sum of right ideals which are right $R$-groups of type-$g\nu$, $\nu \in \{0, 1, 2\}$. A finite right $g2$-primitive near-ring $R$ with $eRe$ a non-ring is a near-ring of matrices over a near-field (which is isomorphic to $eRe$), where $e$ is a right $g2$-primitive idempotent in $R$.

Near-rings considered are right near-rings (not necessarily zero-symmetric) and $R$ is a near-ring. Now we present some definitions and results of [6] and [7].

A group $(G, +)$ is called a right $R$-group if there is a mapping $((g, r) \to gr)$ of $G \times R$ into $G$ such that (1) $(g + h)r = gr + hr$, (2) $g(rs) = (gr)s$ for all $g, h \in G$ and $r, s \in R$. A subgroup (normal subgroup) $H$ of a right $R$-group $G$ is called an $R$-subgroup (ideal) of $G$ if $hr \in H$ for all $h \in H$ and $r \in R$.

Let $G$ be a right $R$-group. An element $g \in G$ is called a generator of $G$ if $gR = G$ and $g(r + s) = gr + gs$ for all $r, s \in R$. $G$ is said to be monogenic if $G$ has a generator. $G$ is said to be simple if $G \neq \{0\}$ and $G$, and $\{0\}$ are the only ideals of $G$.

A monogenic right $R$-group $G$ is said to be a right $R$-group of type-$0$ if $G$ is simple.

A right $R$-group $G$ of type-$0$ is said to be of type-$1$ if $G$ has exactly two $R$-subgroups, namely $\{0\}$ and $G$.

A right $R$-group $G$ of type-$0$ is said to be of type-$2$ if $gR = G$ for all $0 \neq g \in G$.

Note that a right $R$-group of type-$2$ is of type-$1$ and a right $R$-group of type-$1$ is of type-$0$.

Let $\nu \in \{0, 1, 2\}$. A right modular right ideal $K$ of $R$ is called right $\nu$-modular if $K/K$ is a right $\nu$-group of type-$\nu$.

An ideal $P$ of $R$ is called right $\nu$-primitive if $P$ is the largest ideal of $R$ contained in a right $\nu$-modular right ideal of $R$. $R$ is called a right $\nu$-primitive near-ring if $\{0\}$ is a right $\nu$-primitive ideal of $R$.

Now we present some definitions of [11] and [5].

Let $G$ be a right $R$-group of type-$\nu$, $\nu \in \{0, 1, 2\}$. Suppose that $G0 = \{0\}$ for $\nu = 0$ and $P$ is the largest ideal of $R$ contained in $(0 : G) = \{r \in R \mid Gr = \{0\}\}$. Then $G$ is said to be a right $R$-group of type-$\nu(e)$ if $0 \neq g \in G, r_1, r_2 \in R$ and $gxr_1 = gxr_2$ for all $x \in R$ implies $r_1 = r_2 \in P$.

A right modular right ideal $K$ of $R$ is called $\nu(e)$-modular if $R/K$ is a right $R$-group of type-$\nu(e)$.

Let $G$ be a right $R$-group of type-$\nu(e)$. Then $(0 : G)$ is an ideal of $R$ and is called a right $\nu(e)$-primitive ideal of $R$. 

A near-ring \( R \) is called right \( \nu(\epsilon) \)-primitive if \( \{0\} \) is a right \( \nu(\epsilon) \)-primitive ideal of \( R \).

A near-ring \( R \) is called an equiprime near-ring \([2]\) if \( 0 \neq a \in R \), \( x, y \in R \) and \( ax = ay \) for all \( r \in R \), \( x \) implies \( x = y \). An ideal \( I \) of \( R \) is called equiprime if \( R/I \) is an equiprime near-ring. Moreover, an equiprime near-ring is zero-symmetric.

It is known that a near-ring \( R \) is equiprime if and only if \([2]\)

1. \( x, y \in R \) and \( xRy = \{0\} \) implies \( x = 0 \) or \( y = 0 \).
2. If \( \{0\} \neq I \) is an invariant subnear-ring of \( R \), \( x, y \in R \) and \( ax = ay \) for all \( a \in I \) implies \( x = y \).

In \([1]\), G. L. Booth and N. J. Groenewald defined special radicals for near-rings. A class \( E \) consisting of equiprime near-rings is called a special class if it is hereditary and closed under left invariant essential extensions. If \( \mathcal{R} \) is the upper radical in the class of all near-rings determined by a special class of near-rings, then \( \mathcal{R} \) is called a special radical. A class of near-rings \( E \) is said to satisfy condition \( F_1 \) if \( J < I < R \) and \( I \) is left invariant in \( R \) and \( I/J \in \mathcal{E} \) implies \( J < R \). We need the following theorem:

**Theorem 1.1.** ([12]) Let \( \mathcal{E} \) be a class of zero-symmetric near-rings. If \( \mathcal{E} \) is regular, closed under essential left invariant extensions and satisfies condition \( F_1 \), then \( \mathcal{R} := 1\mathcal{E} \) is a \( c \)-hereditary radical class in the variety of all near-rings, \( 8\mathcal{R} = \mathcal{E} \) and \( 8\mathcal{R} \) is hereditary. So, \( \mathcal{R}(R) = \cap \{I < R \mid R/I \in \mathcal{E} \} \) for any near-ring \( R \).

### 2. Right Jacobson Radicals of Type-\( g_0 \)

Let \( G \) be a right \( R \)-group and \( T \) be a subset of \( G \). Then \( (0 : T) := \{r \in R \mid tr = 0 \) for all \( t \in T \)\). By Proposition 3.7 of \([11]\), if \( G \) is a right \( R \)-group of type-0 and \( G0 = \{0\} \), then there is a largest ideal of \( R \) contained in \( (0 : G) \). Moreover, by Proposition 3.1 of \([5]\), if \( G \) is a right \( R \)-group of type-\( \nu \), then \( G0 = \{0\}, \nu \in \{1, 2\} \).

**Definition 2.1.** Let \( \nu \in \{0, 1, 2\} \). Let \( G \) be right \( R \)-group of type-\( \nu \) and \( G0 = \{0\} \) for \( \nu = 0 \), and \( T \) be the set of all generators of the right \( R \)-group \( G \). Then \( G \) is said to be a right \( R \)-group of type-\( g_0 \), if \( (0 : T) = P \), where \( P \) is the largest ideal of \( R \) contained in \( (0 : G) \).

We present an example of a right \( R \)-group of type-\( g_0 \) which is not of type-\( g_1 \).

**Example 2.2.** Let \( (G, +) \) be a finite non-abelian simple group. Since \( \{0\} \) is the maximal normal subgroup of \( (G, +) \), \( \{0\} \) is the maximal right ideal of \( M_0(G) \) and hence \( M_0(G) \) is a right \( M_0(G) \)-group of type-0. This example was considered in \([7]\) and it was shown that \( M_0(G) \) is not a right \( M_0(G) \)-group of type-1. Each \( 0 \neq h \in G \) give rise to the inner automorphism \( t_h \) of \( G \) defined by \( t_h(x) = h + x - h \) for all \( x \in G \). Clearly, a generator of the right \( M_0(G) \)-group \( M_0(G) \) is an automorphism of \( (G, +) \). Let \( T \) be the set of all automorphisms of \( G \). Suppose that for some \( t \in M_0(G) \) and \( 0 \neq h \in G \), \( t_ht = 0 \). Now \( 0 = (t_{-h})t_h = (t_h)^{-1}t_ht = t \). Therefore \( \{0\} = (0 : t_h) = (0 : T) \). Since the largest ideal of \( M_0(G) \) contained in \( (0 : M_0(G)) \) is \( \{0\} \), \( M_0(G) \) is a right \( M_0(G) \)-group of type-\( g_0 \) but not of type-\( g_1 \).

Now we present an example of a right \( R \)-group of type-\( g_1 \) which is not of type-\( g_2 \).
Example 2.3. Let \((G, +)\) be a finite cyclic group of prime order \(p\), where \(p \neq 2\). Since \(\{0\}\) is the only proper subgroup of \(G\), \(\{0\}\) is the only proper right \(M_0(G)\)-subgroup of \(M_0(G)\). Therefore, \(M_0(G)\) is a right \(M_0(G)\)-group of type-1. Clearly, \(M_0(G)\) is not a right \(M_0(G)\)-group of type-2, as \(M_0(G)\) is not a near-field. This example was considered in [7]. A generator of the right \(M_0(G)\)-group \(M_0(G)\) an is automorphism \((G, +)\). We know that \(G\) has \(p - 1\) automorphisms. Let \(T\) be the set of all these automorphisms. Suppose that for some \(s \in M_0(G)\) and \(t \in T\), \(ts = 0\). Now \(0 = (t^{-1})ts = s\). So \(\{0\} = (0 : t) = (0 : T)\). Since the largest ideal of \(M_0(G)\) contained in \(0 : M_0(G)\) is \(\{0\}\), \(M_0(G)\) is a right \(M_0(G)\)-group of type-\(g_1\) but not of type-\(g_2\).

The following are examples of right \(R\)-groups of type-\(g_2\).

Example 2.4. Let \(R\) be a near-field. Then \(R\) is a right \(R\)-group of type-2. Clearly, \(R\) is also a right \(R\)-group of type-\(g_2\).

Example 2.5. Let \((R, +)\) be a group and let \(K\) be a subgroup of \((R, +)\) of index 2. The trivial multiplication on \((R, +)\) determined by \(R \setminus K\) is given by \(a.b = a\) if \(b \in R \setminus K\) and 0 if \(b \in K\). Now \((R, +)\) is a near-ring. It is clear that \(K\) is a maximal (right) ideal of \(R\). Let \(a \in R \setminus K\). Now \(R = K \cup a + K\). It can be easily verified that \(a + K\) is the generator of the right \(R\)-group \(R/K\). So \(R/K\) is a right \(R\)-group of type-2 and \((0 : a + K) = (0 : R/K)\) is the largest ideal of \(R\) contained in \((0 : R/K)\). Hence \(R/K\) is a right \(R\)-group of type-\(g_2\).

Now we introduce some notions related to the right \(R\)-groups of type-\(g_\nu\).

Definition 2.6. Let \(\nu \in \{0, 1, 2\}\) and \(K\) be a right modular right ideal of \(R\). Then \(K\) is said to be right \(g_\nu\)-modular right ideal of \(R\) if \(R/K\) is a right \(R\)-group of type-\(g_\nu\).

Definition 2.7. Let \(\nu \in \{0, 1, 2\}\). An ideal \(P\) of \(R\) is called a right \(g_\nu\)-primitive ideal of \(R\) if \(P\) is the largest ideal of \(R\) contained in \((0 : G) := \{r \in R \mid Gr = \{0\}\}\) for some right \(R\)-group \(G\) of type-\(g_\nu\).

Definition 2.8. Let \(\nu \in \{0, 1, 2\}\). A near-ring \(R\) is called a right \(g_\nu\)-primitive near-ring if \(\{0\}\) is a right \(g_\nu\)-primitive ideal of \(R\).

Definition 2.9. Let \(\nu \in \{0, 1, 2\}\). The intersection of all right \(g_\nu\)-primitive ideals of \(R\) is called the right Jacobson radical of \(R\) of type-\(g_\nu\), and is denoted by \(J_{g_\nu}(R)\). If \(R\) has no right \(g_\nu\)-primitive ideals, then \(J_{g_\nu}(R)\) is defined to be \(R\).

Note that if \(R\) is a ring then \(J_{g_\nu}(R) = J(R)\), where \(J\) is the Jacobson radical of \(R\).

By Proposition 3.1 of [11], for a right \(R\)-group \(G\), \(G0 = \{0\}\) if and only if \(GR_0 = \{0\}\). Since for a right \(R\)-group \(G\) of type-\(g_\nu\), \(G0 = \{0\}\), \(R_0\) is contained in \((0 : g)\) for every generator \(g\) of \(G\). So \(R_0 \subseteq P\) for every right \(g_\nu\)-primitive ideal \(P\) of \(R\). Hence a right \(g_\nu\)-primitive ideal \(P\) of \(R\) is invariant. This shows that a right \(g_\nu\)-primitive near-ring is zero-symmetric.
Proposition 2.10. Let $\nu \in \{0, 1, 2\}$. An ideal $P$ of $R$ is a right $g_{\nu}$-primitive ideal of $R$ if and only if $P$ is the largest ideal of $R$ contained in a right $g_{\nu}$-modular right ideal of $R$.

Proof. Let $P$ be a right $g_{\nu}$-primitive ideal of $R$. There is a right $R$-group $G$ of type-$g_\nu$ such that $P$ is the largest ideal of $R$ contained in $(0 : G)$. Let $g_0$ be a generator of the right $R$-group $G$. The mapping $r \to g_0r$ is a right $R$-homomorphism of $R$ on to $G$ with kernel $K := (0 : g_0)$. So $R/K$ is right $R$-isomorphic to $G$ (as right $R$-groups). Now $K$ is a right $g_\nu$-modular right ideal of $R$ and $P$ is contained in $K$. Let $Q$ be the largest ideal of $R$ contained in $K$. Now $GQ = \{0\}$, that is, $Q \subseteq (0 : G)$ as $RQ \subseteq Q$, $Q$ being invariant ideal of $R$. Since $P$ is the largest ideal of $R$ contained in $(0 : G)$, $Q \subseteq P$. Now $P \subseteq Q$ as $Q$ is the largest ideal of $R$ contained in $K$. Therefore $P = Q$, that is, $P$ is the largest ideal of $R$ contained in $K$. On the other hand suppose that $P$ is the largest ideal of $R$ contained in a right $g_{\nu}$-modular right ideal $K$ of $R$. Now $G := R/K$ is a right $R$-group of type-$g_\nu$. We have $(0 : G) = (0 : R/K) = (K : R)$ and $RP \subseteq P$ as $P$ is an invariant ideal of $R$. So $P \subseteq (K : R)$. Let $T$ be the largest ideal of $R$ contained in $(K : R) = \{r \in R \mid Rr \subseteq K\}$. Since $P$ is an invariant ideal of $R$, and $P \subseteq T$, $T$ is an invariant ideal of $R$. So $RT \subseteq T$. Let $K$ be right modular by $e$. Now $r - er \in K$ for all $r \in R$. We have $t - et \in K$ for all $t \in T$. Since $RT \subseteq T, T \subseteq K$. Since $P$ is the largest ideal of $R$ contained in $K$, $T \subseteq P$. So $T = P$. Now $P$ is the largest ideal of $R$ contained in $(K : R)$ and hence $P$ is a right $g_{\nu}$-primitive ideal of $R$.  

Proposition 2.11. Let $\nu \in \{0, 1, 2\}$. An ideal $P$ of $R$ is a right $g_{\nu}$-primitive ideal of $R$ if and only if $R/P$ is a right $g_{\nu}$-primitive near-ring.

Proof. Let $\nu \in \{0, 1, 2\}$ and $P$ be an ideal of $R$. Suppose that $P$ is a right $g_{\nu}$-primitive ideal of $R$. So, we get a right $g_{\nu}$-modular right ideal $M$ of $R$ such that $P$ is the largest ideal of $R$ contained in $M$. Now $M/P$ is a right $g_{\nu}$-modular right ideal of $R/P$. Since $P$ is the largest ideal of $R$ contained in $M$, the zero ideal of $R/P$ is the largest ideal of $R/P$ contained in $M/P$. Therefore, $R/P$ is a right $g_{\nu}$-primitive near-ring. Suppose now that $R/P$ is a right $g_{\nu}$-primitive near-ring. So, we get a right $g_{\nu}$-modular right ideal $M/P$ of $R/P$ such that the zero ideal of $R/P$ is the largest ideal of $R/P$ contained in $M/P$. Clearly, $M$ is a right $g_{\nu}$-modular right ideal of $R$. Since the zero ideal of $R/P$ is the largest ideal of $R/P$ contained in $M/P$, $P$ is the largest ideal of $R$ contained in $M$. Therefore, $P$ is a right $g_{\nu}$-primitive ideal of $R$. 

Proposition 2.12. $J'_g$ is the Hoehnke radical determined by the class of all right $g_{\nu}$-primitive near-rings, $\nu \in \{0, 1, 2\}$.

Theorem 2.13. Let $G$ be a right $R$-group of type-$g_{\nu}$ and $S$ be an invariant subnear-ring (and right ideal for $\nu = 0$) of $R$ with $GS \neq \{0\}$. Then $G$ is a right $S$-group of type-$g_{\nu}$, $\nu \in \{0, 1, 2\}$. 
Proof. If \( G \) is a right \( R \)-group of type-0 and \( S \) is an invariant subnear-ring and right ideal of \( R \) with \( GS \neq \{0\} \), then under the restriction of \( G \) to \( S \), by Theorem 3.2 of [9], \( G \) is a right \( S \)-group type-0. Also if \( G \) be a right \( R \)-group of type-\( \nu \) and \( S \) is an invariant subnear-ring of \( R \) with \( GS \neq \{0\} \), then under the restriction of \( G \) to \( S \), by Theorems 3.1 and 3.2 of [10], \( G \) is a right \( S \)-group type-\( \nu \), where \( \nu \in \{1, 2\} \). Therefore \( G \) is a right \( S \)-group of type-\( \nu \), \( \nu \in \{1, 2\} \). Let \( A \) be the set of generators of the right \( R \)-group \( G \) and \( P \) be the largest ideal of \( R \) contained in \( (0 : G)_R := \{r \in R \mid Gr = \{0\}\} \). A generator of the right \( R \)-group \( G \) is also a generator of the right \( S \)-group \( G \). From the proof of Theorem 3.10 of [9] (and Theorems 3.9 and 3.10 of [10] for \( \nu \in \{1, 2\} \)) as the extension of \( G \) from \( S \) to \( R \) coincides with the action of \( G \) on \( R \), it follows that a generator of the right \( S \)-group \( G \) is also a generator of the right \( R \)-group \( G \). So \( A \) is the set of generators of the right \( S \)-group \( G \). We have \( P = (0 : A) = \{r \in R \mid ar = 0 \text{ for all } a \in A\} \). Now \( P \cap S = (0 : A) \cap S = \{s \in S \mid As = \{0\}\} \). Let \( Q \) be the largest ideal of \( S \) contained in \( (0 : G)_S := \{s \in S \mid Gs = \{0\}\} = (0 : G) \cap S \). Clearly \( P \cap S \subseteq (0 : G)_S \). By the definition of \( Q \), \( P \cap S \subseteq Q \). Since \( AQ = \{0\} \), \( Q \subseteq P \). So \( Q \subseteq P \cap S \). Therefore \( Q = P \cap S \). Hence \( G \) is a right \( S \)-group of type-\( g_{\nu} \). \[\square\]

Proposition 2.14. A right \( R \)-group of type-\( g_{\nu} \) is an \( R \)-group of type-\( \nu(e) \), \( \nu \in \{0, 1, 2\} \).

Proof. Let \( G \) be a right \( R \)-group of type-\( g_{\nu} \), \( \nu \in \{0, 1, 2\} \). So \( G \) is a right \( R \)-group of type-\( \nu \). In view of Remark 3.9 of [11] \( G \) is a right \( R \)-group of type-\( \nu(e) \) if \( r, s \in R \) and \( gr = gs \) for all \( g \in G \), then \( r - s \in P \) where \( P \) is the largest ideal of \( R \) contained in \( (0 : G)_R := \{r \in R \mid Gr = \{0\}\} \). Let \( gr = gs \) for all \( g \in G \), \( r, s, h \in R \) and \( P \) be the largest ideal of \( R \) contained in \( (0 : G) \). Let \( A \) be the set of all generators of the right \( R \)-group \( G \). Now \( ar = as \) for all \( a \in A \). Since each \( a \in A \) is distributive, \( a(r - s) = 0 \) for all \( a \in A \). Therefore \( r - s \in P \) as \( P = (0 : A) \). Hence \( G \) is a right \( R \)-group of type-\( \nu(e) \). \[\square\]

Remark 2.15. If \( G \) is a right \( R \)-group of type-\( \nu(e) \), then by Proposition 3.12 of [11], \( (0 : G) := \{r \in R \mid Gr = \{0\}\} \) is an ideal of \( R \). Also, by Theorem 3.24 of [11], a right \( g_{\nu} \)-primitive near-ring is an equiprime near-ring.

Definition 2.16. Let \( G \) be a right \( R \)-group of type-\( g_{\nu} \), \( \nu \in \{0, 1, 2\} \). Then \( G \) is called faithful if \( (0 : G) = \{0\} \).

Theorem 2.17. Let \( G \) be a faithful right \( S \)-group of type-\( g_{\nu} \) and \( S \) be an essential left invariant ideal of \( R \). Then \( G \) is a faithful right \( R \)-group of type-\( g_{\nu} \), \( \nu \in \{0, 1, 2\} \).

Proof. Let \( h_0 \) be a generator of the right \( S \)-group \( G \). From the proof of Theorem 3.10 of [9], for \( h \in H, r \in R \) the operation defined by \( hr := h_0(sr) \) if \( h = h_0s, s \in S \), makes \( G \) a right \( R \)-group and is an extension the action of \( G \) on \( S \) to \( R \). Moreover, Theorem 3.10 of [9] and Theorems 3.9 and 3.10 of [10], \( G \) is a right \( R \)-group of type-\( \nu \), for \( \nu \in \{1, 2\} \). Since \( G \) is a right \( R \)-group of type-\( \nu(e) \), by Theorem 3.33 of [11] and Theorem of [5], \( G \) is a faithful \( R \)-group of type-\( \nu(e) \). Let \( A \) be the set of
all generators of the right $S$-group $G$. Now $(0 : G)_S := \{s \in S \mid Gs = \{0\}\} = \{0\}$. We have $\{0\} = (0 : A)_S := \{s \in S \mid As = \{0\}\}$. Since $G$ is a faithful right $R$-group, $(0 : G)_R := \{r \in R \mid Gr = \{0\}\} = \{0\}$. From the proof of Theorem 3.10 of [9], it can be easily seen that a generator of the right $S$-group $G$ is also a generator of the right $R$-group $G$. So $A$ is the set of generators of the right $R$-group $G$. Suppose that $r \in (0 : A)$. Now $Ar = \{0\}$. So $\{0\} = (Ar)S = A(rS)$ and hence $rS = \{0\}$ as $rS \subseteq S$. Since $S$ is an ideal, $KS = \{0\}$ and $S$ is a prime near-ring, we have $K = \{0\}$, where $K$ is the ideal of $R$ generated by $r$. Therefore $r = 0$ and hence $(0 : A)_R = \{0\}$. So $G$ is a faithful right $R$-group of type-$g$.

From the above theorem we have:

**Theorem 2.18.** The class of all right $g_\nu$-primitive near-rings is closed under essential left invariant extensions, $\nu \in \{0, 1, 2\}$.

In view of Theorem 1.1, we have the following:

**Theorem 2.19.** Let $\nu \in \{0, 1, 2\}$. Let $E$ be the class of all right $g_\nu$-primitive near-rings and $UE$ be the upper radical class determined by $E$. Then $UE$ is a c-hereditary Kurosh-Amitsur radical class in the variety of all near-rings with hereditary semisimple class $SUE = E$. So, $J^r_{g_\nu}$ is a Kurosh-Amitsur radical in the class of all near-rings and for any ideal $I$ of $R$, $J^r_{g_\nu}(I) \subseteq J^r_{g_\nu}(R) \cap I$ with equality, if $I$ is left invariant.

**Theorem 2.20.** $J^r_{g_\nu}$ is an ideal-hereditary Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.

**Theorem 2.21.** $J^r_{g_\nu}$ is a special radical in the class of all near-rings.

3. Examples

In this section we present some examples of near-rings $R$ and their right $R$-groups to show that the present right Jacobson radicals are distinct from the known right Jacobson radicals of near-rings. Now we present an example of a right $R$-group of type-$\nu(e)$ which is not of type-$g_\nu$, $\nu \in \{0, 1, 2\}$.

**Proposition 3.1.** If $G$ be a finite group and $G$ has a subgroup of index two, then $M_0(G)$ is a right 2(e)-primitive near-ring.

**Proof.** Let $G$ be a finite group and $H$ be a subgroup of $G$ of index 2. So $H$ is a normal subgroup of $G$. Let $R = M_0(G)$. Then $R/K$ is a right $R$-group of type-$2(e)$, where $K = (H : G) = \{r \in R \mid r(g) \in H, \text{ for all } g \in G\}$. To show this we consider the two distinct cosets $H$ and $H + a$ of $H$ in $G$. Now $G = H \cup H + a$, $H$ and $H + a$ are disjoint sets. $K$ is a right ideal of $R$ which is right modular by the identity element of $R$. So $R/K$ is a monogenic right $R$-group. Now we show that $R/K$ is a right $R$-group of type-2. Let $0 \neq r + K \in R/K$. $(r + K)R = R/K$ if and only if
Example 3.2. Let $G$ be the non-abelian group of order 6. Let $N$ be the subgroup of $G$ of order 3. By Proposition 3.1, $M_0(G)/\langle N : G \rangle$ is a right $M_0(G)$-group of type-2(e) and $M_0(G)$ is a right 2(e)-primitive near-ring. Since $N$ is the maximal (normal) subgroup of $G$, $(N : G)$ is the only proper (maximal) right ideal of $M_0(G)$. So a right $M_0(G)$-group of type-0 is $M_0(G)$-isomorphic to $M_0(G)/\langle N : G \rangle$. Therefore, if $f + \langle N : G \rangle$ is a generator of the right $M_0(G)$-group $M_0(G)/\langle N : G \rangle$, then $\langle 0 : f + \langle N : G \rangle \rangle = (N : G) \neq 0$. Note that as $M_0(G)$ is a simple near-ring, $\{0\}$ is the largest ideal of $M_0(G)$ contained in $\langle 0 : M_0(G)/\langle N : G \rangle \rangle$. Hence $M_0(G)/\langle N : G \rangle$ is a right $M_0(G)$-group of type-$\nu$, $\nu \in \{0, 1, 2\}$.

Now we present another example to show that there are right $R$-groups of type-$\nu(e)$ which are not of type-$\nu$. The following example was considered in [3] and [11].

Example 3.3. Consider $G := \mathbb{Z}_8$, the group of integers under addition modulo 8. Now $T : G \to G$ defined by $T(g) = 5g$ for all $g \in G$ is an automorphism of $G$. $T$ fixes 0, 2, 4, 6 and maps 1 to 5, 3 to 7 and 7 to 3. Now $A := \{I, T\}$ is an automorphism group of $G$ and $\{0\}, \{2\}, \{4\}, \{6\}, \{1, 5\}$ and $\{3, 7\}$ are the orbits. Let $R$ be the centralizer near-ring $M_4(G)$, the near-ring of all self maps of $G$ which fix 0 and commute with $T$. An element of $R$ is completely determined by its action on $\{0, 1, 2, 3, 4, 6\}$. Note that for $f \in R$ we have $f(2), f(4), f(6)$ are arbitrary in $2G$ and $f(1), f(3)$ are arbitrary in $G$. In [3] shown that $I := \langle 0 : 2G \rangle = \{f \in R \mid f(h) = 0, \forall h \in 2G\}$ is the only non-trivial ideal of $R$. Let $K := \langle 2G : G \rangle = \{t \in R \mid t(G) \subseteq 2G\} \neq R$. Let $t_0$ be the identity element in $R$. Now $t_0 + K$ is a generator of the right $R$-group $R/K$. Let $h \in R - K$. We show now that $(h + K)R = R/K$. Since $h \not\in K$, there is an $a \in G - 2G$ such that $b := h(a) \not\in 2G$. We construct an element $s \in R$ such that $s(1) = s(3) = a$, so that $s(5) = s(7) = a + 4$, and $s = 0$ on $2G$. Since $s$ maps $G - 2G$ to $G - 2G$, we get that $t_0 - hs \in K$ and hence $(h + K)s = t_0 + K$. So $(h + K)R = R/K$. Therefore, $R/K$ is a right $R$-group of type-$\nu$. Moreover, $(R/K)I \neq \{K\}$. Therefore, $\{0\}$ is the largest ideal of $R$ contained in $\langle K : R \rangle$ and hence $J^\nu(R) = \{0\}$. Consider $s_1, s_1 \in R$, where $s_1(1) = 1$ and 0 on $G \setminus \{1, 5\}$ and
Proposition 3.4. Let R be the near-ring considered in the Example 3.3 and let K be a right ideal of R. Then $H := \{f(g) \mid f \in K, g \in G\} \subseteq G$ and $H_2 := \{f(g) \mid f \in K, g \in 2G\} \subseteq 2G$ are (normal) subgroups of G and 2G respectively.

Proof. We show that $H_1$ is a subgroup of G. Since $0 \in H_1$, $H_1$ is non-empty. Let $h_1, h_2 \in H_1$. We get $f_1, f_2 \in K$ and $g_1, g_2 \in G$ such that $h_1 = f_1(g_1)$ and $h_2 = f_2(g_2)$. Clearly, $-h_1 = (-f_1)(g_1) \in H_1$ as $-f_1 \in K$. Suppose that one of the $g_i$ is in G - 2G. With out loss of generality, suppose that $g_1 \in G$ and $h_1, h_2 \in 2G$. If $g_1 = 0$, then $h_1 - h_2 = -h_2 \in H_1$. Suppose that $g_1 \neq 0$. So, we get $f_4 \in R$ such that $f_4(g_1) = g_2$. Now $f_1 - f_2 f_4 \in K$ and $h_1 - h_2 = (f_1 - f_2 f_4)(g_1) \in H_1$. Therefore, $H_1$ is a subgroup of G. Similarly, we get that $H_2$ is a subgroup of 2G.

Proposition 3.5. Let R, K, $H_1$ and $H_2$ be as defined in Proposition 3.4. If $H_1 = G$ and $H_2 = 2G$, then $K = R$.

Proof. Suppose that $H_1 = G$ and $H_2 = 2G$. We have 1, 3 $\in H_1$. So, for $i \in \{1, 3\}$, we get $f_i \in K$ such that $f_i(g_i) = i$, where $g_i \in \{1, 3, 5, 7\} = \text{G - 2G}$. For $i = 1, 3$ we also get $m_i \in R$ such that $m_i(i) = g_i, m_1(i + 4) = g_i + 4$ and $m_1 = 0$ on G - $\{i, i + 4\}$. Now $m_1, m_2 \in K, i = 1, 3$. Clearly, $f_1 m_1 + f_3 m_3$ fixes all the elements of G - 2G and maps all the elements of 2G to 0. We have 2, 4, 6 $\in H_2 = 2G = \{0, 2, 4, 6\}$. For $i = 2, 4, 6$ we get $f_i \in K$ such that $f_i(g_i) = i, g_i \in 2G$. So, for $i = 2, 4, 6$ we get $m_1 \in R$ such that $m_i(i) = g_i, m_i = 0$ on G - $\{i\}$. Now $f_i m_i \in K, i = 2, 4, 6$. $f_2 m_2 + f_4 m_4 + f_6 m_6 \in K$. Hence, $K = R$.

Proposition 3.6. Let R, K, $H_1$ and $H_2$ be as defined in Proposition 3.4. If K is a maximal right ideal of R, then K = (2G : G) = $\{f \in R \mid f(G) \subseteq 2G\}$ or (4G : 2G) = $\{f \in R \mid f(2G) \subseteq 4G\}$

Proof. Suppose that K is a maximal right ideal of R. Clearly, if H and T are (normal) subgroups of G and 2G respectively, then H : G = $\{f \in R \mid f(G) \subseteq H\}$ and T : 2G = $\{f \in R \mid f(2G) \subseteq T\}$ are right ideals of R. Now 2G and 4G are the maximal (normal) subgroups of G and 2G respectively. We have K $\subseteq (H_1 : G)$ and K $\subseteq (H_2 : 2G)$. Since K is a maximal right ideal of R, by Proposition 3.5, either $H_1 \neq G$ or $H_2 \neq 2G$.

Case(i) Suppose that $H_2 \neq 2G$. Since K is a maximal right ideal of R and K $\subseteq (H_2 : 2G) \neq R$, we get that $H_2 = 4G$ and K = (4G : 2G).

Case(ii) Suppose that $H_1 \neq G$. Since K is a maximal right ideal of R and K $\subseteq (H_1 : G) \neq R$, we get that $H_1 = 2G$ and K = (2G : G).

Therefore, either K = (2G : G) or (4G : 2G).
**Proposition 3.7.** Let $R$ be the near-ring considered in Example 3.3. Let $U = (4G : 2G) = \{ f \in R \mid f(2G) \subseteq 4G \}$. Then $U$ is a maximal right ideal of $R$ and $R/U$ is a right $R$-group of type-2(e).

**Proof.** Clearly, $U$ is a right ideal of $R$. Consider the right $R$-group $R/U$. We prove that $R/U$ is a right $R$-group of type-2. Since $R$ has identity $1$, $I + U$ is a generator of the right $R$-group $R/U$ and hence $R/U$ is a monogenic right $R$-group. Let $0 \neq f + U \in R/U$. So, $f \notin U$. We get $0 \neq a \in 2G$ such that $b := f(a) \notin 4G$. So, $2G = \{0, b, 2b, 3b\}$ as 2 and 6 are generators of 2G. Construct $r \in R$ by $r(b) = a$, $r(2b) = 0$, $r(3b) = a$ and $r = 0$ on $G - \{0, 1, 3, 5, 7\}$. Now $(I - fr)(x) \in 4G$ for all $x \in 2G$. Therefore, $I - fr \in U$ and hence $(f + U)r = I + U$. This shows that $(f + U)R = R/U$. So, $R/U$ is a right $R$-group of type-2. We know that $P := (0 : 2G)$ is the only non-trivial ideal of $R$. Therefore, $P$ is the largest ideal of $R$ contained in $U = (4G : 2G)$ and hence $P$ is the largest ideal of $R$ contained in $(0 : R/U) = (U : R) = \{ f \in R \mid Rf \subseteq U \}$. Let $0 \neq s + U \in R/U$ and $f, h \in R$. Suppose that $(s + U)rf = (s + U)r(h(a) = 0)$ for all $r \in R$. So, $srf - srh \in R$ for all $r \in R$. If possible, suppose that $f - h \notin P$. We get $0 \neq a \in 2G$ such that $(f - h)(a) = f(a) - h(a) \neq 0$ with $h(a) \neq 0$. Let $s(c) \notin \{0, 4\}$ for some $c \in 2G$. Choose $r \in R$ such that $r(f(a)) = 0$ and $r(h(a)) = c$. Now $(srf)(a) = 0$ and $(srh)(a) = s(c)$. So, $(srf - srh)(a) = 0 - s(c) \notin \{0, 4\}$, a contradiction to the fact that $srf - srh \in U$. Therefore, $f(a) = h(a)$ for all $a \in 2G$. Hence $f - h \in P$. So, $R/U$ is a right $R$-group of type-2(e). \qed

**Proposition 3.8.** Let $R$ be the near-ring considered in Example 3.3. Then $J_{\nu}^*(R) = \{0\}$ and $J_{\nu(e)}^*(R) = (0 : 2G) \neq \{0\}$.

**Proof.** We know that $\{0\}$ and $I := (0 : 2G) = \{ f \in R \mid f(2G) = \{0\} \}$ are the only proper ideals of $R$. Let $K_1 := (2G : G) = \{ f \in R \mid f(G) \subseteq 2G \}$ and $K_2 := (4G : 2G) = \{ f \in R \mid f(2G) \subseteq 4G \}$. By Proposition 3.6, a maximal right ideal of $R$ is either $K_1$ or $K_2$. So, a right $R$-group of type-0 is isomorphic to $R/K_1$ or $R/K_2$. By Example 3.3, $R/K_1$ is a right $R$-group of type-2 but not of type-2(e). Since $\{0\}$ is the largest ideal of $R$ contained in $K_1$, $\{0\}$ is a right $2$-primitive ideal of $R$ but not a right $2(e)$-primitive ideal of $R$. By Proposition 3.7, $R/K_2$ is a right $R$-group of type-2(e). Since $I = (0 : 2G)$ is the largest ideal of $R$ contained in $K_2$, $I$ is a right $2(e)$-primitive ideal of $R$. Therefore, $J_{\nu}^*(R) = \{0\}$ and $J_{\nu(e)}^*(R) = (0 : 2G)$. \qed

**Proposition 3.9.** Let $R$ be the near-ring considered in Example 3.3. Then $J_{\nu}^*(R) = R$, $\nu \in \{0, 1, 2\}$.

**Proof.** Let $R$ be the near-ring considered in the Example 3.3 and $K = (2G : G)$, $U = (4G : 2G)$. As seen above $K$, $U$ are the only maximal right ideals of $R$ and $R/K$ is a right $R$-group of type-2 but not of type-2(e), where as $R/U$ is a right $R$-group of type-2(e). If $f + K$ is a generator of the right $R$-group $R/K$, then the maximal right ideal $(0 : f + K)$ must be either $K$ or $U$. Since $0(K) = 2^{10} \neq 2^9 = 0(U)$, and $R/(0 : f + K)$ is right $R$-isomorphic $R/K$, $(0 : f + K) = K$. Hence $R/K$ is not a right $R$-group of type-$g_\nu$, as $\{0\}$, $(0 : 2G)$ and $R$ are the only ideals of $R$. By a similar argument we get that $R/U$ is not a right $R$-group of type-$g_\nu$. So $J_{\nu}^*(R) = R$. \qed
4. $J_{g_\nu}^*$-semisimple Near-rings, $\nu \in \{0, 1, 2\}$

In this section we present structure theorems for $J_{g_\nu}^*$-semisimple near-rings.

**Proposition 4.1.** Let $R (\neq \{0\})$ be a $J_{g_\nu}^*$-semisimple near-ring satisfying DCC on right ideals of $R$, $\nu \in \{0, 1, 2\}$. Then $R$ is a finite direct sum of minimal right ideals which are right $R$-groups of type-$g_\nu$.

*Proof.* Let $P_i, i \in I$ be the collection of right $g_\nu$-primitive ideals of $R$. Since $R$ is a $J_{g_\nu}^*$-semisimple near-ring, $\bigcap\{P_i \mid i \in I\} = \{0\}$. We get a right $R$-group $G_i$ of type-$g_\nu$ such that $P_i = (0 : G_i) := \{r \in R \mid G_ir = \{0\}\}, i \in I$. Let $A_i$ be the set of generators of $G_i, i \in I$. Now $P_i = (0 : A_i) := \{r \in R \mid A_ir = \{0\}\}$. Note that for each $a \in A_i, (0 : a) := \{r \in R \mid ar = 0\}$ is a right $g_\nu$-modular right ideal of $R$ and the right $R$-group $R/(0 : a)$ is right $R$-isomorphic to $G_i, i \in I$. Since each $P_i$ is an intersection of right $g_\nu$-modular right ideals of $R$ and $\bigcap\{P_i \mid i \in I\} = \{0\}$, the intersection of all right $g_\nu$-modular right ideal of $R$ is zero. We get a finite number of right $g_\nu$-modular right ideals $K_1, K_2, ..., K_n$ of $R$ such that $\bigcap\{K_j \mid j = 1, 2, ..., n\} = \{0\}$. Let $T_i := K_i \cap K_2 \cap ... \cap K_{i-1} \cap K_{i+1} \cap ... \cap K_n, i = 1, 2, ..., n$. We may assume that $T_i \neq \{0\}$ for all $i = 1, 2, ..., n$. Now by Proposition 3.12[(2)] of [8], $R = T_1 \oplus T_2 \oplus ... \oplus T_n$, a direct sum of minimal right ideals $T_i$ of $R$ which are right $R$-groups of type-$g_\nu$. \qed

In [8](Definition 3.5), if $R$ is a direct sum of n minimal right ideals of $R$, then the dimension of $R$ is defined as $n$ and is denoted by $\text{dim } R$.

**Definition 4.2.** A distributive idempotent $e$ of $R$ is called right $g_\nu$-primitive if $eR$ is a right $R$-group of type-$g_\nu$, $\nu \in \{0, 1, 2\}$.

**Theorem 4.3.** Let $R$ be a right $g_\nu$-primitive near-ring satisfying DCC on right ideals of $R$, $\nu \in \{0, 1, 2\}$. Then $R$ is a simple near-ring with identity and $R$ has a subnear-ring which is isomorphic to the matrix near-ring $M_n(S)$, where $S = eRe$, $e$ is a right $g_\nu$-primitive idempotent and $n = \text{dim } R$. If, in addition, $R$ is distributively generated, then $R$ is isomorphic to $M_n(S)$.

*Proof.* $R$ satisfies the hypothesis of Theorem 4.3 of [8] and hence the conclusion follows from it. \qed

**Theorem 4.4.** Let $R$ be a finite right $g_2$-primitive near-ring and $eRe$ be a non-ring. Then $R$ is (isomorphic to) the matrix near-ring $M_n(F)$, where $n = \text{dim } R$, $F := eRe$ is a near-field and $e$ is a right $g_2$-primitive idempotent in $R$.

*Proof.* Proof follows from Theorem 4.16 of [8]. \qed

**Theorem 4.5.** Let $R (\neq \{0\})$ be a $J_{g_\nu}^*$-semisimple near-rings satisfying DCC on right ideals of $R$, $\nu \in \{0, 1, 2\}$. Then $R$ is a direct sum of minimal ideals which are simple right $g_\nu$-primitive near-rings with identity.
Proof. Let \( P_i, i \in I \) be the collection of right \( g_{nu} \)-primitive ideals of \( R, \nu \in \{0,1,2\} \). Now \( \cap \{P_i \mid i \in I\} = \{0\} \). Since \( R \) has DCC on right ideals of \( R \), we get a finite number of right \( g_{nu} \)-primitive ideals of \( P_1, P_2, \ldots, P_n \) of \( R \) such that \( P_1 \cap P_2 \cap \cdots \cap P_n = \{0\} \). We may assume that \( K_j := P_1 \cap P_2 \cap \cdots \cap P_{j-1} \cap P_{j+1} \cap \cdots \cap P_n \neq \{0\}, j = 1,2,\ldots,n \). By Theorem 4.3, \( R/P_i \) is a simple near-ring with identity as \( R/P_i \) is a right \( g_{nu} \)-primitive near-ring with DCC on right ideals. Now by Theorem 2.50 of Pilz [4], \( R = K_1 \oplus K_2 \oplus \cdots \oplus K_n \), \( K_i \) are minimal ideals of \( R \) and are simple right \( g_{nu} \)-primitive near-rings with identity.

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