Interval-Valued Fuzzy Ideals of a Ring

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Abstract

We introduce the notions of interval-valued fuzzy prime ideals, interval-valued fuzzy completely prime ideals and interval-valued fuzzy weakly completely prime ideals. And we give a characterization of interval-valued fuzzy ideals and establish relationships between interval-valued fuzzy completely prime ideals and interval-valued fuzzy weakly completely prime ideals.


1. Introduction and Preliminaries


In this paper, we introduce the notions of interval-valued fuzzy prime ideals, interval-valued fuzzy completely prime ideals and interval-valued fuzzy weakly completely prime ideals. And we give a characterization of interval-valued fuzzy ideals and establish relationships between interval-valued fuzzy completely prime ideals and interval-valued fuzzy weakly completely prime ideals.

Now, we will list some basic concepts and well-known results which are needed in the later sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \ldots$, and note that $M = [M^L, M^U]$, where $M^L$ and $M^U$ are the lower and the upper end points respectively. Especially, we denoted $0 = [0, 0], 1 = [1, 1], \alpha = [a, a]$ for every $a \in (0, 1)$. We also note that

(i) $(\forall M, N \in D(I))(M = N \iff M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))(M \subseteq N \iff M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the complement of $M$, denoted by $M^c$, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$. (See [8]).

Definition 1.1 [8, 11]. A mapping $A : X \rightarrow D(I)$ is called an interval-valued fuzzy set (in short, IVS) in $X$ and is denoted by $A = [A^L, A^U]$. Thus for each $x \in X$, $A(x) = [A^L(x), A^U(x)]$, where $A^L(x)$[resp. $A^U(x)$]$x$ is called the lower[resp. upper] end point of $x$ to $A$. For any $[a, b] \in D(I)$, the interval-valued fuzzy set $A$ in $X$ defined by $A(x) = [a, b]$ for each $x \in X$ is denoted by $[a, b]$ and if $a = b$, then the IVS $[a, b]$ is denoted by simply $a$. In particular, $\tilde{0}$ and $\tilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in $X$, respectively.

We will denote the set of all IVSs in $X$ as $D(I)^X$. It is clear that set $A = [A^L, A^U] \in D(I)^X$ for each $A \in I^X$.

Definition 1.2 [8]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subseteq D(I)^X$. Then

(a) $A \subseteq B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(c) $A^c = [1 - A^U, 1 - A^L]$.

(d) $A \cup B = [A^L \lor B^L, A^U \lor B^U]$.

(d’) $\bigcup_{\alpha \in \Gamma} A_\alpha = [\bigvee_{\alpha \in \Gamma} A^L_\alpha, \bigvee_{\alpha \in \Gamma} A^U_\alpha]$.

(e) $A \cap B = [A^L \land B^L, A^U \land B^U]$.
In this case, if

are called the

and

Definition 1.3 [7]. Let be an IVS in a set and let \( [\lambda, \mu] \in D(I) \). Then the set \( A^{x,\mu} \{ x \in X : A^L(x) \geq \lambda \text{ and } A^U(x) \geq \mu \} \) is called a \([\lambda, \mu]\)-level subset of \( A \).

Definition 1.4 [8]. Let \( [\lambda, \mu] \in D(I) \). Then an interval-valued fuzzy point (in short, IVP) \( x_{[\lambda, \mu]} \) of \( X \) is the IVS in \( X \) defined as follows: For each \( y \in X \),

\[
x_{[\lambda, \mu]}(y) = \begin{cases} 
[\lambda, \mu], & \text{if } y = x; \\
0, & \text{otherwise.}
\end{cases}
\]

In this case, \( x \) is called the support of \( x_{[\lambda, \mu]} \) and \( \lambda \) and \( \mu \) are called the value and nonvalue of \( x_{[\lambda, \mu]} \), respectively. In particular, if \( \lambda = \mu \), then it is also denoted by \( x_A \). An IVP \( x_M \) is said to belong to an IVS \( A \) in \( X \), denoted by \( x_M \in A \), if \( M^L \leq A^L(x) \) and \( M^U \leq A^U(x) \).

It is clear that \( A = \bigcup_{x_M \in A} x_M \) and \( x_M \in A \) if and only if \( x_M \in A^L \) and \( x_M \in A^U \), for each \( A \in P(I)^X \).

We will denote the set of all IVPs in \( X \) as IVP(X).

The following is the immediate result of Definition 1.2 and 1.4.

Theorem 1.5. Let \( A, B \in D(I)^X \). Then \( A \subseteq B \) if and only if for each \( x_M \in \text{IVP}(X), x_M \in A \) implies \( x_M \in B \).

Definition 1.6 [7]. Let \( (X, \cdot) \) be a groupoid and let \( A, B \in D(I)^X \). Then the interval-valued fuzzy product of \( A \) and \( B \), \( A \circ B \) is defined as follows: For each \( x \in X \),

\[
A \circ B(x) = \begin{cases} 
\bigvee_{x=yz} (A^L(y) \land B^L(z)), & \text{if } x = yz; \\
\bigvee_{x=yz} (A^U(y) \land B^U(z)), & \text{otherwise.}
\end{cases}
\]

Result 1.1B [7, Proposition 3.2]. Let \( (X, \cdot) \) be a groupoid, let "\( \circ \)" be the same as above, let \( x_M, y_N \in \text{IVP}(X) \) and let \( A, B \in D(I)^X \). Then
   (a) \( x_M \circ y_N = (xy)_{M \cap N} \).
   (b) \( A \circ B = \bigcup_{x_M \in A, y_N \in B} x_M \circ y_N \).

Definition 1.7 [1]. Let \( G \) be a group and let \( A \in D(I)^G \). Then \( A \) is called an interval-valued fuzzy subgroup (in short, IVG) of \( G \) if it satisfies the following conditions:
   (a) \( A^L(xy) \geq A^L(x) \land A^L(y) \) and \( A^U(xy) \geq A^U(x) \lor A^U(y) \) for any \( x, y \in G \).
   (b) \( A^L(x^{-1}) \geq A^L(x) \) and \( A^U(x^{-1}) \geq A^U(x) \) for each \( x \in G \).

We will denote the set of all IVGs as IVG(G).

Result 1.1C [1, Proposition 3.1]. Let \( A \) be an IVG of a group \( G \) with identity \( e \). Then \( A(x^{-1}) = A(x) \) and \( A^L(x) \geq A^L(e) \) and \( A^U(x) \geq A^U(e) \) for each \( x \in G \).

Definition 1.8 [7]. Let \( (R, +, \cdot) \) be a ring and let \( 0 \neq A \in D(I)^R \). Then \( A \) is called an interval-valued fuzzy subring (in short, IVR) of \( R \) if it satisfies the following conditions:
   (a) \( A \) is an IVG with respect to the operation "\( + \)."
   (b) \( A^L(xy) \geq A^L(x) \land A^L(y) \) and \( A^U(xy) \geq A^U(x) \lor A^U(y) \) for any \( x, y \in R \).

We will denote the set of all IVRs as IVR(R).

2. Interval-valued fuzzy ideals

Definition 2.1 [7]. Let \( A \) be a non-empty IVR of a ring \( R \). Then \( A \) is called an:
   (i) interval-valued fuzzy left ideal (in short, IVLI) of \( R \) if \( A^L(xy) \geq A^L(y) \) and \( A^U(xy) \geq A^U(y) \) for any \( x, y \in R \).
   (ii) interval-valued fuzzy right ideal (in short, IVRI) of \( R \) if \( A^L(xy) \geq A^L(x) \) and \( A^U(xy) \geq A^U(x) \) for any \( x, y \in R \).
   (iii) interval-valued fuzzy ideal (in short, IVRI) of \( R \) if it is an IVLI and an IVRI of \( R \).

We will denote the set of all IVLIs [resp. IVLIs and IVIs] of \( R \) as IVRI(R) [resp. IVLI(R) and IVRI(R)].

Result 2.1A [7, Proposition 6.6]. Let \( R \) be a ring. Then \( A \) is an ideal [resp. a left ideal and a right ideal] of \( R \) if and only if \( \chi_A \in \text{IVRI}(R) \) [resp. \( \text{IVLI}(R) \) and \( \text{IVRI}(R) \)].

Result 2.1B [7, Proposition 6.5]. Let \( R \) be a ring and let \( 0 \neq A \in D(I)^R \). Then \( A \in \text{IVRI}(R) \) [resp. \( \text{IVLI}(R) \) and \( \text{IVRI}(R) \) if and only if for any \( x, y \in R \),
Similarly, we have

(i) \( A^L(x - y) \geq A^L(x) \land A^L(y) \) and \( A^U(x - y) \geq A^U(x) \land A^U(y) \).

(ii) \( A^L(xy) \geq A^L(x) \land A^L(y) \) and \( A^U(xy) \geq A^U(x) \land A^U(y) \) [resp. \( A^L(xy) \geq A^L(y) \land A^U(x) \) and \( A^U(xy) \geq A^U(y) \land A^U(x) \)].

Lemma 2.2. Let \( R \) be a ring and let \( A, B \in D(I, R) \).

(a) If \( A, B \in IVLI(R) \) [resp. IVRI(R) and IVI(R)], then \( A \cap B \in IVLI(R) \) [resp. IVRI(R) and IVI(R)].

(b) If \( A \in IVRI(R) \) and \( B \in IVLI(R) \), then \( A \circ B \subseteq A \cap B \).

Proof. (a) Suppose \( A, B \in IVLI(R) \) and let \( x, y \in R \). Then

\[
(A \cap B)^L(x - y) = A^L(x - y) \land B^L(x - y) \\
\geq (A^L(x) \land A^L(y)) \land (B^L(x) \land B^L(y)) \\
= (A \cap B)^L(x) \land (A \cap B)^L(y).
\]

Similarly, we have \( (A \cap B)^U(x - y) \geq (A \cap B)^U(x) \land (A \cap B)^U(y) \). Also

\[
(A \cap B)^L(xy) = A^L(xy) \land B^L(xy) \\
\geq A^L(y) \land B^L(y) \quad \text{(Since } A, B \in IVLI(R)) \\
= (A \cap B)^L(y).
\]

Similarly, we have \( (A \cap B)^U(xy) \geq (A \cap B)^U(y) \). Hence, by Result 2.B, \( A \cap B \in IVLI(R) \). Similarly, we can easily see the rest.

(b) Let \( x \in G \) and suppose \( A \circ B(x) = [0, 0] \). Then there is nothing to show. Suppose \( A \circ B(x) \neq [0, 0] \). Then

\[
A \circ B(x) = \bigvee_{y \in yG} (A^L(y) \land B^L(z)) \quad \text{and} \quad \bigvee_{z \in zG} (A^U(y) \land B^U(z))
\]

where \( A \in IVRI(R) \) and \( B \in IVLI(R) \).

Thus

\[
A^L(xy) \leq A^L(xz) = A^L(x) \land A^U(yz) = A^U(x) \land B^L(yz) = B^L(xz) \leq B^L(z) \leq B^L(yz) = B^U(yz) = B^U(xz).
\]

Thus

\[
(A \circ B)^L(x) = \bigvee_{x \in xG} (A^L(y) \land B^L(z)) \\
\leq A^L(x) \land B^L(x) = (A \cap B)^L(x).
\]

Similarly, we have \( (A \circ B)^U(x) \leq (A \cap B)^U(x) \). Hence \( A \circ B \subseteq A \cap B \). This completes the proof.

A ring \( R \) is said to be regular if for each \( a \in R \) there exists an \( x \in R \) such that \( a = axa \).

Result 2.C [2, Theorem 9.4]. A ring \( R \) is regular if and only if \( JM = J \cap M \) for each right ideal \( J \) and left ideal \( M \) of \( R \).

Theorem 2.3. A ring \( R \) is regular if and only if for each \( A \in IVRI(R) \) and each \( B \in IVLI(R) \), \( A \circ B = A \cap B \).

Proof. (\( \Rightarrow \)) Suppose \( R \) is regular. From Lemma 2.2(b), it is clear that \( A \circ B \subseteq A \cap B \). Thus it is sufficient to show that \( A \cap B \subseteq A \circ B \). Let \( a \in R \). Then, by the hypothesis, there exists an \( x \in R \) such that \( a = axa \). Thus

\[
A^L(a) = A^L(axa) \geq A^L(ax) \geq A^L(a) \quad \text{and} \quad A^U(a) = A^U(axa) \geq A^U(ax) \geq A^U(a).
\]

So \( A(ax) = A(a) \).

On the other hand,

\[
A \circ B)^L(a) = \bigvee_{a \in aG} (A^L(y) \land B^L(z)) \\
\geq A^L(ax) \land B^L(a) \quad \text{(Since } a = axa) \\
= A^L(a) \land B^L(a) = (A \cap B)^L(a).
\]

Similarly, we have \( (A \circ B)^U(a) \geq (A \cap B)^U(a) \). Thus \( A \cap B \subseteq A \circ B \). Hence \( A \circ B = A \cap B \).

(\( \Leftarrow \)) Suppose the necessary condition holds. Let \( J \) and \( M \) be right and left ideals of \( R \), respectively. Then, by Result 2.A, \( \langle x_j, x_j \rangle \in IVRI(R) \) and \( \langle x_M, x_M \rangle \in IVLI(R) \).

Let \( a \in J \cap M \) and let \( A = \langle x_j, x_j \rangle, B = \langle x_M, x_M \rangle \). Then, by the hypothesis, \( (A \circ B)(a) = (A \cap B)(a) = [1, 1] \). Thus

\[
(A \circ B)^L(a) = \bigvee_{a \in aG} (A^L(a_1) \land B^L(a_2)) \\
= \bigvee_{a \in aG1 \circ aG2} (x_j(a_1) \cap x_M(a_2)) \\
= 1.
\]

Similarly, we have \( (A \circ B)^U(a) = 1 \). So there exist \( b_1, b_2 \in R \) such that \( x_M(b_1) = 1 \) and \( x_M(b_2) = 1 \) with \( a = b_1 b_2 \). Thus \( a \in JM \), i.e., \( J \cap M \subseteq JM \). Since \( JM \subseteq J \cap M \), \( JM = J \cap M \). Hence, by Result 2.C, \( R \) is regular. This completes the proof.

\[\Box\]

3. Interval-valued fuzzy prime ideals

Definition 3.1. Let \( P \) be an IV of a ring \( R \). Then \( P \) is said to be prime if \( P \) is not a constant mapping and for any \( A, B \in IVRI(R) \), \( A \circ B \subseteq P \) implies either \( A \subseteq P \) or \( B \subseteq P \).

We will denote the set of all interval-valued fuzzy prime ideals of \( R \) as IVPI(R).

Theorem 3.2. Let \( J \) be an ideal of a ring \( R \) such that \( J \neq R \). Then \( J \) is a prime ideal of \( R \) if and only if
Thus either such that \( \chi \) and \( R \) of Definition 3.1. Hence \( P \) is a prime ideal of \( R \). This contradicts \( \chi \) and \( R \) of Definition 3.1. Hence \( P \) is a prime ideal of \( R \). Thus \( P \) is a prime ideal of \( R \). Hence \( P \) is a prime ideal of \( R \).

On the other hand, \( A \circ B(a) \neq [0,0] \). Then clearly \( A \circ B(a) \neq [0,0] \). This contradicts (3.1). So \( P \) satisfies the second condition of Definition 3.1. Hence \( P = [\chi_J, \chi_J] \in \text{IVPI}(R) \).

\( (\Rightarrow) \): Suppose \( P = [\chi_J, \chi_J] \in \text{IVPI}(R) \). Since \( P \) is not a constant mapping on \( R \), \( J \neq R \). Let \( A \) and \( B \) be two ideals of \( R \) such that \( AB \subset J \). Let \( \tilde{A}, \tilde{B} \in \text{IVRI}(R) \). Consider the product \( A \circ B \). Let \( x \in R \).

Suppose \( A \circ B(x) \neq [0,0] \). Then clearly \( \tilde{A} \circ \tilde{B} \subset R \). Suppose \( \tilde{A} \circ \tilde{B}(x) \neq [0,0] \). Then \( \tilde{A} \circ \tilde{B} \neq [0,0] \). Similarly, we have \( \tilde{A} \circ \tilde{B} \neq [0,0] \). Thus there exist \( y, z \in R \) with \( x = yz \) such that \( \chi_A(y) \neq 0 \) and \( \chi_B(z) \neq 0 \). So \( \chi_A(y) = 1 \) and \( \chi_B(z) = 1 \). This implies \( y \in A \) and \( z \in B \). Thus \( x = yz \in AB \subset J \). So \( \chi_J(x) = 1 \). It follows that \( \tilde{A} \circ \tilde{B} \subset R \). Since \( \tilde{A} \circ \tilde{B} \subset R \), either \( A \subset P \) or \( B \subset P \). Thus either \( A \subset J \) or \( B \subset J \). Hence \( J \) is a prime ideal of \( R \). This completes the proof.

Remark 3.4. Let \( P \in \text{IVI}(Z) \). Then, by Proposition 3.3, \( R_P \) is an ideal of \( Z \). Hence there exists an integer \( n \geq 0 \) such that \( R_P = nZ \).

Proposition 3.5. Let \( P \in \text{IVI}(Z) \) with \( R_P = nZ \). Then \( P \) can take at most \( r \) values, where \( r \) is the number of distinct prime divisors of \( n \).

Proof. Let \( a \in Z \) and let \( d = (a, n) \). Then there exist \( r, s \in Z \) such that \( d = ar + ns \). Thus
\[
P^L(d) = P^L(ar + ns) \geq P^L(ar) \cap P^L(ns) \geq P^L(a) \cap P^L(n).
\]

Similarly, we have \( P^U(d) \geq P^U(a) \cap P^U(n) \). Since \( r \in nZ \), by Result 1.C,
\[
P^L(n) = P^L(0) \geq P^L(a) \cap P^L(n) = P^L(0) \geq P^L(a).
\]

Thus \( P^L(d) \geq P^L(a) \) and \( P^U(d) \geq P^U(a) \). Since \( d = \text{a divisor of } a \), there exists a \( t \in Z \) such that \( a = dt \). Then
Thus, $P(a) = P(d)$. Moreover, by Result 1.C, $P(x) = P(-x)$ for each $x \in R$. Hence for each $a \in \mathbb{Z}$ there exists a positive divisor $d$ of $n$ such that $P(a) = P(d)$. This completes the proof. \hfill \square

The following result gives a complete characterization of interval-valued fuzzy prime ideals of $\mathbb{Z}$:

**Theorem 3.6.** Let $P \in \text{IVPI}(\mathbb{Z})$ with $\mathbb{Z}_P \neq \{0\}$. Then $P$ has two distinct values. Conversely, if $P \in D(I)^\mathbb{Z}$ such that $P(n) = [\lambda_1, \mu_1]$ when $p \mid n$ and $P(n) = [\lambda_2, \mu_2]$ when $p \nmid n$, where $p$ is a fixed prime, $\lambda_1 > \lambda_2$ and $\mu_1 > \mu_2$, then $P \in \text{IVPI}(\mathbb{Z})$ with $\mathbb{Z}_P \neq \{0\}$.

Proof. Suppose $P \in \text{IVPI}(\mathbb{Z})$ with $\mathbb{Z}_P = n\mathbb{Z} \neq \{0\}$. Then, by Proposition 3.3, $\mathbb{Z}_P$ is a prime ideal of $\mathbb{Z}$. Thus $n$ is a prime integer. Since $n$ has two distinct positive integers, by Proposition 3.5, $P$ has at most two distinct values. On the other hand, an interval-valued fuzzy prime ideals cannot be a constant mapping. Hence $P$ has two distinct values.

Conversely, let $P$ be an IVS in $\mathbb{Z}$ satisfying the given conditions. Let $a, b \in \mathbb{Z}$.

Case(i): Suppose $p \mid (a - b)$. Then $P((a - b) = [\lambda_1, \mu_1]$. Thus $\lambda_1 = P^L((a - b) \geq P^L(a) \land P^L(b)$ (Since $\lambda_1 > \lambda_2$) and $\mu_1 = P^U((a - b) \geq P^U(a) \land P^U(b)$ (Since $\mu_1 > \mu_2$).

Case(ii): Suppose $p \mid a$ and $p \nmid b$. Then either $P(a) = [\lambda_2, \mu_2]$ or $P(b) = [\lambda_2, \mu_2]$. So $\lambda_2 = P^L(a - b) \geq P^L(a) \land P^L(b)$ and $\mu_2 = P^U(a - b) \geq P^U(a) \land P^U(b)$.

Case(iii): Suppose $p \mid ab$. Then clearly $P^L(ab) \geq P^L(b)$ and $P^U(ab) \geq P^U(b)$.

Case(iv): Suppose $p \mid ab$. Then $p \mid a$ and $p \nmid b$. Thus $P^L(ab) \geq P^L(a)$ and $P^U(ab) \geq P^U(b)$. Consequently, by Result 1.C, $P \in \text{IVPI}(\mathbb{Z})$ with $\mathbb{Z}_P = n\mathbb{Z} \neq \{0\}$.

Moreover, by the similar arguments of the proof of Proposition 3.2, we can see that $P \in \text{IVPI}(\mathbb{Z})$. This completes the proof. \hfill \square

**Proposition 3.7.** Let $R$ be a ring with 1. If every IVI of $R$ has finite values, then $R$ is a Noetherian ring.

Proof. Let $\{J_i\}_{i \in \mathbb{Z}}$ be a sequence of ideals of $R$ such that $J_1 \subset J_2 \subset J_3 \subset \cdots$ and let $J = \bigcup_{i \in \mathbb{Z}} J_i$. Then clearly $J$ is an ideal of $R$. We define a mapping $P : R \to D(I)$ as follows: For each $x \in R$,

$$P(x) = \begin{cases} 0, & \text{if } x \not\in J, \\ \left[\frac{1}{i_1}, \frac{1}{i_2}\right], & \text{if } x \in J, \end{cases}$$

where $i_1$ = minimum of $i$ such that $x \in J_i$. Then it is clear that $P \in \text{IVI}(R)$ from the definition of $P$. Moreover, we can easily see that $P \in \text{IVI}(R)$. If the chain does not terminate, then $P$ takes infinitely many values. This contradicts the hypothesis. Thus the chain terminates. Hence $R$ is a Noetherian ring. This completes the proof. \hfill \square

**Proposition 3.8.** Let $A : \mathbb{Z} \to D(I)$ be the mapping such that

(a) $A(x) = A(-x)$ for each $x \in \mathbb{Z}$.

(b) $A^L(x + y) \geq A^L(x) \land A^L(y)$ and $A^U(x + y) \geq A^U(x) \land A^U(y)$ for any $x, y \in \mathbb{Z}$.

If there exists a non-zero integer $m$ such that $A(m) = A(0)$, then $A$ can take at most finitely many values.

Proof. It is clear that $A \in D(I)^\mathbb{Z}$ from the definition of $A$. Moreover, we can easily show that $A \in \text{IVI}(\mathbb{Z})$ such that $Z_A \neq 0$. Hence, by Proposition 3.5, $A$ can take at most finitely many values. \hfill \square

## 4. Interval-valued fuzzy completely prime ideals

**Definition 4.1.** Let $P$ be an IVI of a ring $R$. Then $P$ is called an interval-valued fuzzy completely prime ideals (in short, $\text{IVCPI}$) of $R$ if it satisfies the following conditions:

(a) $P$ is not a constant mapping.

(b) For any $x, y \in \text{IVP}(R)$, $x \cdot y \in P$ implies either $x \in P$ or $y \in P$.

We will denote the set of all IVCPIs of $R$ as $\text{IVCPI}(G)$.

**Proposition 4.2.** (a) Let $R$ be a ring. Then $\text{IVCPI}(R) \subseteq \text{IVPI}(R)$.

(b) Let $R$ be a commutative ring. Then $\text{IVCPI}(R) \subseteq \text{IVPI}(R)$. Hence $\text{IVCPI}(R) = \text{IVPI}(R)$.

Proof. (a) Let $P \in \text{IVCPI}(R)$ and let $A, B \in \text{IVI}(R)$ such that $A \circ B \in P$. Suppose $A \not\subseteq P$. Then, by Theorem 1.5, there exists an $x_{[\lambda_{[\mu]]]} \in \text{IVP}(R)$ such that $x_{[\lambda_{[\mu]]]} \in P$ but $x_{[\lambda_{[\mu]}]} \notin P$. Let $y_{[\lambda_{[\mu]}]} \in B$. Then, by Result 1.B(a), $x_{[\lambda_{[\mu]]]} \cdot y_{[\lambda_{[\mu]}]} = (xy)_{[\lambda_{[\mu]}]}$. On the other hand,

$$P^L(xy) = (A \circ B)^L(xy) \geq A^L(x) \land B^L(y) = \lambda \land t = (x_{[\lambda_{[\mu]}]} \circ y_{[\lambda_{[\mu]}]})^L(xy).$$

Similarly, we have $P^U(xy) \geq (x_{[\lambda_{[\mu]}]} \circ y_{[\lambda_{[\mu]}]})^U(xy)$.

Let $z \in R$ such that $x \not= xy$. Then clearly $[x_{[\lambda_{[\mu]]]} \circ y_{[\lambda_{[\mu]}]}](z) = [0, 0]$. Thus $x_{[\lambda_{[\mu]}]} \circ y_{[\lambda_{[\mu]}]} \in P$. Since $P \in \text{IVCPI}(R)$, $x_{[\lambda_{[\mu]}]} \in P$ or $y_{[\lambda_{[\mu]}]} \in P$. Since $x_{[\lambda_{[\mu]}]} \notin P$, $y_{[\lambda_{[\mu]}]} \in P$. So, by Theorem 1.5, $B \subseteq P$. Hence $P \in \text{IVPI}(R)$.

(b) Let $P \in \text{IVPI}(R)$ and let $x_{[\lambda_{[\mu]}]} \in \text{IVP}(R)$ such that $x_{[\lambda_{[\mu]}]} \circ y_{[\lambda_{[\mu]}]} \in P$. Then $(x_{[\lambda_{[\mu]}]} \circ y_{[\lambda_{[\mu]}]})^L(xy) \leq P^L(xy)$ and $(x_{[\lambda_{[\mu]}]} \circ y_{[\lambda_{[\mu]}]})^U(xy) \leq P^U(xy)$.
Thus, by Result 1.B(a),
\[ \lambda \land t \leq P^L(xy) \text{ and } \mu \land s \leq P^U(xy). \] (4.1)

We define two mappings \( A, B : R \to D(I) \) as follows:
For each \( z \in R \),
\[
A(z) = \begin{cases} 
[\lambda, \mu], & \text{if } z \in (x); \\
[0, 0], & \text{otherwise.}
\end{cases}
\]
and
\[
B(z) = \begin{cases} 
[t, s], & \text{if } z \in (y); \\
[0, 0], & \text{otherwise.}
\end{cases}
\]
where \( (z) \) is the ideal generated by \( z \). Then clearly \( A, B \in D(I)^R \) from the definitions of \( A \) and \( B \). It is easily seen that if \( z \) is not expressible in the form \( z = uv \) for some \( u \in (x) \) and \( v \in (y) \), then \( A \circ B(z) = [0, 0] \). Suppose there exist \( u \in (x) \) and \( v \in (y) \) such that \( z = uv \). Then
\[
(A \circ B)^L(z) = \bigvee_{z=uv, u \in (x), v \in (y)} (A^L(u) \land B^L(v)) = \lambda \land t
\]
and
\[
(A \circ B)^U(z) = \bigvee_{z=uv, u \in (x), v \in (y)} (A^U(u) \land B^U(v)) = \mu \land s.
\]
Since \( R \) is commutative and \( u \in (x) \), there exist \( n \in \mathbb{Z} \) and \( b \in R \) such that \( u = nx + xb \). Since \( v \in (y) \), there exist \( m \in \mathbb{Z} \) and \( c \in R \) such that \( v = my + yc \). Since \( R \) is commutative, \( uv = (nx + xb)(my + yc) = xyd + mnx \) for some \( d \in R \). Then
\[
P^L(uv) \geq P^L(xy) \quad \text{(Since } P \in \text{IVI}(R)) \quad\text{(By (4.1))}
\]
Similarly, we have that
\[
P^U(uv) \geq P^U(xy) \geq \mu \land s.
\]
Thus \( z_{[\lambda, \mu]} \cap [t, s] \in P \). So, in all, \( A \circ B \in P \). On the other hand, from the definitions of \( A \) and \( B \), we can easily prove that \( A, B \in \text{IVI}(R) \). Since \( P \in \text{IVPI}(R) \), either \( A \in P \) or \( B \in P \). Thus either \( x[\lambda, \mu] \in P \) or \( y[t, s] \in P \). Hence \( P \in \text{IVCPI}(R) \). This completes the proof. \( \square \)

**Proposition 4.3.** Let \( P \) be a non-constant IVI of a ring \( R \).
(a) If \( P \) is an IVI [resp. IVCPI] of \( R \), then
(i) \( R_P \) is a prime [resp. completely prime] ideal of \( R \).
(ii) \( \text{Im} P \) consists of exactly two points of \( D(I) \).
(b) If \( P(0) = [1, 1] \) and \( P \) satisfies the conditions (i) and (ii), then \( P \in \text{IVPI}(R) \) [resp. IVCPI(\( R \))].

**Proof.** (a) We shall confirm our proof to the case of interval-valued fuzzy prime ideals. An analogous proof can be given by for interval-valued fuzzy completely prime ideals. Suppose \( P \in \text{IVPI}(R) \). Then, by Proposition 3.3, \( R_P \) is a prime ideal of \( R \). Assume that \( \text{Im} P \) contains exactly two values. Then there exist \( x, y \in R \setminus R_P \) such that \( P(x) \neq P(y) \). Suppose without loss of generality that \( P^L(x) < P^L(y) \) and \( P^U(x) < P^U(y) \). Since \( P \in \text{IVPI}(R) \) and \( A(y) \neq A(0) \), by Result 1.C, \( P^L(y) < P^L(0) \) and \( P^U(y) < P^U(0) \). Let \( [\lambda, \mu], [t, s] \in D(I) \) be chosen such that
\[
P^L(x) < \lambda < P^L(y) < t < P^L(0)
\]
and
\[
P^U(x) < \mu < P^U(y) < s < P^U(0).
\]
Let \( (x) \) and \( (y) \) denote respectively the ideals generated by \( x \) and \( y \). We define two mappings \( A, B : R \to D(I) \) as follows: \( A = [\lambda x, \mu x(y)] \) and \( B = [t y, s y(y)] \). Then it is easily seen that \( A, B \in \text{IVI}(R) \) from the definitions of \( A \) and \( B \). Let \( z \in R \) which cannot be expressed in the form \( z = uv \) for \( u \in (x) \) and \( v \in (y) \). Then \( A \circ B(z) = [0, 0] \). Thus \( A \circ B \in P \). Now let \( z \in R \). Suppose there exist \( u \in (x) \) and \( v \in (y) \) such that \( z = uv \) for some \( u \in (x) \) and \( v \in (y) \). Then
\[
(A \circ B)^L(z) = \bigvee_{z=uv, u \in (x), v \in (y)} (A^L(u) \land B^L(v)) = \lambda \land t
\]
and
\[
(A \circ B)^U(z) = \bigvee_{z=uv, u \in (x), v \in (y)} (A^U(u) \land B^U(v)) = \mu \land s.
\]
Similarly, we have \( (A \circ B)^U(z) = \mu. \) Since \( u \in (x) \), there exist \( m \in \mathbb{Z} \) and \( r_1 \in R(i=1, 2, 3, 4) \) such that \( u = mx + r_1x + r_2x + r_4x \). Similarly, there exist \( n \in \mathbb{Z} \) and \( s_1 \in R(i=1, 2, 3, 4) \) such that \( v = ny + s_1y + s_2y + s_3y + s_4y \). Since \( P \in \text{IVPI}(R) \), by Result 1.C,
\[
P^L(z) = P^L(uv) \geq P^L(x) \land P^L(y) > \lambda
\]
and
\[
P^U(z) = P^U(uv) \geq P^U(x) \land P^U(y) > \mu.
\]
Thus \( (A \circ B)^L(z) \leq P^L(z) \) and \( (A \circ B)^U(z) \leq P^U(z) \). So \( A \circ B \in P \). Since \( P \in \text{IVPI}(R) \), either \( A \subset P \) or \( B \subset P \). Then either \( A^L(x) = \lambda \leq P^L(x) \), \( A^U(x) = \mu \leq P^U(x) \) or \( B^L(y) = t \leq P^L(y) \), \( B^U(y) = s \leq P^U(y) \). This contradicts (4.2). Hence \( \text{Im} P \) consists of exactly two points of \( D(I) \).

(b) Suppose \( P(0) = [1, 1] \) and \( P \) satisfies the conditions (i) and (ii). Then, by the similar arguments of proof of Theorem 3.2, we can see that \( P \in \text{IVPI}(R) \). This completes the proof. \( \square \)

**Corollary 4.3.** Let \( P \) be an interval-valued fuzzy completely prime ideal of a ring \( R \). Then for any \( x, y \in R \),
\[
P(xy) = [P^L(x) \land P^L(y), P^U(x) \land P^U(y)]
\]

**Remark 4.4.** Proposition 4.3 generalizes Proposition 3.5.

**Definition 4.5.** Let \( A \) be a non-constant IVI of a ring \( R \). Then \( A \) is called an interval-valued fuzzy weakly completely prime ideal of \( R \) if for any
\[ x, y \in R, A(xy) = [A^L(x) \land A^L(y), A^U(x) \land A^U(y)]. \]

The following is the immediate result of Definitions 4.1 and 4.5.

**Proposition 4.6.** Let \( A \) be an interval-valued fuzzy weakly completely prime ideal of a ring \( R \). Then for each \( [\lambda, \mu] \in D(I) \), \( x_{[\lambda, \mu]} \circ y_{[t, s]} \in A \) implies that either \( x_{[\lambda, \mu]} \in A \) or \( y_{[t, s]} \in A \). Furthermore, for each \( [\lambda, \mu] \in D(I) \) such that \( \lambda + \mu \leq 1 \), \( \lambda < A^L(0) \) and \( \mu < A^U(0) \), \( A^{[\lambda, \mu]} \) is a completely prime ideal of \( R \). In particular, \( A^{[0,0]} \) is a completely prime ideal of \( R \). Conversely if for each \( [\lambda, \mu] \in D(I) \), \( A^{[\lambda, \mu]} \) is a completely prime ideal then \( A \) is an interval-valued fuzzy weakly completely prime ideal.

The following is the example that an interval-valued fuzzy weakly completely prime ideal need not be an interval-valued fuzzy completely prime ideal.

**Example 4.7.** Let \( R = \mathbb{Z} \times \mathbb{Z} \), let \( S = \{0\} \times \mathbb{Z} \) and let \( T = (2) \times \mathbb{Z} \). We define a mapping \( A : R \rightarrow D(I) \) as follows: For each \( x \in R \),

\[
A(x) = \begin{cases} 
[1, 1], & \text{if } x \in S; \\
\left[\frac{1}{2}, \frac{3}{2}\right], & \text{if } x \in T \setminus S; \\
[0, 0], & \text{if } x \in R \setminus T.
\end{cases}
\]

Then clearly \( A \in D(I)^R \) from the definition of \( A \). Moreover, we can easily show that \( A \) is an interval-valued fuzzy weakly completely prime ideal but, by Proposition 4.2, \( A \) is not an interval-valued fuzzy weakly completely prime ideal.

References


