Structures of Pseudo Ideal and Pseudo Atom in a Pseudo $Q$-Algebra

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Abstract. As a generalization of $Q$-algebra, the notion of pseudo $Q$-algebra is introduced, and some of their properties are investigated. The notions of pseudo subalgebra, pseudo ideal, and pseudo atom in a pseudo $Q$-algebra are introduced. Characterizations of their properties are provided.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: a $BCK$-algebra and a $BCI$-algebra ([7,8]). It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. Q. P. Hu and X. Li [5,6] introduced a wide class of abstract algebra: a $BCH$-algebra. They have shown that the class of $BCI$-algebra is a proper subclass of the class of $BCH$-algebra. $BCK$-algebras have several connections with other areas of investigation, such as: lattice ordered groups, $MV$-algebras, Wajsberg algebras, and implicative commutative semigroups. J. M. Font et al. [3] have discussed Wajsberg algebras which are term-equivalent to $MV$-algebras. D. Mundici [13] proved $MV$-algebras are categorically equivalent to bounded commutative $BCK$-algebra, and J. Meng [11] proved that implicative com-

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In this paper, we introduce the notion of pseudo $Q$-algebra as a generalization of $Q$-algebra and investigate some of their properties. We also define the notions of pseudo subalgebra, pseudo ideal, and pseudo atom in a pseudo $Q$-algebra and provide characterizations of their properties in a pseudo $Q$-algebra.

2. Preliminaries

A $Q$-algebra ([14]) is a non-empty set $X$ with a constant $0$ and a binary operation “$*$” satisfying axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

For brevity we also call $X$ a $Q$-algebra. In $X$ we can define a binary relation “$\leq$” by $x \leq y$ if and only if $x * y = 0$.

In a $Q$-algebra $X$ the following property holds:

(IV) $(x * (x * y)) * y = 0$, for any $x, y \in X$.

A $BCK$-algebra is a $Q$-algebra $X$ satisfying the additional axioms:

(V) $((x * y) * (x * z)) * (z * y) = 0$,

(VI) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(VII) $0 * x = 0$,

for all $x, y, z \in X$.

Definition 2.1.([14]) Let $(X; *, 0)$ be a $Q$-algebra and $\emptyset \neq I \subset X$. $I$ is called a subalgebra of $X$ if

(S) $x * y \in I$ whenever $x \in I$ and $y \in I$.

$I$ is called an ideal of $X$ if it satisfies:

(Q$_0$) $0 \in I$, 

for all $x, y, z \in X$.
Structures of Pseudo Ideal and Pseudo Atom in a Pseudo $Q$-Algebra

$(Q_1)$ $x \ast y \in I$ and $y \in I$ imply $x \in I$.

A $Q$-algebra $X$ is called a $QS$-algebra ([1]) if it satisfies the following identity:

$$(x \ast y) \ast (x \ast z) = z \ast y$$

for any $x, y, z \in X$.

**Example 2.2.** ([1]) Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} := \{nz | z \in \mathbb{Z}\}$, where $n \in \mathbb{Z}$. Then $(\mathbb{Z}; \ast, 0)$ and $(n\mathbb{Z}; \ast, 0)$ are both $Q$-algebras and $QS$-algebras, where “$-$” is the usual subtraction of integers. Also, $(\mathbb{R}; \ast, 0)$ and $(\mathbb{C}; \ast, 0)$ are $Q$-algebras and $QS$-algebras where $\mathbb{R}$ is the set of all real numbers, $\mathbb{C}$ is the set of all complex numbers.

**Example 2.3.**

(1) Let $X = \{0, 1, 2\}$ be a set with the table as follows:

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<th>0</th>
<th>1</th>
<th>2</th>
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<tbody>
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Then $X$ is a $Q$-algebra, but not a $QS/BCI$-algebra, since $(2 \ast 0) \ast (2 \ast 1) = 2 \neq 1 = 1 \ast 0$.

(2) Let $X = \{0, 1, 2\}$ be a set with the table as follows:

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</table>

Then $X$ is both a $Q$-algebra and $QS$-algebra.

(3) Let $X = \{0, 1, 2\}$ be a set with the table as follows:

<table>
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Then $X$ is both a $Q$-algebra and $BCI$-algebra, but not a $QS$-algebra, since $(0 \ast 1) \ast (0 \ast 2) = 0 \neq 1 = 2 \ast 1$.

3. Pseudo Ideal

In the following, let $X$ denote a pseudo $Q$-algebra unless otherwise specified.

**Definition 3.1.** A pseudo $Q$-algebra is a non-empty set $X$ with a constant 0 and two binary operations “$*$” and “$\circ$” satisfying the following axioms: for any $x, y, z \in X$,

(P1) $x \ast x = x \circ x = 0$;

(P2) $x \ast 0 = x = x \circ 0$;
(P3) \((x * y) \circ z = (x \circ z) * y\).

For brevity, we also call \(X\) a pseudo \(BCH\)-algebra. In \(X\) we can define a binary operation \(\preceq\) by \(x \preceq y\) if and only if \(x * y = 0\) if and only if \(x \circ y = 0\). Note that if \((X; *, 0)\) is a \(Q\)-algebra, then letting \(x \circ y := x * y\), produces a pseudo \(Q\)-algebra \((X; *, \circ, 0)\). Hence every \(Q\)-algebra is a pseudo \(Q\)-algebra in a natural way.

**Definition 3.2.** Let \((X; *, \circ, 0)\) be a pseudo \(Q\)-algebra and let \(\emptyset \neq I \subseteq X\). \(I\) is called a pseudo subalgebra of \(X\) if \(x * y, x \circ y \in I\) whenever \(x, y \in I\). \(I\) is called a pseudo ideal of \(X\) if it satisfies
\begin{align*}
(\text{PI1}) & \quad 0 \in I; \\
(\text{PI2}) & \quad x * y, x \circ y \in I \text{ and } y \in I \text{ imply } x \in I \text{ for all } x, y \in X.
\end{align*}

**Example 3.3.** Let \(X := \{0, a, b, c\}\) be a set with the following Cayley tables:

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</table>

Then \((X; *, 0)\) and \((X; \circ, 0)\) are not \(Q\)-algebras, since \((b * a) * c = a \neq 0 = (b * c) * a\) and \((b \circ a) \circ c = 0 \neq c = (b \circ c) \circ a\). It is easy to check that \((X; *, \circ, 0)\) is a pseudo \(Q\)-algebra. Let \(I := \{0, a\}\). Then \(I\) is both a pseudo subalgebra of \(X\) and a pseudo ideal of \(X\). Let \(J := \{0, a, c\}\). Then \(J\) is a pseudo subalgebra of \(X\), but it is not a pseudo ideal of \(X\) since \(b \circ c = c \in J\) and \(b * c = a \in J\), but \(b \notin J\).

**Proposition 3.4.** Let \(I\) be a pseudo ideal of a pseudo \(Q\)-algebra \(X\). If \(x \in I\) and \(y \preceq x\), then \(y \in I\).

**Proof.** Assume that \(x \in I\) and \(y \preceq x\). Then \(y \ast x = 0\) and \(y \circ x = 0\). By (PI1) and (PI2), we have \(y \in I\). \(\square\)

**Proposition 3.5.** If \(X\) is a pseudo \(Q\)-algebra satisfying \(a \ast b = a \ast c\) and \(a \circ b = a \circ c\) for all \(a, b, c \in X\), then \(0 \ast b = 0 \ast c\) and \(0 \circ b = 0 \circ c\).

**Proof.** For any \(a, b, c \in X\), we have
\[
0 \ast b = (a \circ a) \ast b = (a \ast b) \circ a = (a \ast c) \circ a = (a \circ a) \ast c = 0 \ast c
\]
and
\[
0 \circ b = (a \ast a) \circ b = (a \ast b) \ast a = (a \circ c) \ast a = (a \ast a) \circ c = 0 \circ c.
\]
This concludes the proof. \(\square\)

**Proposition 3.6.** Let \((X; *, \circ, 0)\) be a pseudo \(Q\)-algebra. Then the following hold:
for all \(x, y, z \in X\).

(i) \(x \ast (x \circ y) \preceq y, x \circ (x \ast y) \preceq y\).
(ii) \( x * y \leq z \iff x \circ z \leq y \),

(iii) \( 0 *(x*y) = (0 \circ x) \circ (0 * y) \),

(iv) \( 0 \circ (x \circ y) = (0 * x) * (0 \circ y) \),

(v) \( 0 * x = 0 \circ x \).

**Proof.** (i) By (P1) and (P3), we obtain \([x * (x \circ y)] \circ y = (x \circ y) * (x \circ y) = 0 \) and \([x \circ (x \circ y)] * y = (x \circ y) \circ (x \circ y) = 0 \). Hence \( x * (x \circ y) \leq y \) and \( x \circ (x \circ y) \leq y \).

(ii) \( x * y \leq z \iff (x \circ y) \circ z = 0 \iff (x \circ z) * y = 0 \iff x \circ z \leq y \).

(iii) and (iv) For any \( x, y \in X \), by (P1) and (P3) we have

\[
(0 \circ x) \circ (0 * y) = [(x * y) * (x \circ y)] \circ (0 * y) \\
= [(x * y) \circ (x * y)] \circ (0 * y) \\
= [(x \circ x) * y] \circ (x \circ y) \circ (0 * y) \\
= [(0 * y) \circ (0 * y)] \circ (x \circ y) \\
= 0 * (x \circ y)
\]

and

\[
(0 * x) * (0 \circ y) = [(x \circ y) \circ (x \circ y)] * (0 \circ y) \\
= [(x \circ y) * x] * (x \circ y) \circ (0 \circ y) \\
= [(x * x) \circ y] \circ (x \circ y) \circ (0 \circ y) \\
= [(0 \circ y) \circ (0 \circ y)] \circ (x \circ y) \\
= 0 \circ (x \circ y).
\]

(v) For any \( x \in X \), by (P1) and (P3) we obtain \( 0 \circ x = (x \circ x) \circ x = (x * x) \circ x = 0 \circ x \).

**Theorem 3.7.** For any pseudo \( Q \)-algebra \( X \), the set

\[ K(X) := \{ x \in X \mid 0 \leq x \} \]

is a pseudo subalgebra of \( X \).

**Proof.** Let \( x, y \in K(X) \). Then \( 0 \leq x \) and \( 0 \leq y \). Hence \( 0 * x = 0 \circ x = 0 \) and \( 0 * y = 0 \circ y = 0 \). Since \( 0 * (x * y) = (0 \circ x) \circ (0 * y) = 0 \circ 0 = 0 \) and \( 0 \circ (x \circ y) = (0 \circ x) * (0 \circ y) = 0 * 0 = 0 \), we have \( x * y, x \circ y \in K(X) \). Thus \( K(X) \) is a pseudo subalgebra of \( X \).

**Theorem 3.8.** If \( I \) is a pseudo ideal of a pseudo \( Q \)-algebra \( X \), then

(i) \( \forall x, y, z \in X, x, y \in I, z * y \leq x \Rightarrow z \in I \),

(ii) \( \forall a, b, c \in X, a, b \in I, c \circ b \leq a \Rightarrow c \in I \).

**Proof.** (i) Suppose that \( I \) is a pseudo ideal of \( X \) and let \( x, y, z \in X \) be such that \( x, y \in I \) and \( z * y \leq x \). Then \( (z * y) \circ x = 0 \in I \). Since \( x \in I \) and \( I \) is a pseudo ideal
of $X$, we have $z \ast y \in I$. Since $y \in I$ and $I$ is a pseudo ideal of $X$, we obtain $z \in I$. Thus (i) is valid.

(ii) Let $a, b, c \in X$ be such that $a, b \in I$ and $c \circ b \preceq a$. Then $(c \circ b) \ast a = 0 \in I$ and so $c \circ b \in I$. Since $b \in I$ and $I$ is a pseudo ideal of $X$, we have $c \in I$. Thus (ii) is true.

**Theorem 3.9.** Let $I$ be a pseudo subalgebra of a pseudo $Q$-algebra $X$. Then $I$ is a pseudo ideal of $X$ if and only if $\forall x, y \in X, x \in I, y \in X - I \Rightarrow y \ast x \in X - I$ and $y \circ x \in X - I$.

**Proof.** Assume that $I$ is a pseudo ideal of $X$ and let $x, y \in X$ be such that $x \in I$ and $y \in X - I$. If $y \ast x \notin X - I$, then $y \ast x \notin I$. Since $I$ is a pseudo ideal of $X$, we have $y \in I$. This is a contradiction. Hence $y \ast x \in X - I$. Now if $y \circ x \notin X - I$, then $y \circ x \in I$ and so $y \in I$. This is a contradiction, and therefore $y \circ x \in X - I$.

Conversely, assume that $\forall x, y \in X, x \in I, y \in X - I \Rightarrow y \ast x \in X - I$ and $y \circ x \in X - I$. Since $I$ is a pseudo subalgebra, we have $0 \in I$. Let $x \in I, y \in X$ such that $y \ast x, y \circ x \in I$. If $y \notin I$, then $y \ast x, y \circ x \in X - I$ by assumption. This is a contradiction. Hence $y \in I$. Thus $I$ is a pseudo ideal of $X$.

**Proposition 3.10.** Let $A$ be a pseudo ideal of a pseudo $Q$-algebra $X$. If $B$ is a pseudo ideal of $A$, then $B$ is a pseudo ideal of $X$.

**Proof.** Since $B$ is a pseudo ideal of $A$, we have $0 \in B$. Let $y, x \ast y, x \circ y \in B$ for some $x \in X$. If $x \in A$, then $x \in B$ since $B$ is a pseudo ideal of $A$. If $x \in X - A$, then $y, x \ast y, x \circ y \in B \subseteq A$ and so $x \in A$ because $A$ is a pseudo ideal of $X$. Thus $x \in B$ since $B$ is a pseudo ideal of $A$. This completes the proof.

**Proposition 3.11.** Let $I$ be a pseudo ideal of a pseudo $Q$-algebra $X$. Then

$$(\forall x \in X)(x \in I \Rightarrow 0 \ast (0 \circ x) \in I \text{ and } 0 \circ (0 \ast x) \in I).$$

**Proof.** Let $x \in I$. Then

$$0 = (0 \circ x) \ast (0 \circ x) = (0 \ast (0 \circ x)) \circ x$$

and

$$0 = (0 \ast x) \circ (0 \ast x) = (0 \circ (0 \ast x)) \ast x$$

which imply that $0 \ast (0 \circ x), 0 \circ (0 \ast x) \in I$. This completes the proof.

**Theorem 3.12.** Let $I$ be a pseudo ideal of a pseudo $Q$-algebra $X$ and let

$$I^\# := \{x \in X | 0 \ast (0 \circ x), 0 \circ (0 \ast x) \in I\}.$$ 

Then $I^\#$ is a pseudo ideal of $X$ and $I \subseteq I^\#$.

**Proof.** Obviously, $0 \in I^\#$. Let $a \in X, y \in I^\#$ such that $a \ast y, a \circ y \in I^\#$. Then
0 ∗ (0 ∗ (a ∗ y)), 0 ∗ (0 ∗ (a ∗ y)), 0 ∗ (0 ∗ (a ∗ y)) ∈ I, and 0 ∗ (0 ∗ (a ∗ y)) ∈ I. Using Proposition 3.6 (iii) and (iv), we have

\[(0 ∗ (0 ∗ a)) ∗ (0 ∗ (0 ∗ y)) = 0 ∗ ((0 ∗ a) ∗ (0 ∗ y)) = 0 ∗ (0 ∗ (a ∗ y)) ∈ I\]

and

\[(0 ∗ (0 ∗ a)) ∗ (0 ∗ (0 ∗ y)) = 0 ∗ ((0 ∗ a) ∗ (0 ∗ y)) = 0 ∗ (0 ∗ (a ∗ y)) ∈ I.\]

Since 0 ∗ (0 ∗ y), 0 ∗ (0 ∗ y) ∈ I, it follows from (PI2) that 0 ∗ (0 ∗ a), 0 ∗ (0 ∗ a) ∈ I. Hence a ∈ I#. Thus I# is a pseudo ideal of X. By Proposition 3.11, we know that I ⊆ I#. This completes the proof.

Let X be a pseudo Q-algebra. For any non-empty subset S of X, we define

\[G(S) := \{x ∈ S|0 ∗ x = x = 0 ∗ x\}.\]

In particular, if S = X then we say that G(X) is the G-part of X.

**Proposition 3.13.** If X is a pseudo Q-algebra, then a left cancellation law holds in G(X).

**Proof.** Let x, y, z ∈ G(X) satisfy x ∗ y = x ∗ z and x ∗ y = x ∗ z. By Proposition 3.5, we have 0 ∗ y = 0 ∗ z and 0 ∗ y = 0 ∗ z. Since y, z ∈ G(X), it follows that y = z. □

**Proposition 3.14.** Let X be a pseudo Q-algebra. Then x ∈ G(X) if and only if 0 ∗ x ∈ G(X) and 0 ∗ x ∈ G(X).

**Proof.** If x ∈ G(X), then 0 ∗ x = x = 0 ∗ x and 0 ∗ (0 ∗ x) = 0 ∗ x and 0 ∗ (0 ∗ x) = 0 ∗ x. Hence 0 ∗ x ∈ G(X) and 0 ∗ x ∈ G(X). Conversely, assume that 0 ∗ x ∈ G(X) and 0 ∗ x ∈ G(X). Then 0 ∗ (0 ∗ x) = 0 ∗ x and 0 ∗ (0 ∗ x) = 0 ∗ x. By applying Proposition 3.13, we obtain 0 ∗ x = x = 0 ∗ x. Therefore x ∈ G(X). □

**Proposition 3.15.** Let X be a pseudo Q-algebra. If x, y ∈ G(X), then x ∗ y = y ∗ x.

**Proof.** If x, y ∈ G(X), then 0 ∗ x = x = 0 ∗ x and 0 ∗ y = y = 0 ∗ y. Using (P3), we have

\[x ∗ y = (0 ∗ x) ∗ (0 ∗ y) = (0 ∗ (0 ∗ y)) ∗ x = (0 ∗ y) ∗ x = y ∗ x.\]

This completes the proof. □

**Theorem 3.16.** If a pseudo Q-algebra X satisfies x ∗ (x ∗ y) = x ∗ y or x ∗ (x ∗ y) = x ∗ y for all x, y ∈ X, then it is a trivial algebra.

**Proof.** Assume that X satisfies x ∗ (x ∗ y) = x ∗ y for all x, y ∈ X. Putting x = y in the equation x ∗ (x ∗ y) = x ∗ y, we obtain

\[x = x ∗ 0 = x ∗ (x ∗ x) = x ∗ x = 0.\]
Concerning the other case, the argument is similar. Hence $X$ is a trivial algebra. □

**Theorem 3.17.** Let $X$ be a pseudo $Q$-algebra such that

$$x \circ (y \diamond z) = (x \circ y) \diamond z \quad (\text{resp., } x \diamond (y \circ z) = (x \diamond y) \circ z)$$

for all $x, y, z \in X$. Then $X$ is a group under operation $\circ$(resp.$\diamond$).

**Proof.** Putting $x = y = z$ in (3.1) and using (P1) and (P2), we obtain $x = x \circ 0 = 0 \circ x$(resp., $x = x \diamond 0 = 0 \star x$). This means that 0 is the zero element of $X$ with respect to the operation $\circ$(resp., $\star$). Since $x \diamond x = 0 = x \star x$, we know that the inverse of $x$ is itself with respect to the operation $\diamond$(resp., $\star$). Hence $X$ is a group under operation $\circ$(resp., $\star$). □

**Definition 3.18.** A Pseudo $Q$-algebra $X$ is said to be $\circ$-medial if it satisfies the following identity

$$\text{(M1)} \quad (x \circ y) \diamond (z \circ u) = (x \circ z) \diamond (y \circ u), \forall x, y, z, u \in X.$$

**Proposition 3.19.** Every $\circ$-medial pseudo $Q$-algebra $X$ satisfies the following identities: for any $x, y \in X$

(i) $x \circ y = 0 \circ (y \star x),$

(ii) $0 \circ (0 \star x) = x,$

(iii) $x \circ (x \star y) = y.$

**Proof.** (i) For any $x, y \in X$, we have $x \circ y = (x \circ y) \circ 0 = (x \circ y) \circ (x \star x) = (x \star x) \circ (y \star x) = 0 \circ (y \star x)$.

(ii) If we put $y = 0$ in (i), then we have (ii).

(iii) Using (ii), (P1) and (P2), we have $x \circ (x \circ y) = (x \circ 0) \circ (x \circ y) = (x \circ x) \circ (0 \star y) = 0 \circ (0 \star y) = y.$ □

4. Pseudo Atom

**Definition 4.1.** An element $a$ of a pseudo $Q$-algebra $X$ is called a pseudo atom of $X$ if for every $x \in X$, $x \preceq a$ implies $x = a$.

Obviously, 0 is a pseudo atom of $X$. Let $L(X)$ denote the set of all pseudo atoms of $X$, we call it the center of $X$.

**Theorem 4.2.** Let $X$ be a pseudo $Q$-algebra. Then the following are equivalent: for all $x, y, z, w, u \in X$

(i) $w$ is a pseudo atom,

(ii) $w = x \circ (x \circ w)$ and $w = x \star (x \star w)$

(iii) $(x \circ y) \diamond (x \circ w) = w \star y$ and $(x \circ y) \star (x \circ w) = w \circ y$

(iv) $w \circ (x \diamond y) = y \circ (x \star w)$ and $w \star (x \star y) = y \star (x \star w)$,
(v) $0 \circ (y \ast w) = w \ast y$ and $0 \ast (y \circ w) = w \circ y$.

(vi) $0 \circ (0 \ast w) = w$ and $0 \ast (0 \circ w) = w$.

(vii) $0 \circ (0 \ast (w \circ z)) = w \circ z$ and $0 \ast (0 \circ (w \ast z)) = w \ast z$.

(viii) $z \circ (z \ast (w \circ u)) = w \circ u$ and $z \ast (z \circ (w \ast u)) = w \ast u$.

Proof. (i) $\Rightarrow$ (ii): Let $w$ be a pseudo atom of $X$. Since $x \circ (x \ast w) \preceq w$ and $x \ast (x \circ w) \preceq w$ by Proposition 3.6 (i), we have $w = x \circ (x \ast w)$ and $w = x \ast (x \circ w)$.

(ii) $\Rightarrow$ (iii): For every $x \in X$, we obtain $(x \ast y) \circ (x \ast w) = (x \circ (x \ast w)) \ast y = w \ast y$ and $(x \circ y) \ast (x \circ w) = (x \ast (x \circ w)) \circ y = w \circ y$ by (P3) and (ii).

(iii) $\Rightarrow$ (iv): Replacing $y$ by $x \circ y$ in (iii), we get

$$w \ast (x \circ y) = (x \ast (x \circ y)) \circ (x \ast w) = (x \circ (x \ast w)) \ast (x \circ y) = y \circ (x \ast w)$$

and

$$w \circ (x \ast y) = (x \circ (x \ast y)) \ast (x \circ w) = (x \ast (x \ast w)) \circ (x \ast y) = y \ast (x \ast w).$$

(iv) $\Rightarrow$ (v): Put $y := 0$ in (iv). Then $w \ast (x \circ 0) = 0 \circ (x \ast w)$ and $w \circ (x \ast 0) = 0 \ast (x \ast w)$. Hence $w \ast x = 0 \circ (x \ast w)$ and $w \circ x = 0 \ast (x \ast w)$ by (P2).

(v) $\Rightarrow$ (vi): Set $y := 0$ in (v). Then $0 \circ (0 \ast w) = w \ast 0 = w$ and $0 \ast (0 \circ w) = w \circ 0 = w$.

(vi) $\Rightarrow$ (vii): For any $w, z \in X$, we have

$$0 \circ (0 \ast (w \circ z)) = 0 \ast (0 \ast (w \circ z)) = 0 \ast (0 \circ (w \circ z)) = 0 \ast [(0 \ast w) \circ (0 \circ z)] = 0 \circ (0 \ast w) \circ (0 \ast (0 \circ z)) = w \circ z$$

and

$$0 \ast (0 \circ (w \ast z)) = 0 \ast (0 \ast (w \ast z)) = 0 \ast [(0 \ast w) \circ (0 \ast z)] = 0 \circ [(0 \ast w) \circ (0 \ast z)] = (0 \ast (0 \ast w)) \ast (0 \ast (0 \ast z)) = w \ast z.$$
(vii) ⇒ (viii): For any \( u, w, z \in X \), we have
\[
\begin{align*}
w \circ u &= 0 \circ (0 \ast (w \circ u)) \\
&= 0 \circ ((z \circ z) \ast (w \circ u)) \\
&= 0 \circ [(z \ast (w \circ u)) \circ z] \\
&= (0 \ast (z \ast (w \circ u))) \ast (0 \circ z) \\
&= (0 \ast (0 \circ z)) \circ (z \ast (w \circ u)) \\
&= (0 \ast (0 \circ (z \circ 0))) \circ (z \ast (w \circ u)) \\
&= (0 \ast (0 \ast (z \circ 0))) \circ (z \ast (w \circ u)) \\
&= (z \circ 0) \circ (z \ast (w \circ u)) \\
&= z \circ (z \ast (w \circ u)).
\end{align*}
\]
By a similar way, we obtain \( z \ast (z \circ (w \circ u)) = w \circ u \).

(viii) ⇒ (i): If \( z \ast x = z \circ x = 0 \), then by (viii) we have \( x = x \circ 0 = z \circ (z \circ (x \circ 0)) = z \ast (z \circ x) = z \circ 0 = z \). This shows that \( z \) is a pseudo atom of \( X \). This completes the proof. □

**Corollary 4.3.** Let \( X \) be a pseudo \( Q \)-algebra. If \( a \) is a pseudo atom of \( X \), then for all \( x \) of \( X \), \( a \ast x \) and \( a \circ x \) are pseudo atoms. Hence \( L(X) \) is a pseudo subalgebra of \( X \).

**Corollary 4.4.** Let \( X \) be a pseudo \( Q \)-algebra. For every \( x \) of \( X \), there is a pseudo atom \( a \) such that \( a \leq x \), i.e., every pseudo \( Q \)-algebra is generated by a pseudo atom.

**Proposition 4.5.** A non-zero element \( a \in X \) is a pseudo atom of a pseudo \( Q \)-algebra \( X \) if \( \{0, a\} \) is a pseudo ideal of \( X \).

**Proof.** Let \( x \preceq a \) for any \( x \in X \). Then \( x \ast a = x \circ a = 0 \in \{0, a\} \). Since \( x \in \{0, a\} \) is a pseudo ideal of \( X \), we have \( x = 0 \) or \( x = a \). Since \( a \neq 0 \), we obtain \( x = a \). Hence \( a \) is a pseudo atom of \( X \). □

**Proposition 4.6.** If non-zero element of a pseudo \( Q \)-algebra \( X \) is a pseudo atom, then any pseudo subalgebra of \( X \) is a pseudo ideal of \( X \).

**Proof.** Let \( S \) be a pseudo subalgebra of \( X \) and let \( x, y \ast x, y \circ x \in S \). By Theorem 4.2, we have \( y = x \ast (x \circ y) = x \ast (0 \ast (y \circ x)) \). Since \( 0, y \circ x \in S \) and \( S \) is a pseudo subalgebra of \( X \), we obtain \( 0 \ast (y \circ x) \in S \). Hence \( y = x \ast (0 \ast (y \circ x)) \in S \). Thus any pseudo subalgebra of \( X \) is a pseudo ideal of \( X \). □

For pseudo atom \( a \) of a Pseudo \( Q \)-algebra \( X \),
\[
V(a) := \{x \in X|a \preceq x\}
\]
is called a pseudo branch of \( X \).

**Theorem 4.7.** Let \( X \) be a pseudo \( Q \)-algebra. Suppose that \( a \) and \( b \) are pseudo atoms of \( X \). Then the following properties hold:
(i) For all \( x \in V(a) \) and all \( y \in V(b) \), \( x \cdot y \in V(a \cdot b) \) and \( x \circ y \in V(a \circ b) \).

(ii) For all \( x \) and \( y \) in \( V(a) \), \( x \circ y, x \cdot y \in K(X) \), where \( K(X) = \{ x \in X \mid 0 \leq x \} \).

(iii) If \( a \neq b \), then for all \( x \in V(a) \) and \( y \in V(b) \), we have \( x \cdot y, x \circ y \in K(X) \).

(iv) For all \( x \in V(b) \), \( a \cdot x = a \star b \) and \( a \circ x = a \circ b \).

(v) If \( a \neq b \), then \( V(a) \cap V(b) = \emptyset \).

**Proof.** (i) For all \( x \in V(a) \) and all \( y \in V(b) \), by Proposition 3.6 and Theorem 4.2 we have

\[
(a \cdot b) \circ (x \cdot y) = [0 \circ (0 \cdot (a \cdot b))] \circ (x \cdot y) \\
= (0 \circ (x \cdot y)) \circ (0 \cdot (a \cdot b)) \\
= (0 \cdot (x \cdot y)) \circ (0 \cdot (a \cdot b)) \\
= (0 \cdot x) \circ (0 \cdot y) \circ (0 \cdot (a \cdot b)) \\
= (0 \cdot (0 \cdot (a \cdot b))) \circ (0 \cdot (a \cdot b)) \\
= (0 \cdot (0 \cdot (a \cdot b))) \circ (0 \cdot (a \cdot b)) \\
= (0 \cdot (a \cdot b)) \circ (0 \cdot (a \cdot b)) \\
= (0 \cdot (a \cdot b)) \circ (0 \cdot (a \cdot b)) \\
= 0 \circ (b \cdot y) = 0 \circ 0 = 0
\]

and

\[
(a \circ b) \cdot (x \circ y) = [0 \circ (0 \cdot (a \cdot b))] \cdot (x \circ y) \\
= (0 \cdot (x \circ y)) \cdot (0 \circ (a \cdot b)) \\
= (0 \cdot (x \circ y)) \cdot (0 \circ (a \cdot b)) \\
= (0 \cdot (0 \cdot (a \cdot b))) \circ (0 \cdot (a \cdot b)) \\
= (0 \cdot (0 \cdot (a \cdot b))) \circ (0 \cdot (a \cdot b)) \\
= (0 \cdot (a \cdot b)) \circ (0 \cdot (a \cdot b)) \\
= (0 \cdot (a \cdot b)) \circ (0 \cdot (a \cdot b)) \\
= (0 \cdot (a \cdot b)) \circ (0 \cdot (a \cdot b)) \\
= 0 \circ (b \cdot y) = 0 \circ 0 = 0.
\]

Hence \( x \cdot y \in V(a \cdot b) \) and \( x \circ y \in V(a \circ b) \).

(ii) and (iii) are simple consequences of (i).

(iv) For all \( x \in V(b) \), by Theorem 4.2 we have \( (a \cdot x) \circ (a \cdot b) = (a \circ (a \cdot b)) \circ x = b \circ x = 0 \).

Moreover, \( a \cdot b \) is a pseudo atom by Corollary 4.3. Therefore \( a \cdot x = a \cdot b \). Also we
get \((a \odot x) \ast (a \odot b) = (a \ast (a \odot b)) \ast x = b \ast x = 0\). Moreover, \(a \odot b\) is a pseudo atom by Corollary 4.3. Therefore \(a \odot x = a \odot b\).

(v) Let \(a \neq b\) and \(V(a) \cap V(b) \neq \emptyset\). Then there exists \(c \in V(a) \cap V(b)\). By (i), we have \(0 = c \ast c = c \odot c \in V(a \ast b), V(a \odot b)\). Hence \(a \ast b = a \odot b = 0\), which is a contradiction. Thus (v) is true. \(\square\)

References