Forced Oscillation Criteria for Nonlinear Hyperbolic Equations via Riccati Method

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Abstract. In this paper, we consider the nonlinear hyperbolic equations with forcing term. Some sufficient conditions for the oscillation are derived by using integral averaging method and a generalized Riccati technique.

1. Introduction

We shall provide oscillation results of solution of the hyperbolic equation

\[
\frac{\partial}{\partial t} \left( r(t) \frac{\partial}{\partial t} u(x, t) \right) + p(t) \frac{\partial}{\partial t} u(x, t) - a(t) \Delta u(x, t) - \sum_{i=1}^{k} b_i(t) \Delta u(x, \tau_i(t)) + \sum_{i=1}^{m} q_i(x, t) \phi_i(u(x, \sigma_i(t))) = f(x, t), \quad (x, t) \in \Omega \equiv G \times (0, \infty),
\]

where $\Delta$ is the Laplacian in $\mathbb{R}^n$ and $G$ is a bounded domain of $\mathbb{R}^n$ with piecewise smooth boundary $\partial G$. Recently, the oscillation of solution of hyperbolic equation via Riccati method has been investigated by many authors, see for example [2], [6], [7]. In particular, Shoukaku [6] established the oscillation results of solution of the equation (E). In the work of [6], restriction is imposed on forcing term $f(x, t)$ to be oscillatory function.

Gaef and Spikes [3], Wong and Agarwal [8], Li [4] and Agawal, et al [1] obtained several oscillation results for second order nonlinear differential equations. Their results used the different assumption of forcing term from the work of [6].

Motivated by the work of [1], in this paper we will obtain the oscillation results
of the hyperbolic equation (E), and remove the assumption of the forcing term such as the work [6].

We assume throughout this paper that:

\( (H1) \) \( r(t) \in C^1([0, \infty); (0, \infty)), p(t) \in C([0, \infty); \mathbb{R}), \)
\( a(t), b_i(t) \in C([0, \infty); [0, \infty)) (i = 1, 2, \ldots, k), \)
\( q_i(x, t) \in C(\overline{\Omega}; [0, \infty)) (i = 1, 2, \ldots, m), f(x, t) \in C(\overline{\Omega}; \mathbb{R}); \)

\( (H2) \) \( \tau_i(t) \in C([0, \infty); \mathbb{R}), \lim_{t \to \infty} \tau_i(t) = \infty (i = 1, 2, \ldots, k), \)
\( \sigma_i(t) \in C^1([0, \infty); \mathbb{R}), \lim_{t \to \infty} \sigma_i(t) = \infty (i = 1, 2, \ldots, m); \)

\( (H3) \) \( \varphi_i(s) \in C^1(\mathbb{R}; \mathbb{R}) (i = 1, 2, \ldots, m) \) are convex on \([0, \infty), \) and \( \varphi_i(s) \geq 0 \) and \( \varphi_i(-s) = -\varphi_i(s) \) for \( s \geq 0. \)

We consider the following Dirichlet and Robin boundary conditions

\( (B1) \) \( u = \psi \) on \( \partial G \times [0, \infty), \)
\( (B2) \) \( \frac{\partial u}{\partial \nu} + \mu u = \tilde{\psi} \) on \( \partial G \times [0, \infty), \)

where \( \nu \) denotes the unit exterior normal vector to \( \partial G \) and \( \psi, \tilde{\psi} \in C(\partial G \times (0, \infty); \mathbb{R}), \)
\( \mu \in C(\partial G \times (0, \infty); [0, \infty)). \)

**Definition 1.** By a solution of Eq. (E) we mean a function \( u \in C^2(\overline{G} \times [t_-, \infty)) \cap C(\overline{G} \times [t_-, \infty)) \) which satisfies (E), where
\[
 t_-= \min \left\{ 0, \min_{1 \leq i \leq k} \left\{ \inf_{t \geq 0} \tau_i(t) \right\} \right\}, \quad \tilde{t}_-= \min \left\{ 0, \min_{1 \leq i \leq m} \left\{ \inf_{t \geq 0} \sigma_i(t) \right\} \right\}.
\]

**Definition 2.** A solution \( u \) of Eq. (E) is said to be oscillatory in \( \Omega \) if \( u \) has a zero in \( G \times (t, \infty) \) for any \( t > 0. \) That is, there exists a point \( t_1 > t \) such that \( u(x, t_1) = 0. \)

**Definition 3.** We say that functions \( H_1, H_2 \) belong to a function class \( \mathbb{H}, \) denoted by \( H_1, H_2 \in \mathbb{H}, \) if \( H_1, H_2 \in C(D; [0, \infty)) \) satisfy
\[
 H_i(t, t) = 0, \quad H_i(t, s) > 0 (i = 1, 2) \quad \text{for} \ t > s,
\]
where \( D = \{(t, s) : 0 < s \leq t < \infty\}. \) Moreover, the partial derivatives \( \partial H_1 / \partial t \) and \( \partial H_2 / \partial s \) exist on \( D \) such that
\[
 \frac{\partial H_1}{\partial t}(s, t) = h_1(s, t)H_1(s, t) \quad \text{and} \quad \frac{\partial H_2}{\partial s}(t, s) = -h_2(t, s)H_2(t, s),
\]
where \( h_1, h_2 \in C_{loc}(D; \mathbb{R}). \)

**2. Reduction to One-Dimensional Problems**

In this section we reduce the multi-dimensional oscillation problems for (E) to
one-dimensional oscillation problems. It is known that the first eigenvalue \( \lambda_1 \) of the eigenvalue problem

\[
-\Delta w = \lambda w \quad \text{in} \quad G,
\]

\[
w = 0 \quad \text{on} \quad \partial G
\]

is positive, and the corresponding eigenfunction \( \Phi(x) \) can be chosen so that \( \Phi(x) > 0 \) in \( G \). Now we define

\[
q_i(t) = \min_{x \in G} q_i(x, t).
\]

The following notation will be used:

\[
U(t) = K_i \int_G u(x, t)\Phi(x)dx,
\]

\[
\tilde{U}(t) = \frac{1}{|G|} \int_{\partial G} u(x, t)dx,
\]

\[
F(t) = K_i \int_G f(x, t)\Phi(x)dx,
\]

\[
\tilde{F}(t) = \frac{1}{|G|} \int_{\partial G} f(x, t)dx,
\]

\[
\Psi(t) = K_i \int_{\partial G} \psi \frac{\partial \Phi}{\partial \nu}(x)dS,
\]

\[
\tilde{\Psi}(t) = \frac{1}{|G|} \int_{\partial G} \tilde{\psi}dS,
\]

where \( K_i = (\int_G \Phi(x)dx)^{-1} \) and \( |G| = \int_G dx \).

**Theorem 1.** If every eventually positive solution \( y(t) \) of the functional differential inequalities

\[
(r(t)y'(t))' + p(t)y'(t) + \sum_{i=1}^{m} q_i(t)\phi_i(y(\sigma_i(t))) \leq \pm G(t)
\]

satisfies \( \liminf_{t \to \infty} y(t) = 0 \), then every solution \( u(x, t) \) of the problem (E), (B1) is oscillatory in \( \Omega \) or satisfies

\[
\liminf_{t \to \infty} |U(t)| = 0,
\]

where

\[
G(t) = F(t) - a(t)\Psi(t) - \sum_{i=1}^{k} b_i(\tau_i(t))\Psi(\tau_i(t)).
\]

**Proof.** Suppose to the contrary that there is a nonoscillatory solution \( u \) of the problem (E), (B1) which does not satisfy (2). Without loss of generality we may assume that \( u(x, t) > 0 \) in \( G \times [t_0, \infty) \) for some \( t_0 > 0 \) because the case where \( u(x, t) < 0 \) can be treated similarly. Since (H2) holds, we see that \( u(x, \tau_i(t)) > 0 \) \( (i = 1, 2, \ldots, k) \) and \( u(x, \sigma_i(t)) > 0 \) \( (i = 1, 2, \ldots, m) \) in \( G \times [t_1, \infty) \) for some
\( t_1 \geq t_0 \). Multiplying (E) by \( K_\Phi \Phi(x) \) and integrating over \( G \), we obtain

\[(3) \quad \left( r(t)U'(t) \right)' + p(t)U'(t) - a(t)K_\Phi \int_G \Delta u(x, t) \Phi(x) dx - \sum_{i=1}^{k} b_i(t)K_\Phi \int_G \Delta u(x, \tau_i(t)) \Phi(x) dx + \sum_{i=1}^{m} K_\Phi \int_G q_i(x, t) \varphi_i(u(x, \sigma_i(t))) \Phi(x) dx = F(t), \quad t \geq t_1. \]

From Green’s formula it follows that

\[(4) \quad K_\Phi \int_G \Delta u(x, t) \Phi(x) dx \leq -\Psi(t), \quad t \geq t_1, \]
\[(5) \quad K_\Phi \int_G \Delta u(x, \tau_i(t)) \Phi(x) dx \leq -\Psi(\tau_i(t)), \quad t \geq t_1. \]

An application of Jensen’s inequality shows that

\[(6) \quad \sum_{i=1}^{m} K_\Phi \int_G q_i(x, t) \varphi_i(u(x, \sigma_i(t))) \Phi(x) dx \geq \sum_{i=1}^{m} q_i(t) \varphi_i(U(\sigma_i(t))), \quad t \geq t_1. \]

Combining (3)–(6) yields

\[ (r(t)U'(t))' + p(t)U'(t) + \sum_{i=1}^{m} q_i(t) \varphi_i(U(\sigma_i(t))) \leq \tilde{G}(t), \quad t \geq t_1. \]

Therefore \( U(t) \) is a positive solution of (1) which does not satisfy (2). This contradicts the hypothesis and completes the proof. \( \square \)

**Theorem 2.** If every eventually positive solution \( y(t) \) of the functional differential inequalities

\[(7) \quad (r(t)y'(t))' + p(t)y'(t) + \sum_{i=1}^{m} q_i(t) \varphi_i(y(\sigma_i(t))) \leq \pm \tilde{G}(t) \]

satisfies \( \lim \inf_{t \to \infty} y(t) = 0 \), then every solution \( u(x, t) \) of the problem (E), (B2) is oscillatory in \( \Omega \) or satisfies

\[(8) \quad \lim \inf_{t \to \infty} |\tilde{U}(t)| = 0, \]

where

\[ \tilde{G}(t) = \tilde{F}(t) + a(t)\tilde{\Psi}(t) + \sum_{i=1}^{k} b_i(\tau_i(t))\tilde{\Psi}(\tau_i(t)). \]
Proof. Suppose to the contrary that there is a nonoscillatory solution \( u \) of problem (E), (B2) which does not satisfy (8). Without loss of generality we may assume that \( u(x,t) > 0 \) in \( G \times [t_0, \infty) \) for some \( t_0 > 0 \). Since (H2) holds, we see that \( u(x, \tau_i(t)) > 0 \) (\( i = 1, 2, \ldots, k \)) and \( u(x, \sigma_i(t)) > 0 \) (\( i = 1, 2, \ldots, m \)) in \( G \times [t_1, \infty) \) for some \( t_1 \geq t_0 \). Dividing (E) by \( |G| \) and integrating over \( G \), we obtain

\[
(r(t)\tilde{U}'(t))' + p(t)\tilde{U}'(t) - \frac{a(t)}{|G|} \int_G \Delta u(x,t)dx - \sum_{i=1}^{k} \frac{b_i(t)}{|G|} \int_G \Delta u(x,\tau_i(t))dx + \frac{1}{|G|} \sum_{i=1}^{m} \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx = \tilde{F}(t), \quad t \geq t_1.
\]

It follows from Green’s formula that

\[
\frac{1}{|G|} \int_G \Delta u(x,t)dx \leq \tilde{\Psi}(t), \quad t \geq t_1,
\]

\[
\frac{1}{|G|} \int_G \Delta u(x,\tau_i(t))dx \leq \tilde{\Psi}(\tau_i(t)), \quad t \geq t_1.
\]

Applying Jensen’s inequality, we observe that

\[
\frac{1}{|G|} \sum_{i=1}^{m} \int_G q_i(x,t)\varphi_i(u(x,\sigma_i(t)))dx \geq \sum_{i=1}^{m} q_i(t)\varphi_i(U(\sigma_i(t))), \quad t \geq t_1.
\]

Combining (9)–(12) yields

\[
(r(t)\tilde{U}'(t))' + p(t)\tilde{U}'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(U(\sigma_i(t))) \leq \tilde{G}(t), \quad t \geq t_1.
\]

Hence, \( \tilde{U}(t) \) is a positive solution of (7) which does not satisfy (8). This contradicts the hypothesis and completes the proof. \( \square \)

3. Second Order Functional Differential Inequality

We obtain the sufficient conditions for every positive solution \( y(t) \) of the functional differential inequality

\[
(r(t)y'(t))' + p(t)y'(t) + \sum_{i=1}^{m} q_i(t)\varphi_i(y(\sigma_i(t))) \leq f(t)
\]

to satisfy \( \liminf_{t \to \infty} y(t) = 0 \), where \( f(t) \in C([0, \infty); \mathbb{R}) \). We assume the following hypotheses:

(H4) \( \varphi'_j(t) > 0, \varphi'_j(t) \) is nondecreasing for \( t > 0 \) and some \( j \in \{1, 2, \ldots, m\} \);
(H5) there exists a positive constant $\sigma$ such that

$$\sigma_j'(t) \geq \sigma \quad \text{and} \quad \sigma_j(t) \leq t;$$

(H6) there exists a positive constant $K$ such that

$$q_j(t) \geq K|f(t)|.$$

**Theorem 3.** If the Riccati inequalities for $i = 1, 2$

$$x'(t) + \frac{1}{2} p_i(t) x(t) \leq -q(t)$$

have no solution on $[T, \infty)$ for all large $T$, then eventually positive solution of (13) satisfies

$$\liminf_{t \to \infty} y(t) = 0,$$

where

$$p_1(t) = \tilde{K} e^{R(t)}, \quad p_2(t) = p_1(\sigma_j(t)), \quad R(t) = \log r(t) + \int_{t_0}^{t} \frac{p(s)}{r(s)} ds,$$

$$q(t) = \frac{e^{R(t)}}{r(t)} \{q_j(t) - K|f(t)|\}$$

for every positive constant $\tilde{K}$.

**Proof.** Suppose that $y(t)$ is an eventually positive solution of (13) on $[t_0, \infty)$ for some $t_0 > 0$, and $\liminf_{t \to \infty} y(t) > 0$. Hence, there exists $k_1 > 0$ such that $y(t) \geq k_1$, $t \geq t_1$ for some $t_1 \geq t_0$. It follows from (13) that

$$\left( e^{R(t)} y'(t) \right)' + q_j(t) e^{R(t)} \frac{e^{R(t)}}{r(t)} \varphi_j(y(\sigma_j(t))) \leq e^{R(t)} \frac{e^{R(t)}}{r(t)} f(t), \quad t \geq t_1.$$

Since $\varphi_j(y(\sigma_j(t))) > \varphi_j(k_1) \equiv K_1$, $t \geq t_2$ for some $t_2 \geq t_1$, we can see from (H6) that

$$\left( e^{R(t)} y'(t) \right)' \leq -e^{R(t)} \frac{e^{R(t)}}{r(t)} \{K_1 q_j(t) - |f(t)|\} \leq 0, \quad t \geq t_2.$$

Then we consider $y'(t) < 0$ or $y'(t) \geq 0$ for $t \geq t_2$.

**Case 1.** $y'(t) < 0$ for $t \geq t_2$. Setting

$$z(t) = \frac{e^{R(t)} y'(t)}{\varphi_j(y(t))},$$
then

\[ (17) \quad \dot{z}(t) = \frac{(e^{R(t)}y'(t))'}{\varphi_j(y(t))} - e^{R(t)}y'(t) \frac{y'(t)\varphi_j'(y(t))}{\varphi_j^2(y(t))} \]

\[ \leq -q_j(t) \frac{e^{R(t)}}{r(t)} \frac{\varphi_j(y(\sigma_j(t)))}{\varphi_j(y(t))} + \frac{e^{R(t)}|f(t)|}{r(t)\varphi_j(y(t))} - e^{-R(t)}\varphi_j'(y(t))z^2(t) \]

\[ \leq -q_j(t) \frac{e^{R(t)}}{r(t)} + \frac{e^{R(t)}|f(t)|}{r(t)\varphi_j(k_1)} - e^{-R(t)}\varphi_j'(k_1)z^2(t) \]

\[ \leq - \frac{e^{R(t)}}{r(t)} \left\{ q_j(t) - \frac{|f(t)|}{K_1} \right\} - e^{-R(t)}\varphi_j'(k_1)z^2(t) \]

which contradicts the fact that \( z(t) \) is a negative solution of (14).

Case 2. \( y'(t) \geq 0 \) for \( t \geq t_2 \). Since \( y(t) > 0 \), \( y'(t) \geq 0 \) eventually, we see that \( y(\sigma_j(t)) \geq k_1 \) for some \( k_1 > 0 \). Let

\[ w(t) = \frac{e^{R(t)}y'(t)}{\varphi_j(y(\sigma_j(t)))}. \]

By using \( e^{R(t)}y'(t) \) is nonincreasing, we have

\[ (18) \quad w'(t) = \frac{(e^{R(t)}y'(t))'}{\varphi_j(y(\sigma_j(t)))} - e^{R(t)}y'(t) \frac{\sigma_j'(t)y'(\sigma_j(t))\varphi_j'(y(\sigma_j(t)))}{\varphi_j^2(y(\sigma_j(t)))} \]

\[ \leq -q_j(t) \frac{e^{R(t)}}{r(t)} + \frac{e^{R(t)}|f(t)|}{r(t)\varphi_j(y(\sigma_j(t)))} \]

\[ -e^{-R(\sigma_j(t))}\varphi_j'(y(\sigma_j(t)))\sigma_j'(t)w^2(t) \]

\[ \leq - \frac{e^{R(t)}}{r(t)} \left\{ q_j(t) - \frac{|f(t)|}{\varphi_j(k_1)} \right\} \]

\[ -e^{-R(\sigma_j(t))}\varphi_j'(k_1)\sigma y^2(t), \quad t \geq t_2. \]

Therefore \( w(t) \) is a positive solution of (14). This contradicts the hypothesis and completes the proof.

Theorem 4. If for some \( T \geq 0 \) and for \( i = 1, 2 \), there exist \( H_1, H_2 \in \mathbb{R} \) and some \( c \in (a, b) \) such that \( T \leq a < b \) and

\[ (19) \quad \frac{1}{H_1(c, a)} \int_a^c H_1(s, a) \left\{ q(s) - \frac{1}{2}\lambda_1^2(s, a)p_1(s) \right\} \phi(s)ds \]

\[ + \frac{1}{H_2(b, c)} \int_c^b H_2(b, s) \left\{ q(s) - \frac{1}{2}\lambda_2^2(b, s)p_1(s) \right\} \phi(s)ds > 0, \]

then eventually positive solution of (13) satisfies \( \liminf_{t \to \infty} y(t) = 0 \), where \( \phi(t) \in C^1([T, \infty); (0, \infty)) \) and

\[ \lambda_1(s, t) = \frac{\phi'(s)}{\phi(s)} + h_1(s, t), \quad \lambda_2(t, s) = \frac{\phi'(s)}{\phi(s)} - h_2(t, s). \]
Proof. Suppose that \( y(t) \) is a positive solution of (13) on \([t_0, \infty)\) for some \( t_0 > 0 \), and \( \liminf_{t \to \infty} y(t) > 0 \). At first, we assume that \( y(t) > 0 \) on \((a, b)\) for \( a, b \geq t_0 \). Proceeding as the same proof of Theorem 3, we have the inequality (14). Multiplying (14) by \( H_2(t, s) \) and \( \phi(s) \), integrating over \([c, t]\) for \( t \in [c, b) \) and letting \( t \to b^+ \), we see easily that

\[
(20) \quad \frac{1}{H_2(b, c)} \int_c^b H_2(b, s) \left( q(s) - \frac{1}{2} \lambda_2^2(b, s)p_i(s) \right) \phi(s) ds \leq x(c)\phi(c).
\]

Similarly, multiplying (14) by \( H_1(s, t) \) and \( \phi(s) \), integrating over \([t, c]\) for \( t \in (a, b)\) and letting \( t \to a^+ \), we have

\[
(21) \quad \frac{1}{H_1(c, a)} \int_a^c H_1(s, a) \left( q(s) - \frac{1}{2} \lambda_1^2(s, a)p_i(s) \right) \phi(s) ds \leq -x(c)\phi(c).
\]

Adding (20) and (21), we can lead to the contradiction. Pick up a sequence \( \{T_i\} \subset [t_0, \infty) \) such that \( T_i \to \infty \) as \( i \to \infty \). By assumptions, for each \( i \in \mathbb{N} \), there exists \( a_i, b_i, c_i \in [0, \infty) \) such that \( T_i \leq a_i < c_i < b_i \), and (19) holds with \( a, b, c \) replaced by \( a_i, b_i, c_i \), respectively. From that, every nontrivial solution \( y(t) \) of (13) has no zero \( t_i \in (a_i, b_i) \). Noting that \( t_i > a_i \geq T_i, \) \( i \in \mathbb{N} \), we see that \( y(t) \) is a eventually positive solution of (13). This contradiction proves that Theorem 4 holds.  

\[\square\]

Theorem 5. For some functions \( H_1, H_2 \in \mathbb{H} \), each \( T \geq 0 \) and for \( i = 1, 2 \), if

\[
(22) \quad \limsup_{t \to \infty} \int_T^t H_1(s, T) \left( q(s) - \frac{1}{2} \lambda_1^2(s, T)p_i(s) \right) \phi(s) ds > 0
\]

and

\[
(23) \quad \limsup_{t \to \infty} \int_T^t H_2(t, s) \left( q(s) - \frac{1}{2} \lambda_2^2(t, s)p_i(s) \right) \phi(s) ds > 0,
\]

then eventually positive solution of (13) satisfies \( \liminf_{t \to \infty} y(t) = 0 \), where \( \phi(t) \in C^1([T_0, \infty); (0, \infty)) \) for some \( T_0 > 0 \).

Proof. For any \( T \geq t_0 \), let \( a = T \). In (22) we choose \( T = a \). Then there exists \( c > a \) such that for \( t \in [a, c] \)

\[
(24) \quad \int_a^c H_1(s, a) \left( q(s) - \frac{1}{2} \lambda_1^2(s, a)p_i(s) \right) \phi(s) ds > 0
\]

(cf. [9, Theorem 8.8.5]). In (23) we choose \( T = c \). Then there exists \( b > c \) such that for \( t \in [c, b) \)

\[
(25) \quad \int_c^b H_2(b, s) \left( q(s) - \frac{1}{2} \lambda_2^2(b, s)p_i(s) \right) \phi(s) ds > 0.
\]
Combining (22) and (23) we obtain (19). The conclusion come from Theorem 4, and the proof is completed.

4. Oscillation Criteria for Eq. (E)

4.1. Oscillation results by Riccati inequality

We are going to use the following lemma which is due to Usami [5].

Lemma. If there exists a function \( \phi(t) \in C^1([T_0, \infty); (0, \infty)) \) such that
\[
\int_{T_1}^\infty \left( \frac{\bar{p}(t) |\phi'(t)|^\beta}{\phi(t)} \right)^{\frac{1}{\beta-1}} dt < \infty, \quad \int_{T_1}^\infty \frac{1}{\bar{p}(t)(\phi(t))^{\beta-1}} dt = \infty,
\]
\[
\int_{T_1}^\infty \phi(t) \bar{q}(t) dt = \infty
\]
for some \( T_1 \geq T_0 \), then the Riccati inequality
\[
x'(t) + \frac{1}{\beta \bar{p}(t)} |x(t)|^\beta \leq -\bar{q}(t),
\]
where \( \beta > 1 \), \( \bar{p}(t) \in C([T_0, \infty); (0, \infty)) \) and \( \bar{q}(t) \in C([T_0, \infty); \mathbb{R}) \), has no solution on \([T, \infty)\) for all large \( T \).

Combining Theorems 1-3, we obtain following theorems.

Theorem 6. Assume that (H1)–(H5) hold, and that
(H7) there exists a positive constant \( K \) such that
\[
q_j(t) \geq K|\bar{G}(t)|.
\] If for \( i = 1, 2 \),
\[
\int_{T_1}^\infty \left( \frac{p_i(t) |\phi'(t)|^2}{\phi(t)} \right) dt < \infty, \quad \int_{T_1}^\infty \frac{1}{p_i(t)\phi(t)} dt = \infty,
\]
\[
\int_{T_1}^\infty \phi(t)Q(t) dt = \infty,
\]
then every solution \( u(x, t) \) of (E), (B1) is oscillatory in \( \Omega \) or satisfies (2), where
\[
Q(t) = \frac{e^{R(t)}}{r(t)} \left\{ q_j(t) - K|\bar{G}(t)| \right\}.
\]

Theorem 7. Assume that (H1)–(H5) hold, and that
(H8) there exists a positive constant \( K \) such that
\[
q_j(t) \geq K|\tilde{G}(t)|.
\]
If for $i = 1, 2$,
\[
\int_{T_i}^{\infty} \left( \frac{p_i(t)\phi'(t)^2}{\phi(t)} \right) dt < \infty, \quad \int_{T_i}^{\infty} \frac{1}{p_i(t)\phi(t)} dt = \infty, \\
\int_{T_i}^{\infty} \phi(t)\tilde{Q}(t)dt = \infty,
\]
then every solution $u(x, t)$ of (E), (B2) is oscillatory in $\Omega$ or satisfies (8), where
\[
\tilde{Q}(t) = \frac{e^{R(t)}}{r(t)} \left\{ q_j(t) - K|\tilde{G}(t)| \right\}.
\]

**Example 1.** We consider the problem
\[
\begin{align*}
&\frac{\partial}{\partial t} \left( e^{t} \frac{\partial}{\partial t} u(x, t) \right) + e^{t} \frac{\partial}{\partial t} u(x, t) - \left( e^{t} + e^{\frac{t}{2}} \right) \Delta u(x, t) \\
&\quad + 2e^{t} u \left( x, t - \frac{\pi}{2} \right) = e^{\frac{t}{2}} \sin x \sin t, \quad (x, t) \in (0, \pi) \times (0, \infty), \\
&u(0, t) = u(\pi, t) = 0, \quad t > 0.
\end{align*}
\]
Here $n = k = m = 1$, $r(t) = e^{t}$, $p_1(t) = e^{2t}$, $p_2(t) = e^{2t-\pi/2}$, $q_1(x, t) = 2e^{t}$, $\sigma_1(t) = t - \pi/2$ and $f(x, t) = e^{t} \sin x \sin t$. It is easily verified that $\Phi(x) = \sin x$ and
\[
q_1(t) \equiv 2e^{t} \geq \frac{\pi}{4} |e^{\frac{t}{2}} \sin t| \equiv |\tilde{G}(t)|.
\]
By choosing $\phi(t) = e^{-3t}$, the conditions of Theorem 6 are satisfied. Therefore, we conclude that every solution $u$ of the problem (26), (27) is oscillatory in $(0, \pi) \times (0, \infty)$ or satisfies (2). For example, $u = \sin x \sin t$ is such a solution.

**Example 2.** Consider the problem
\[
\begin{align*}
&\frac{\partial}{\partial t} \left( e^{-t} \frac{\partial}{\partial t} u(x, t) \right) + 2e^{-t} \frac{\partial}{\partial t} u(x, t) - \Delta u(x, t) \\
&\quad + e^{\frac{t}{2}} u \left( x, \frac{t}{2} \right) = \left( e^{-t} + 1 \right) \cos x, \quad (x, t) \in \left( 0, \frac{\pi}{2} \right) \times (0, \infty), \\
&-u_x(0, t) = 0, \quad u_x \left( \frac{\pi}{2}, t \right) = -e^{-t}, \quad t > 0.
\end{align*}
\]
Here $n = k = m = 1$, $r(t) = e^{-t}$, $p_1(t) = e^{t}$, $p_2(t) = e^{t/2}$, $q_1(x, t) = 2e^{-t}$, $a(t) = 1$, $\sigma_1(t) = t/2$ and $f(x, t) = \left( e^{-t} + 1 \right) \cos x$. A simple calculation yields $\tilde{G}(t) = 2/\pi$ and
\[
q_1(t) \equiv e^{\frac{t}{2}} \geq \frac{\pi}{2} \equiv |\tilde{G}|.
\]
By choosing $\phi(t) = e^{-\frac{t}{2}}$ we note that the conditions of Theorem 7 holds. Therefore, every solution $u$ of the problem (28), (29) is oscillatory in $(0, \pi) \times (0, \infty)$ or satisfies
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4.2. Interval oscillation results

Combining Theorems 1–2 and 4, we have following theorems.

**Theorem 8.** Assume that (H1)–(H5) and that (H7) hold. If for some \( T \geq 0 \) and for \( i = 1, 2 \), there exist \( H_1, H_2 \in \mathbb{H} \) and some \( c \in (a, b) \) such that \( T \leq a < b \), (19) with \( q(s) \) replaced by \( Q(s) \), then every solution \( u(x, t) \) of (E), (B1) is oscillatory in \( \Omega \) or satisfies (2).

**Theorem 9.** Assume that (H1)–(H5) and (H8) hold. If for some \( T \geq 0 \) and for \( i = 1, 2 \), there exist \( H_1, H_2 \in \mathbb{H} \) and some \( c \in (a, b) \) such that \( T \leq a < b \), (19) with \( q(s) \) replaced by \( \tilde{Q}(s) \), then every solution \( u(x, t) \) of (E), (B2) is oscillatory in \( \Omega \) or satisfies (8).

Combining Theorems 1–2 and 5, we obtain two theorems.

**Theorem 10.** Assume that (H1)–(H5) and (H7) hold. For some functions \( H_1, H_2 \in \mathbb{H} \), some \( T \geq 0 \) and for \( i = 1, 2 \), if (22) and (23) with \( q(s) \) replaced by \( Q(s) \) hold, then every solution \( u(x, t) \) of (E), (B1) is oscillatory in \( \Omega \) or satisfies (2).

**Theorem 11.** Assume that (H1)–(H5) and (H8) hold. For some functions \( H_1, H_2 \in \mathbb{H} \), some \( T \geq 0 \) and for \( i = 1, 2 \), if (22) and (23) with \( q(s) \) replaced by \( \tilde{Q}(s) \) hold, then every solution \( u(x, t) \) of (E), (B2) is oscillatory in \( \Omega \) or satisfies (8).

**Remark.** Our results in this paper hold without the hypotheses (H5) and (H6), if condition \( \sigma_j(t) = t \) satisfied.

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**References**


