On set-valued Choquet integrals and convergence theorems(II)

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1. Introduction.

It is well-known that closed set-valued functions had been used repeatedly in many papers [1, 2, 5, 6, 7, 8, 9, 13, 15, 16]. We studied closed set-valued Choquet integrals in [7, 8] and convergence theorems under some sufficient conditions, for examples: (i) convergence theorems for monotone convergent sequences of Choquet integrably bounded closed set-valued functions (see [7]), (ii) convergence theorems for the upper limit and the lower limit of a sequence of Choquet integrably bounded closed set-valued functions (see [9]).

The aim of this paper is to prove convergence theorem for convergent sequences of Choquet integrably bounded interval number-valued functions in the metric $\Delta_{\alpha}$ (see Definition 3.4). In section 2, we list various definitions and notations which are used in the proof of convergence theorem and discuss some properties of measurable interval number-valued functions. In section 3, using these definitions and properties, we investigate main results.

2. Definitions and preliminaries.

Definition 2.1 [8, 12] (1) A fuzzy measure on a measurable space $(X, \mathcal{I})$ is an extended real-valued function $\mu : \mathcal{I} \to [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(A) \leq \mu(B)$, whenever $A, B \in \mathcal{I}$, $A \subset B$.

(2) A fuzzy measure $\mu$ is said to be autocontinuous from above (resp., below) if $\mu(A \cup B) \to \mu(A)$ (resp., $\mu(A \cap B) \to \mu(A)$) whenever $A \in \mathcal{I}$, $(B_n) \subset \mathcal{I}$ and $\mu(B_n) \to 0$.

(3) If $\mu$ is autocontinuous both from above and from below, it is said to be autocontinuous.

Recall that a function $f : X \to [0, \infty]$ is said to be measurable if $(x, \mu(x)) = \mu(x)$ for all $x \in (0, \infty)$.

Definition 2.2 [12] (1) A sequence $(f_n)$ of measurable functions is said to converge to $f$ in measure, in symbols $f_n \to f$ if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \mu_\varepsilon(f_n(x) - f(x)) = 0.$$ 

(2) A sequence $(f_n)$ of measurable functions is said to converge to $f$ in distribution, in symbols $f_n \to_d f$ if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \mu_{f_n}(r) = \mu(r), \quad e.c.,$$

where $\mu_{f_n}(r) = \mu((x, f_n(x)))$ and "e.c." stands for "except at most countably many values of $r$".

Definition 2.3 [10, 11, 12] (1) The Choquet integral of a
measurable function $f$ with respect to a fuzzy measure $\mu$ is defined by
\[
(C) \int f \, d\mu = \int_0^\infty \mu_r(r) \, dr
\]
where the integral on the right-hand side is an ordinary one.

(2) A measurable function $f$ is called integrable if the Choquet integral of $f$ can be defined and its value is finite.

Throughout the paper, $R^+$ will denote the interval $[0, \infty)$,
\[
(I) (R^+) = \{ [a, b] | a, b \in R^+ \text{ and } a \leq b \}.
\]

Then a element in $I(R^+)$ is called an interval number. On the interval number set, we define:

for each pair $[a, b], [c, d] \in I(R^+)$ and $\delta \in R^+$,
\[
[a, b] + [c, d] = [a + c, b + d],
\]
\[
[a, b] \cdot [c, d] = [a \cdot c, b \cdot d],
\]
\[
[a, b] = \{ ka, kb \}.
\]

Then $(R^+, d_H)$ is a metric space, where $d_H$ is the Hausdorff metric defined by
\[
d_H(A, B) = \max \{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y) \}
\]
for all $A, B \in I(R^+)$. By the definition of the Hausdorff metric, we have immediately the following proposition.

Proposition 2.4 For each pair $[a, b], [c, d] \in I(R^+),
\[
d_H([a, b], [c, d]) = \max \{ |a - c|, |b - d| \}.
\]

Let $(R^+)$ be the class of closed subsets of $R^+$. Throughout this paper, we consider a closed set-valued function $F : X \to (R^+) \setminus \{ \emptyset \}$ and an interval number-valued function $F : X \to (R^+) \setminus \{ \emptyset \}$. We denote that $d_H - \lim_{\omega} A_\omega = A$ if and only if
\[
\lim_{\omega} d_H(A_\omega, A) = 0, \text{ where } A \in (R^+) \text{ and } (A_\omega) \subset (R^+).
\]

Definition 2.5 [6,7] A closed set-valued function $F$ is said to be measurable if for each open set $O \subset R^+$,
\[
F^{-1}(O) = \{ x \in X | F(x) \cap (O \setminus \emptyset) \} \in \mathcal{F}.
\]

Definition 2.6 [1] Let $F$ be a closed set-valued function. A measurable function $f : X \to R^+$ satisfying
\[
f(x) = F(x) \text{ for all } x \in X
\]
is called a measurable selection of $F$.

We say $f : X \to R^+$ is in $L^1(\mu)$ if and only if $f$ is measurable and $(C) \int f \, d\mu < \infty$. We note that "$x \in X, \mu - a.e."$ stands for "$x \in X, \mu$-almost everywhere". The property $F(x)$ holds for $x \in X, \mu - a.e.$ means that there is a measurable set $A$ such that $\mu(A) = 0$ and the property $F(x)$ holds for all $x \in A^c$, where $A^c$ is the complement of $A$.

Definition 2.7 [6,7] (1) Let $F$ be a closed set-valued function and $A \in \mathcal{F}$. The Choquet integral of $F$ on $A$ is defined by
\[
(C) \int_A F \, d\mu = (C) \int_A f \, d\mu \text{ if } f \in S_c(F)
\]
where $S_c(F)$ is the family of $\mu - a.e$. Choquet integrable selections of $F$, that is,
\[
S_c(F) = \{ f \in L^1(\mu) | f(x) \in F(x) \text{ for all } x \in X, \mu - a.e. \}
\]
(2) A closed set-valued function $F$ is said to be Choquet integrable if $(C) \int F \, d\mu \neq \emptyset$.
(3) A closed set-valued function $F$ is said to be Choquet integrably bounded if there is a function $g \in L^1(\mu)$ such that
\[
\| F(x) \| = \sup_{f \in F(x)} \| f \| \leq g(x) \text{ for all } x \in X.
\]

Instead of $(C) \int F \, d\mu$, we will write $(C) \int F \, d\mu$. Let us discuss some basic properties of measurable closed set-valued functions. Since $R^+ = [0, \infty)$ is a complete separable metric space in the usual topology, using Theorem 8.1.3(1) and Theorem 1.0(2) of [5], we have the following two theorems.

Theorem 2.8 [1,5] A closed set-valued function $F$ is measurable if and only if there exists a sequence of measurable selections $(f_n)$ of $F$ such that
\[
F(x) = cl(\{ f_n(x) \}) \quad \text{for all } x \in X.
\]

Theorem 2.9 [1,5] If $F$ is a measurable closed set-valued function and Choquet integrably bounded, then it is Choquet integrable.

3. Main results.

In this section, we prove convexity of interval number-valued Choquet integrals and discuss the concepts of convergent sequences of measurable interval number-valued functions in the metric $\Delta_k$.

Since $(X, \mathcal{F})$ is a measurable space and $R^+ = (0, \infty)$ is a separable metric space, Theorem 1.0(2) of [5] implies the following theorem. Recall that a measurable closed set-valued function is said to be convex-valued if $F(x)$
is convex for all \( x \in X \) and that a set \( A \) is an interval number if and only if it is closed and convex.

**Theorem 3.1** If \( F \) is a measurable closed set-valued function and Choquet integrably bounded, then there exists a sequence \( \{ f_n \} \) of Choquet integrable functions \( f_n : X \to R^+ \) such that \( F(x) = cl(\{ f_n(x) \}) \) for all \( x \in X \).

**Proof.** By Theorem 1.0 (2) ([5]), there exists a sequence \( \{ f_n \} \) of measurable functions \( f_n : X \to R^+ \) such that \( F(x) = cl(\{ f_n(x) \}) \) for all \( x \in X \). Since \( F \) is Choquet integrably bounded, there is a measurable function \( g \in L^1(\mu) \) such that

\[
\| F(x) \| = \sup(\{ r | r \in F(x) \} \leq g(x)) \text{ for all } x \in X
\]

Since \( f_n(x) \in F(x) \) for all \( x \in X \) and all \( n = 1, 2, \ldots \), \( f_n(x) \leq g(x) \) for all \( x \in X \). By Proposition 3.2([11]),

\[
(C) \int f_n d\mu \leq (C) \int g d\mu < \infty , \text{ for all } n = 1, 2, \ldots
\]

So, \( f_n \) is Choquet integrable for all \( n = 1, 2, \ldots \). The proof is complete.

**Theorem 3.2** If \( F \) is a measurable closed set-valued function and Choquet integrably bounded and if we define

\[
f^*(x) = \sup(\{ r | r \in F(x) \})
\]

and

\[
f_*(x) = \inf(\{ r | r \in F(x) \})
\]

for all \( x \in X \), then \( f^* \) and \( f_* \) are Choquet integrable selections of \( F \).

**Proof.** Since \( F \) is Choquet integrably bounded, there exists a function \( g \in L^1(\mu) \) such that \( \| F(x) \| \leq g(x) \) for all \( x \in X \). Theorem 3.1 implies that there is a sequence \( \{ f_n \} \) of Choquet integrable selections of \( F \) such that

\[
F(x) = cl(\{ f_n(x) \}) \text{ for all } x \in X.
\]

Then

\[
f^*(x) = \sup(\{ r | r \in F(x) \}) = \sup f_n(x)
\]

and

\[
f_*(x) = \inf(\{ r | r \in F(x) \}) = \inf f_n(x).
\]

Since the supremum and the infimum of a sequence \( \{ f_n \} \) of measurable functions are measurable, \( f^* \) and \( f_* \) are measurable. And also, we have

\[
0 \leq f_*(x) \leq f^*(x) = \| F(x) \| \leq g(x) \text{ for all } x \in X.
\]

Since \( g \in L^1(\mu) \), \( f^* \) and \( f_* \) belong to \( L^1(\mu) \). By the closedness of \( F(x) \) for all \( x \in X \), \( f_n(x) \in F(x) \) and \( f^*(x) \in F(x) \) for all \( x \in X \). Therefore, \( f^* \) and \( f_* \) are Choquet integrable selections of \( F \).

**Assumption (A)** For each pair \( f, g \in S_2(F) \), there exists \( h \in S_2(F) \) such that \( f \sim h \) and

\[
(C) \int g d\mu = (C) \int h d\mu.
\]

We consider the following classes of interval number-valued functions:

\[
\mathcal{J} = \{ F : F : X \to (R^+) \text{ is measurable and Choquet integrably bounded} \}
\]

and

\[
\mathcal{J}_1 = \{ F \in \mathcal{J} | F \text{ satisfies the assumption(A)} \}.
\]

**Theorem 3.3** If \( F \in \mathcal{J}_1 \), then we have

1. \( cF \in \mathcal{J}_1 \) for all \( c \in R^+ \),

2. \( (C) \int F d\mu \) is convex,

3. \( (C) \int F d\mu = \{(C) \int f d\mu, (C) \int f d\mu \} \).

**Proof.** (1) The proof of (1) is trivial.

(2) If \( (C) \int F d\mu \) is a single point set, then it is convex. Otherwise, let \( y_1, y_2 \in (C) \int F d\mu \) and \( y_1 < y_2 \). Then, there exist \( f_1, f_2 \in S_2(F) \) such that

\[
y_1 = (C) \int f_1 d\mu \text{ and } y_2 = (C) \int f_2 d\mu.
\]

Further, let \( y = (C) \int y d\mu \) we need to a selection \( f \in S_2(F) \) with \( y = (C) \int f d\mu \). Since \( y = (y_1, y_2) \), there exists \( \lambda \in (0, 1) \) such that \( y = \lambda y_1 + (1 - \lambda) y_2 \). For above two selections \( f_1, f_2 \in S_2(F) \), the assumption (A) implies that there exists \( g \in S_2(F) \) such that \( f_1 \sim g \) and \( (C) \int g d\mu = (C) \int f_2 d\mu \). We define a function \( f = \lambda f_1 + (1 - \lambda) g \) and note that \( \lambda f_1 \sim (1 - \lambda) g \). Since \( F \) is interval number-valued, it is convex and hence

\[
f(x) = \lambda f_1(x) + (1 - \lambda) g(x) \in F(x)
\]

for \( x \in X \) \( \mu \)-a.e. By Theorem 5.6 ([11]) and Proposition 3.2 (2)[11],

\[
y = \lambda y_1 + (1 - \lambda) y_2 = (C) \int f_1 d\mu + (C) \int (1 - \lambda) f_2 d\mu
\]

\[
= \lambda (C) \int f_1 d\mu + (1 - \lambda) (C) \int f_2 d\mu
\]

\[
= \lambda (C) \int f_1 d\mu + (1 - \lambda) (C) \int g d\mu
\]

\[
= (C) \int (\lambda f_1 + (1 - \lambda) g) d\mu
\]

\[
= (C) \int f d\mu
\]

325
Thus, we have $f \in S_1(F)$ and

$$y = (C) \int f \, du = (C) \int F \, du.$$  

The proof of (2) is complete.

(3) We note that $f_n \leq f \leq f^*$ for all $f \in S_1(F)$. Thus, by Proposition 3.2(2)(11),

$$(C) \int f \, du \leq (C) \int f^* \, du$$

for all $f \in S_1(F)$. Theorem 3.2 implies

$$(C) \int f \, du, (C) \int f^* \, du \in (C) \int F \, du.$$  

By (2), $(C) \int F \, du$ is convex in $R^+$ and hence

$$(C) \int F \, du = [(C) \int f_\epsilon \, du, (C) \int f^* \, du].$$

We consider a function $\Delta_\epsilon$ on $\mathbb{T}_1$, defined by

$$\Delta_\epsilon(F, G) = \sup_{x \in X} d_{\epsilon, \mu}(F(x), G(x))$$

for all $F, G \in \mathbb{T}_1$. Then, it is easily to show that $\Delta_\epsilon$ is a metric on $\mathbb{T}_1$.

Definition 3.4 Let $F \in \mathbb{T}_1$. A sequence $(F_n) \subset \mathbb{T}_1$ converges to $F$ in the metric $\Delta_\epsilon$, in symbols, $F_n \rightarrow \epsilon, F$ if $\lim_{n \rightarrow \infty} \Delta_\epsilon(F_n, F) = 0$.

Theorem 3.5 (Convergence Theorem) Let $F, G, H \in \mathbb{T}_1$ and $(F_n)$ be a sequence in $\mathbb{T}_1$. If a fuzzy measure $\mu$ is autocontinuous and if $F_n \rightarrow \epsilon, F$ and $G \leq F \leq H$, then we have

$$d_H - \lim_{n \rightarrow \infty} (C) \int F_n \, du = (C) \int F \, du.$$  

Proof. By Proposition 2.4,

$$d_{\mu}(F_n(x), F(x)) = \max \{ |F_n(x) - f\epsilon(x)|, |f^*_\epsilon(x) - f\epsilon(x)| \}$$

for all $x \in X$, where

$$f\epsilon_n(x) = \inf \{ \epsilon \in F_n(x) \},$$  

$$f^*_\epsilon(x) = \sup \{ \epsilon \in F_n(x) \}$$

for $n = 1, 2, \ldots$,

$$f\epsilon(x) = \inf \{ \epsilon \in F(x) \},$$  

$$f^*(x) = \sup \{ \epsilon \in F(x) \}.$$  

Since $\Delta_\epsilon(F_n, F) \rightarrow 0$ as $n \rightarrow \infty$,

$$\sup_{x \in X} |f\epsilon_n(x) - f\epsilon(x)| \rightarrow 0$$

and

$$\sup_{x \in X} |f^*_\epsilon(x) - f^*(x)| \rightarrow 0.$$  

Given any $\epsilon > 0$, there exist two natural numbers $N_1, N_2$ such that $|F_n(x) - f\epsilon(x)| < \epsilon$ for all $n \geq N_1$ and all $x \in X$, and $|f^*_\epsilon(x) - f^*(x)| < \epsilon$ for all $n \geq N_2$ and all $x \in X$. We put $N = \max\{N_1, N_2\}$. Thus for each $n \geq N$,

$$\mu(x, \{|F_n(x) - f\epsilon(x)| > \epsilon\}) = \mu(\varnothing) = 0$$

and

$$\mu(x, \{|f^*_\epsilon(x) - f^*(x)| > \epsilon\}) = \mu(\varnothing) = 0.$$  

Then, clearly we have that for arbitrary $\epsilon > 0$,

$$\mu(x, \{|F_n(x) - f\epsilon(x)| \geq \epsilon\}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

and

$$\mu(x, \{|f^*_\epsilon(x) - f^*(x)| \geq \epsilon\}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$  

That is, $F_n \rightarrow \epsilon, F$, and $F \rightarrow \epsilon, F^*$ as $n \rightarrow \infty$. It is clearly to show that if $G \leq F \leq H$ then

$$\mu(F) \leq \mu(G) \leq \mu(H)$$

and

$$\mu(F) \leq \mu(G) \leq \mu(H)$$

for all $r \in R^+$,

where

$$g\epsilon(x) = \inf \{ \epsilon \in G(x) \},$$  

$$g^*(x) = \sup \{ \epsilon \in G(x) \},$$  

$$h\epsilon(x) = \inf \{ \epsilon \in H(x) \},$$  

$$h^*(x) = \sup \{ \epsilon \in H(x) \}.$$  

Since $\mu$ is autocontinuous, by Theorem 3.2(12), we have

$$\lim_{n \rightarrow \infty} (C) \int f_n \, du = (C) \int f \, du$$

and

$$\lim_{n \rightarrow \infty} (C) \int f_n \, du = (C) \int f \, du.$$  

Therefore,

$$d_{\mu}(C) \int F_n \, du, (C) \int F \, du$$

$$= \max \{ (C) \int f_n \, du - (C) \int f \, du, (C) \int f_n \, du - (C) \int f \, du \}$$

$$\rightarrow 0$$

as $n \rightarrow \infty$.

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References.